Recall def. of Screw

\[ \Theta = \text{angle of rotation} \]
\[ d = \text{length of translation} \]
\[ l = \text{line of screw axis} \]

\[ \rho = \frac{d}{\Theta} \]

\[ \text{screw} \triangleq \left( \Theta \frac{s_1}{\|s_1\|}, \Theta \frac{s_2}{\|s_2\|} \right) \]

\[ = \frac{1}{\|q\|} \left( \Theta q, \Theta q + dq \right) \]

Consider differential motion occurring over differential time, \( dt \)

\[ \Theta = \omega \, dt \quad \text{where } \omega \text{ is the angular velocity} \]

Plücker coords \( (q, q_0) = (\omega, p \times \omega) \)

\[ \text{pitch} = \frac{\|q\|}{\|\omega\|} \]

Screw coords for stiff twist:

\( (s, s_0) = (\omega, p \times \omega) \)
Screw coords for diff. twist: \( \omega \) plays role of \( q \).

\[
(s, s_0) = (\omega, p \times \omega + \frac{||N||}{||\omega||} \omega)
\]

Note that \( N \) is \( \perp \) to \( \omega \) (by construction).

\[
N = \frac{\omega}{||\omega||}
\]

\[
(s, s_0) = (\omega, p \times \omega + N)
\]

What is the physical interp?

Given a vector \( A \),

\( \omega \times A = \) velocity

or time rate of change of \( A \)

What is the twist? \( \rightarrow (\omega, \omega \times (-\dot{A}) + N) \)

\( \omega \times -p + N \)

= velocity of point attached to moving body that is instantaneously coincident with the origin.
Kinematic Constraints - First Order

Let \( N_p \) be a point of interest
Let \( \hat{u} \) be a direction, then

\[
\hat{u} \cdot N_p = 0
\]

constrains the point to move along the \( \hat{u} \) direction.

\[
\hat{u} \cdot N_p \geq 0
\]

Recall \( N_p = N_o + \omega \times p \)

\[
\hat{u} \cdot N_p = \hat{u} \cdot N_o + (\omega \times p) \cdot \hat{u}
\]

\[
= \hat{u} \cdot N_o + (p \times \hat{u}) \cdot \omega
\]

Looks like a reciprocal product defined for Plücker coordinates

\[
(q, q_o) \times (p, p_o) = q \cdot p_o - p \cdot q_o
\]

Define contact screw

\[
(C, C_o) = (\hat{u}, p \times \hat{u})
\]

Consideration of constraint gives rise to a "contact screw"
Kinematic Constraint can now be written as:

\[(c, c_0) \times (\omega, N_0) = c \cdot N_0 + c_0 \cdot \omega = 0\]

Called virtual product, because contact screw can be 0.

Definition: Two screws are reciprocal, contrary, or repelling if the reciprocal product is 0, < 0, or > 0.

Bilateral constraints must be reciprocal.
Unilateral constraints are reciprocal or repelling.

Note:
Every point on a rigid body has same \(\omega\).
Diff twist represents velocity of pt at origin.
In \((c, c_0)\), \(c\) represents direction of constraint.
\(c_0\) is moment of constraint about origin.
Example: Contact constraints for planar motion

Let contact points be at:

\[ r_1 = (0, 0, 0) \]
\[ r_2 = (1, 0, 0) \]
\[ r_3 = (0, 1, 0) \]

Contact screws, \((C_i, Co_i)\), \(i = 1, 2, 3\)

1) \(0, 0, -1; 0, 0, 0\)  \(s\)  \(r \times s = s_{0}\)
2) \(0, 0, 1; 0, -1, 0\)
3) \(0, 0, -1; -1, 0, 0\)

Recall that moment about origin = moment about each of the axes.

Moment between \(\|\) lines = 0

Moment \(\|\) intersecting ""
How can we determine possible motion of "grasped" triangle?

Let \((t, t_0) = (\omega, N_0)\) be the diff. twist of the \(\Delta\).

Assume bilateral contacts.

\[(t, t_0) \cdot (c_i, C_{oi}) = 0 \quad \forall \ i = 1, 2, 3.\]

\[t \cdot C_{oi} + t_0 \cdot C_{\hat{o}} = 0 \quad \forall \ i = 1, 2, 3\]

\[C_{\hat{o}} \cdot N_0 + C_{oi} \cdot \omega = 0\]

\[
\begin{bmatrix}
C_1 & C_{o1} \\
C_2 & C_{o2} \\
C_3 & C_{o3}
\end{bmatrix}
\begin{bmatrix}
N_0 \\
\omega
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
N_{ox} \\
N_{oy} \\
N_{oz} \\
\omega_x \\
\omega_y
\end{bmatrix}
= 0
\]

\[\text{det} = 1\]

Note that \(N_{ox}, N_{oy}, \omega_z\) are unconstrained! Planar motion.

\[\begin{align*}
&-N_{oz} = 0 \\
&N_{oz} - \omega y = 0 \Rightarrow \omega y = 0 \\
&-N_{oz} - \omega x = 0 \Rightarrow \omega x = 0
\end{align*}\]
1) Twist axis direction
\((t, t_0) = (\omega, \omega_0)\)

\[
\begin{bmatrix}
0 \\
0 \\
\omega_z
\end{bmatrix}
\]

2) Find twist pitch

\((t, t_0) = (\omega, \omega_0)\)

\[
= \begin{bmatrix}
0 \\
\omega_z \\
\omega_y
\end{bmatrix}
\]

3) Find closest point on twist axis to origin (instantaneous center of rotation)

\[
\frac{tx t_0}{t \cdot t} = \frac{\omega x \omega_0}{\omega \cdot \omega} = \begin{bmatrix}
-\frac{\omega_0 y \omega_z}{\omega_0 x \omega_z} \\
\frac{\omega_0 y}{\omega_0 x} \\
0
\end{bmatrix}
\]

Let \(\omega_z = \frac{1}{2}\)

Let \(\omega_0 = \begin{bmatrix}
-1 \\
0 \\
0
\end{bmatrix}\)

Suppose we use a different origin?
Suppose we use a different origin

\( r_1 = (1, 0, 0) \)
\( r_2 = (2, 0, 0) \)
\( r_3 = (1, 1, 0) \)

Contact screws

\( s_1, s_2, s_3 \) are not changed

\( r \times s_i \) is changed by

\[
\begin{pmatrix}
0 & 0 & -1, 0 & 1 & 0 \\
0 & 0 & 1, 0 & -2 & 0 \\
0 & 0 & -1, 0 & 1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
N_{0x} \\
N_{0y} \\
N_{0z} \\
w_x \\
w_y \\
w_z
\end{pmatrix}
\]

Determinant

Still \( \neq 0 \)

\( = 1 \)

Check for freedoms.

Same result as before, \( N_{0x} = w_x = w_y = 0 \)

Freedoms: \( N_{0x}, N_{0y}, w_z \)
1. Check twist axis direction

\[ \omega = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix} \quad \text{Same as before} \]

2. Find pitch

\[ \rho = \frac{t \cdot t_o}{t \cdot t} = \frac{\begin{pmatrix} 0 \\ \omega_z \\ N_y \end{pmatrix} \cdot \begin{pmatrix} 0 \\ N_z \\ 0 \end{pmatrix}}{t \cdot t} = 0 \quad \text{Same as before.} \]

3. Find closest point on twist axis to origin

Only one difference from before.

Since origin was changed,

\[ (N_y, N_z) \]

specifies velocity of \( \cdot \)
Further constraint in the plane

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
N_{ox} \\
N_{oy} \\
S_{ox} \\
S_{oy} \\
\omega_2
\end{bmatrix}
\]

already eliminated by contacts 1, 2, 3.

Can this move?

Assume bilateral constraints

\[
\begin{bmatrix}
0 & 1 & 1 \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
N_{ox} \\
N_{oy} \\
\omega_2
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\text{Det}(A) = \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0
\]

Null Space is 1D

\[
\begin{bmatrix}
N_{ox} \\
N_{oy} \\
\omega_2
\end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \lambda \in \mathbb{R}
\]

Location of center of rotation

\[
\frac{\text{txt}_0}{t \cdot t} = \frac{\omega \times N_S}{\omega \cdot \omega} = \frac{1}{\omega_2} \begin{bmatrix}
-N_{oy} \\
N_{ox}
\end{bmatrix}
\]
Suppose we move frame of representation?

Constraints

\[
\begin{bmatrix}
0 & 1 & 2 \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
N_{0x} \\
N_{0y} \\
\omega_z
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
N_{0x} \\
N_{0y} \\
\omega_z
\end{bmatrix}
= \text{Null Sp}(A) \gamma
\]

\[\text{arbitrary scalar}\]

Matlab gives

\[
\begin{bmatrix}
N_{0x} \\
N_{0y} \\
\omega_z
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-2 \\
1
\end{bmatrix} \gamma
\]

\[\text{Intuitive Check}\]

\[\text{Center of rotation is} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\]
Revisit Reuleaux's Method

Repelling screws

\((c, c_0) \times (\omega, \omega_0) \geq 0\)

\[(\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \omega & 0 & 0 & \omega_0 \\ 0 & 0 & \omega & 0 \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}) \begin{bmatrix} N_{o,x} \\ N_{o,y} \\ N_{o,z} \\ \omega \end{bmatrix} = 0\]

\(\Rightarrow N_{o,y} \geq 0\)

\(\Rightarrow N_{o,x}, \omega_2\) arbitrary

What are IC locations such that \(N_{o,y} \geq 0\)?

\[
\begin{bmatrix}
-\frac{N_{o,y}}{\omega_2} \\
\frac{N_{o,x}}{\omega_2} \\
0
\end{bmatrix}
\]

Must have \(N_{o,y} \geq 0\)

Case 1: \(\omega_2 > 0\), \(N_{o,y} > 0\)

\(\Rightarrow p_x < 0\)

Case 2: \(\omega_2 < 0\), \(N_{o,y} > 0\)

\(\Rightarrow p_x > 0\)
Contact constraint example: grasping

Let \((c_i, co_i)\) be the contact screw for the \(i\)th contact in a grasp/fixture.

Let \((\omega, \nu_i)\) be the differential twist of the object.

Determine instantaneous motions allowed by grasp.

\[
\begin{align*}
S_1 &= (-1, 0, 0) & S_{o1} &= (0, 0, 1) \\
S_2 &= (0, 1, 0) & S_{o2} &= (1, -1, 0) \\
S_3 &= (0, -1, 0) & S_{o3} &= (0, 0, -1) \\
S_4 &= (1, 0, 0) & S_{o4} &= (0, 1, 0) \\
S_5 &= (0, 0, -1) & S_{o5} &= (0, 0, 0) \\
S_6 &= (0, 1, 0) & S_{o6} &= (-1, 0, 0)
\end{align*}
\]

Note: moment about origin = (moment about x-axis, y-axis, z-axis)

Moment between \(\parallel\) lines = 0

Moment between intersecting lines = 0
Assume constraints are bilateral differential
Find legal twists if any exist.

\[
\begin{bmatrix}
-1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
N_{0x} \\
N_{0y} \\
N_{0z} \\
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

⇒ \( N_{0z} = 0 \)

Null space has 4 dimensions

\[
\text{Null}(A) = \begin{bmatrix}
-1 \\
1 \\
0 \\
1 \\
-1
\end{bmatrix}
\Rightarrow \{ \text{corner of cube at origin cannot move up or down} \}
\]

\[
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
\Rightarrow \{ \text{axis of rotation is along line II to cube diagonal between grasp points} \}
\]

Note, if origin had been chosen to lie on axis of rotation, then first 3 elements would be zero

Point closest to origin

\[
\frac{txc}{t \cdot t} = \frac{1}{3} \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
1/3 \\
1/3 \\
2/3
\end{bmatrix}
\]
Points seems wrong.

Test to see if it is on axis.

\[ \vec{p}_2 - \vec{p}_1 = \left( \frac{2}{3}, \frac{2}{3}, -\frac{2}{3} \right) \]

cross with \((1, 1, -1)\) \[
\left\{ \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \right\} = \left[ \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \right]
\]

Yes! It is on the line.
A final point about screw systems.

Choose your coordinate frame to make your own life easy.

Results are

\( (C, c_0, \omega_t, t_0) \) are independent of coordinate frame location & orientation!