Summarize

3D Rigid Body Dynamics

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20.1.1

$$m\,{}^N\dot{v} = {}^N F$$

$${}^N I\,{}^N\dot{\omega} = {}^N N - {}^N\omega \times {}^N I\,{}^N\omega$$

$${}^N\dot{x} = {}^N v$$

$${}^N\dot{e} = \left(B(q)\,{}^B_N R\right){}^N\omega$$

where ${}^N I = {}^N_B R\,{}^B I\,{}^B_N R$ — ${}^N I$ not constant

${}^B I$ is constant

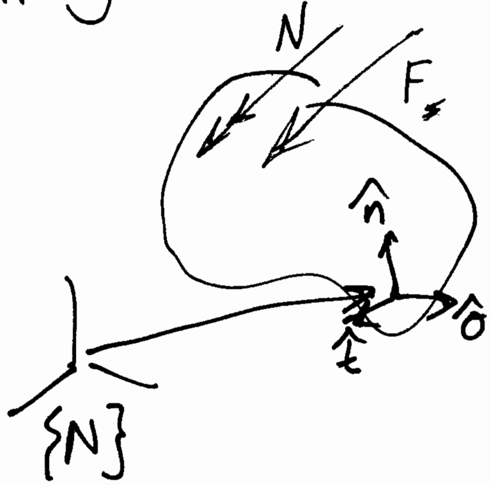
$$M\,{}^N\dot{\nu} = \begin{bmatrix} {}^N F \\ {}^N N \end{bmatrix} + \begin{bmatrix} 0 \\ \hline -{}^N\omega \times {}^N I\,{}^N\omega \end{bmatrix}$$

$$\dot{q} = G(q)\,{}^N\nu$$

$$\|e\| = 1$$

where $M = \left[\begin{array}{ccc|c} m & & & \\ & m & & 0 \\ & & m & \\ \hline & 0 & & {}^N_B R\,{}^B I\,{}^B_N R \end{array}\right]_{(6\times 6)}$, $G(q) = \left[\begin{array}{ccc|c} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ \hline & 0 & & \underset{(4\times 3)}{B(q)}\,\underset{(3\times 3)}{{}^B_N R} \end{array}\right]_{(7\times 6)}$

Now include contact forces
and get back to time stepping



Contact ~~Force~~ Wrenches

$$\begin{bmatrix} {}^{*}F \\ {}^{*}N \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{n} \\ r\times\hat{n} \end{bmatrix}_{(6\times1)} \lambda_n}_{\text{frictionless}} + \overbrace{\begin{bmatrix} \hat{t} \\ r\times\hat{t} \end{bmatrix}_{(6\times1)} \lambda_t + \begin{bmatrix} \hat{o} \\ r\times\hat{o} \end{bmatrix}_{(6\times1)} \lambda_o}^{\text{friction}}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ \hat{n} \end{bmatrix}_{(6\times1)} \beta_n + \begin{bmatrix} 0 \\ \hat{t} \end{bmatrix}_{(6\times1)} \beta_t + \begin{bmatrix} 0 \\ \hat{o} \end{bmatrix}_{(6\times1)} \beta_o}_{\text{moments}}$$

+ other forces

Assume point contacts w/ friction

$$ {}^{N}M\,{}^{N}\dot{\nu} = {}^{N}W\lambda + {}^{N}g_{ext} = {}^{N}W_n\lambda_n + {}^{N}W_t\lambda_t + {}^{N}W_o\lambda_o + {}^{N}g_{ext} $$

$$ {}^{N}\dot{q} = {}^{N}G\,{}^{N}\nu $$

$\boxed{\text{ADD FRICTION CONSTRAINTS}}$

$$ W_n = \begin{bmatrix} {}^{N}\hat{n}_1 & {}^{N}\hat{n}_2 & \cdots & {}^{N}\hat{n}_{n_c} \\ {}^{N}r_1 \times {}^{N}\hat{n}_1 & {}^{N}r_2 \times {}^{N}\hat{n}_2 & & {}^{N}r_{n_c} \times {}^{N}\hat{n}_{n_c} \end{bmatrix} \qquad \lambda_n = \begin{bmatrix} \lambda_{1n} \\ \lambda_{2n} \\ \vdots \\ \lambda_{n_c n} \end{bmatrix} $$

Same for W_t, W_o, λ_t, & λ_o

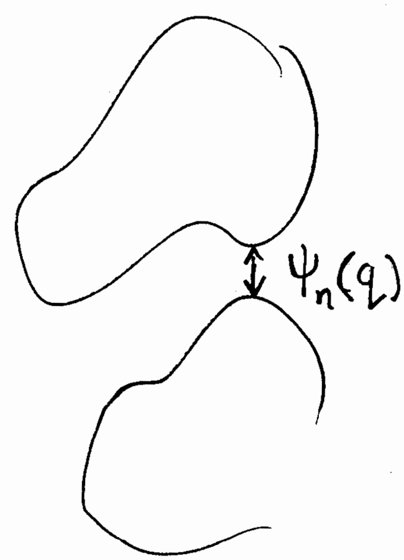
$$ g_{ext} = \begin{bmatrix} \text{gravity,} \\ \text{drag,} \\ \text{etc} \end{bmatrix} + \begin{bmatrix} 0 \\ -{}^{N}\omega \times {}^{N}I\,{}^{N}\omega \end{bmatrix} $$

Now we have to turn this into an LCP time-stepping method.

Assume we can write the gap at potential contact point

$$ \Psi_n(q) \geq 0 $$

possibly explicit $\Psi_n(q,t) \geq 0$



Let $\dot{\nu} \simeq \dfrac{\nu^{l+1} - \nu^{l}}{h}$, $\dot{q} = \dfrac{q^{l+1} - q^{l}}{h}$

$$M \frac{\nu^{l+1} - \nu^{l}}{h} \cong W_n^{l} \lambda_n^{l+1} + W_t^{l} \lambda_t^{l+1} + W_o^{l} \lambda_o^{l+1} + g_{ext}^{l}$$

$$\frac{q^{l+1} - q^{l}}{h} \cong G^{l} \nu^{l+1} \longleftarrow \text{same logic as before to use } \nu^{l+1}$$

$$0 \leq \Psi_n \perp \lambda_n \geq 0$$

And a friction model in similar form

Frictionless First

Aim for consistency at end of time step

$$\boxed{\begin{aligned} \nu^{l+1} &= \nu^{l} + hM^{-1}(W_n \lambda_n^{l+1} + g_{ext}) \\ q^{l+1} &= q^{l} + hG \nu^{l+1} \\ 0 \leq \Psi_n^{l+1} &\perp \lambda_n^{l+1} \geq 0 \end{aligned}}$$

To keep things linear, approx Ψ_n^{l+1} via Taylor Series.

- Include $\psi_n^{l+1} \geq 0$

$$\psi_n^{l+1} = \psi_n^l + \frac{\partial \psi_n^l}{\partial q}(q^{l+1} - q^l) + \frac{\partial \psi_n^l}{\partial t} h \geq 0$$

Divide by h

$$\Rightarrow \psi_n^l / h + \underbrace{\frac{\partial \psi_n^l}{\partial q} G}_{W_n^T} \nu^{l+1} + \frac{\partial \psi_n^l}{\partial t} \geq 0$$

- So now we have a Mixed LCP

$$\cancel{0 \leq W_n^T \nu^{l+1} + \psi_n^l / h + \frac{\partial \psi_n^l}{\partial t} \perp \lambda_n^{l+1} \geq 0}$$

$$\boxed{\begin{aligned} \nu^{l+1} &= \nu^l + h M^{-1}(W_n \lambda_n^{l+1} + g_{ext}) \\ q^{l+1} &= q^l + h G \nu^{l+1} \\ 0 &\leq W_n^T \nu^{l+1} + \psi_n^l / h + \frac{\partial \psi_n^l}{\partial t} \perp \underbrace{h \lambda_n^{l+1}}_{\text{normal impulse}} \geq 0 \end{aligned}}$$

The Path Alg solves this problem readily
CPNET.ORG – Michael Ferris

- But sometimes better to simplify

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Let $p_n^{l+1} = h\lambda_n^{l+1}$ be impulse

Rewrite equations to solve

Mixed C.P.

$$M\nu^{l+1} - W_n p_n^{l+1} - M\nu^l - \cancel{h g_{ext}}^{\,p_{ext}} = 0$$

$$0 \le W_n^T \nu^{l+1} + \frac{\psi_n^l}{h} + \frac{\partial \psi_n^l}{\partial t} \perp p_n^{l+1} \ge 0$$

Solve this first, then solve

$$q^{l+1} = q^l + hG\nu^{l+1}$$

But we're not quite there yet.

$$\Theta(q) = e_0^2 + e_1^2 + e_2^2 + e_3^2 - 1 = 0$$

$$\Theta^{l+1} = 0 = \left(\Theta^l + \frac{\partial \Theta^l}{\partial q}\left(q^{l+1} - q^l\right)\right)/h$$

$$\boxed{\underbrace{\frac{\partial \Theta^l}{\partial q}}_{W_E^T} G\nu^{l+1} + \frac{\Theta^l}{h} = 0}$$

In mechanics there is always duality between constraint and force

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In corporate to give a mixed LCP

$$\begin{bmatrix} 0 \\ 0 \\ \rho^{l+1} \end{bmatrix} = \begin{bmatrix} M\nu^{l+1} - W_E p_E^{l+1} - W_n p_n^{l+1} - M\nu^{l} - \overset{p_{ext}}{\cancel{h\,g_{ext}}} \\ W_E^T \nu^{l+1} + \Theta^l/h \\ W_n^T \nu^{l+1} + \psi_n^l/h + \frac{\partial \psi_n^l}{\partial t} \end{bmatrix} \begin{matrix} \cancel{\geq 0} \\ \cancel{\geq 0} \\ \geq 0 \end{matrix}$$

invert to eliminate ν^{l+1}, ρ^{l+1}

$$\begin{bmatrix} 0 \\ 0 \\ \rho^{l+1} \end{bmatrix} = \underbrace{\begin{bmatrix} M & -W_E & -W_n \\ W_E^T & 0 & 0 \\ W_n^T & 0 & 0 \end{bmatrix}}_{(7+n_c)\times(7+n_c)} \begin{bmatrix} \nu^{l+1} \\ p_E^{l+1} \\ p_n^{l+1} \end{bmatrix} + \underbrace{\begin{bmatrix} -M\nu^{l+1} - \overset{p_{ext}}{\cancel{h\,g_{ext}}} \\ \Theta^l/h \\ \psi_n^l/h + \partial\psi_n^l/\partial t \end{bmatrix}}_{(7+n_c\times 1)}$$

$$0 \leq \underset{(n_c\times 1)}{\rho_n^{l+1}} \perp \underset{(n_c\times 1)}{p_n^{l+1}} \geq 0$$

constraint stabilization

Pathmcp$(z, l, u, \text{cpfunjac})$

find $z \ni$

$l_i \leq z_i < u_i \Rightarrow F_i(z) = 0$

$l_i = z_i \Rightarrow F_i(z) \geq 0$

~~Let's add friction~~

$u_i = z_i \Rightarrow F_i(z) \leq 0$

Let $u \to \infty$ $\quad l = 0$

$\Rightarrow$ standard NCP

if $F_i(z)$ linear, then

$\Rightarrow$ std. LCP

To make MLCP, F_i must be linear and set some lower bounds to $-\infty$

3D Dynamics – Contact Friction

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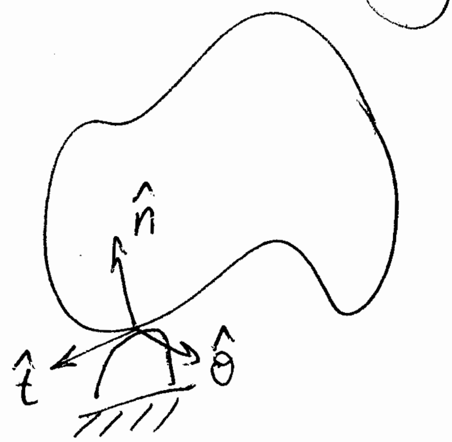
$$f = \hat{n}\lambda_n + \hat{t}\lambda_t + \hat{\theta}\lambda_\theta$$

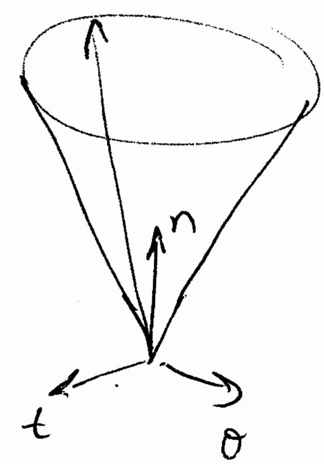
$$r \times f = r \times \hat{n}\lambda_n + r \times \hat{t}\lambda_t + r \times \hat{\theta}\lambda_\theta$$



Friction Model – ~~When Slid~~

Friction acts to maximize rate at which energy is dissipated

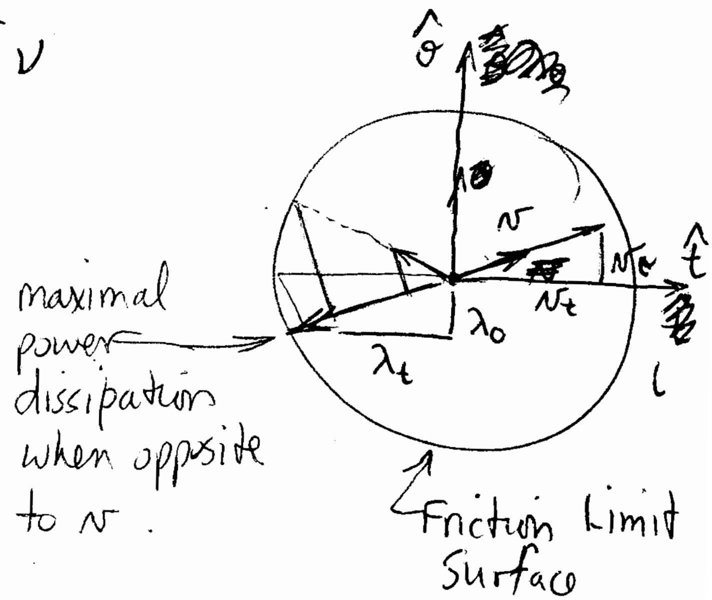
Friction force lies with a cone.



Sliding or Rolling → $\lambda_t^2 + \lambda_\theta^2 \le \mu^2 \lambda_n^2$

Sliding $(v_t, v_\theta) \neq 0$ $\quad (\lambda_t, \lambda_\theta) \in \operatorname{argmax}\{-v_t\lambda_t' - v_\theta\lambda_\theta' :$
$\lambda_t^2 + \lambda_\theta^2 \le \mu^2\lambda_n^2\}$

where $v_t = W_t^T \nu$, $\quad v_\theta = W_\theta^T \nu$



When sliding we can solve for λ_t, λ_o :

$$\lambda_t = \frac{-\mu \lambda_n v_t}{\sqrt{v_t^2 + v_o^2}} \qquad \lambda_o = \frac{-\mu \lambda_n v_o}{\sqrt{v_t^2 + v_o^2}}$$

NONLINEAR CONSTRAINTS

If we knew the approximate sliding direction, then we could linearize with Taylor series

But we don't!

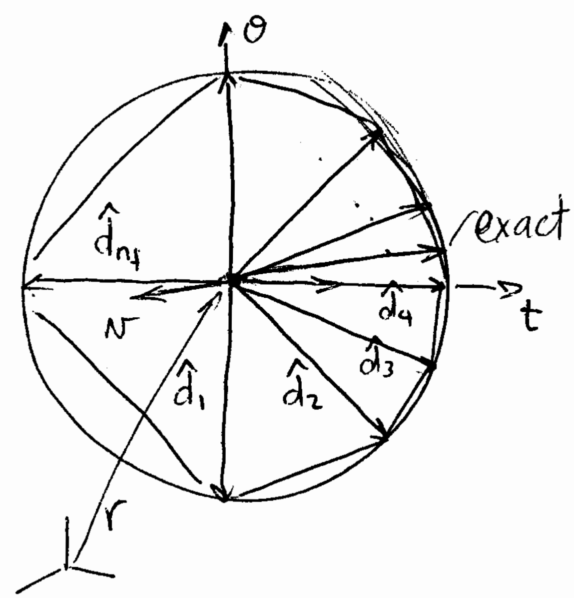
And $\sqrt{v_t^2 + v_o^2}$ can go to zero!

Approximate Friction Limit Surface as a Polygon

Friction force:

$$\hat{d}_1 \lambda_{1f} + \hat{d}_2 \lambda_{2f} + \cdots + \hat{d}_{n_f} \lambda_{n_f f}$$

$$\lambda_{if} \geq 0 \quad \forall\ i$$



Friction moment:

$$r \times \hat{d}_1 \lambda_{1f} + \cdots + r \times \hat{d}_{n_f} \lambda_{n_f f}$$

Friction Wrench

$$W_f \lambda_f \qquad W_f = \underset{(6 \times n_f)}{\begin{bmatrix} \hat{d}_1 & \cdots & \hat{d}_{n_f} \\ r \times \hat{d}_1 & & r \times \hat{d}_{n_f} \end{bmatrix}} \qquad \lambda_f = \underset{(n_f \times 1)}{\begin{bmatrix} \lambda_{1f} \\ \lambda_{2f} \\ \vdots \\ \lambda_{n_f f} \end{bmatrix}} \geq 0$$

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Now sum of $\overset{\text{contact}}{\wedge}$ forces is:

$$W_n \lambda_n + W_f \lambda_f$$

$$\lambda_n, \lambda_f \geq 0$$

$$\lambda_n = \begin{bmatrix} \lambda_{1n} \\ \lambda_{2n} \\ \vdots \\ \lambda_{n_c n} \end{bmatrix} \quad \lambda_f = \begin{bmatrix} \lambda_{1f} \\ \vdots \\ \lambda_{n_c f} \end{bmatrix}$$

where $\lambda_{1f} = \begin{bmatrix} \lambda_{1f_1} \\ \lambda_{1f_2} \\ \vdots \\ \lambda_{1f_{n_d}} \end{bmatrix}$

Over a small time step, impulses

$$W_n p_n + W_f p_f$$

$$p_n, p_f \geq 0 \quad ; \quad p_n = h\lambda_n \ , \ p_f = \lambda_f h$$

How do we write constraints that pick best friction force?

Generalize previous approach

$$0 \leq W_f^T \nu + E s \perp p_f \geq 0$$

$$0 \leq U p_n - E^T p_f \perp s \geq 0$$

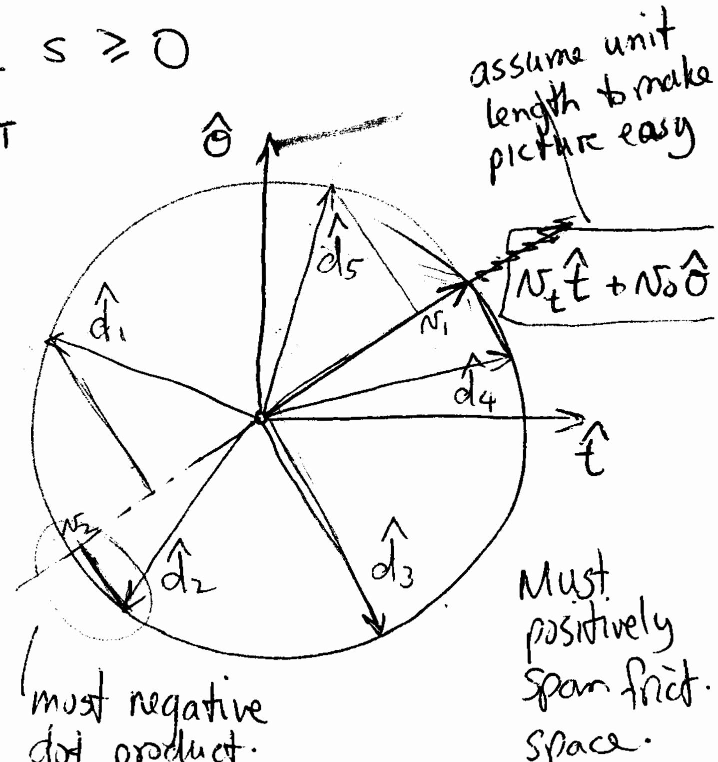
where $E = (1, 1, \ldots, 1)^T$

What is $W_f^T \nu$?

Let $W_t^T \nu = \nu_t$

$W_o^T \nu = \nu_o$

$$W_f^T \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_5 \end{bmatrix}$$



Expand First LC condition

$$\left.\begin{aligned} s &\geq -v_1 \\ s &\geq -v_2 \\ &\vdots \\ s &\geq -v_5 \end{aligned}\right\} \Rightarrow \quad s \geq \max_i (-v_i)$$

Only row(s) ~~with~~ i can have $s = -v_i$

$\therefore \quad p_{jf} = 0 \quad \forall j \neq i$

Expand second condition

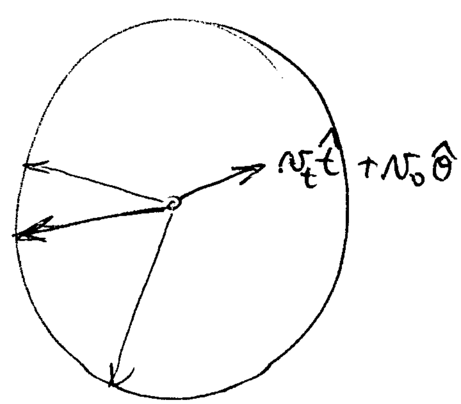
$$0 \leq \mu p_n - p_{1f} - p_{2f} - p_{3f} - p_{4f} - p_{5f} \perp s \geq 0$$

If sliding, $s > 0$, so $\mu p_n = p_{1f} + \cdots + p_{5f}$

$\therefore$ at least one of $p_{if} \neq 0$

Therefore $s = \max_i (-v_i)$, so $p_{if} \geq 0$

So friction force will not oppose sliding direction exactly! But will be close & will have proper magnitude



Implementing Time Stepping

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$$\begin{bmatrix} 0 \\ 0 \\ \rho_n^{l+1} \\ \rho_f^{l+1} \\ \sigma^{l+1} \end{bmatrix} = \begin{bmatrix} M^l & (-W_E)^l & (-W_n)^l & -W_f^l & 0 \\ (W_E^T)^l & 0 & 0 & 0 & 0 \\ (W_n^T)^l & 0 & 0 & 0 & 0 \\ (W_f^T)^l & 0 & 0 & 0 & E^l \\ 0 & 0 & (U)^l & (-E^T)^l & 0 \end{bmatrix} \begin{bmatrix} \nu^{l+1} \\ p_E^{l+1} \\ p_n^{l+1} \\ p_f^{l+1} \\ s^{l+1} \end{bmatrix} + \begin{bmatrix} -M^l \nu^l - p_{ext}^l \\ \Theta^l / h + \partial \Theta^l / \partial t \\ \Psi_n^l / h + \partial \Psi_n^l / \partial t \\ 0 \\ 0 \end{bmatrix}$$

$$0 \leq \begin{bmatrix} \rho_n^{l+1} \\ \rho_f^{l+1} \\ \sigma^{l+1} \end{bmatrix} \perp \begin{bmatrix} p_n^{l+1} \\ p_f^{l+1} \\ s^{l+1} \end{bmatrix} \geq 0$$

Solve for z^{l+1} then substitute into

$$q^{l+1} = q^l + G^l \nu^{l+1}$$

Examples

1. Parking a blimp against ceiling, front, & left walls.

2. Disc spinning & sliding on flat ground

3. Box being pushed along ground by position-controlled pusher

4. Mobile robot with two drive wheels pushing ball

5. Other example relevant to your project

- Determine how to compute:

$$M^l,\ W_E^l,\ W_n^l,\ W_f^l,\ U^l,\ E^l,\ G^l,\ p_{ext}^l,\ \Theta^l,\ \frac{\partial \Theta^l}{\partial t},\ \Phi_n^l,\ \frac{\partial \Phi_n^l}{\partial t}$$

- Use differing numbers of friction direction vectors at the contacts.

Pushing Box

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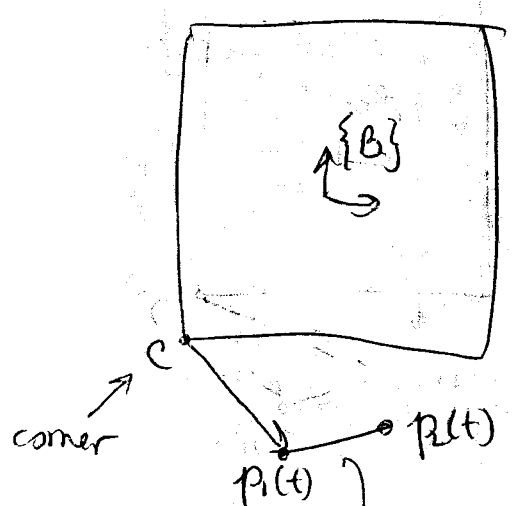
$$\Psi_n(t) = \begin{bmatrix} \hat{n}\cdot(p_1(t) - c) \\ \hat{n}\cdot(p_2(t) - c) \end{bmatrix}$$



trajectory of pusher is given as a fcn of time, $x(t), y(t), \ldots$ i.e. $q(t)$

$\hat{n}, c$ are fcns of q

$$\frac{\partial \Psi_n}{\partial t} = \begin{bmatrix} \hat{n}\cdot \partial p_1/\partial t \\ \hat{n}\cdot \partial p_2/\partial t \end{bmatrix}$$

$$\underset{6\times 2}{W_n} = G^T\left(\frac{\partial \Psi}{\partial q}\right)^T$$

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Mobile Robot with Caster

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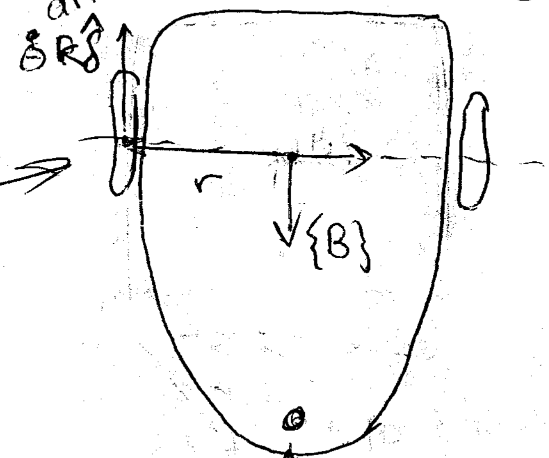
driven velocity of wheel contact point. Speed is $\dot\theta R$ (wheel ang. vel. times radius)



Treat wheels as points on body that are controlled to move relative to ~~body~~ chassis.

treat as contact point with $\mu = 0.1$

Nasty Trick

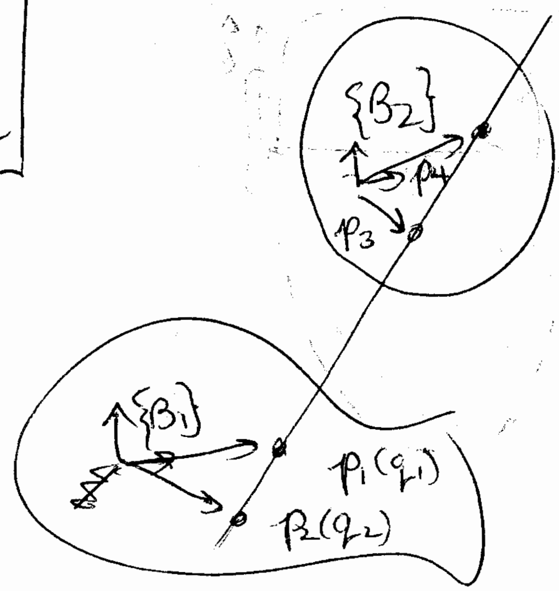
$$0 \le p_f^{l+1} \perp W_f^{l\,T}(\nu^{l+1}) + \cancel{\dot\theta R \hat\delta \cdot \begin{bmatrix} \hat d_1 \\ \hat d_2 \\ \hat d_3 \\ \vdots \\ d_{n_f} \end{bmatrix}} + E s^{l+1} \ge 0$$

$$+ \begin{bmatrix} \hat d_1^T \\ \hat d_2^T \\ \vdots \\ \hat d_{n_f}^T \end{bmatrix} \hat\delta \dot\theta R$$

Prismatic Joint Between Two Bodies

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}$$

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$



Planar Case

$\Theta(q) \Rightarrow 2$ equations

Use $p_1(q_1)$ & $p_2(q_1)$ to write eq. of line ~~as~~ $\mathcal{L}_1$ as function of q_1.

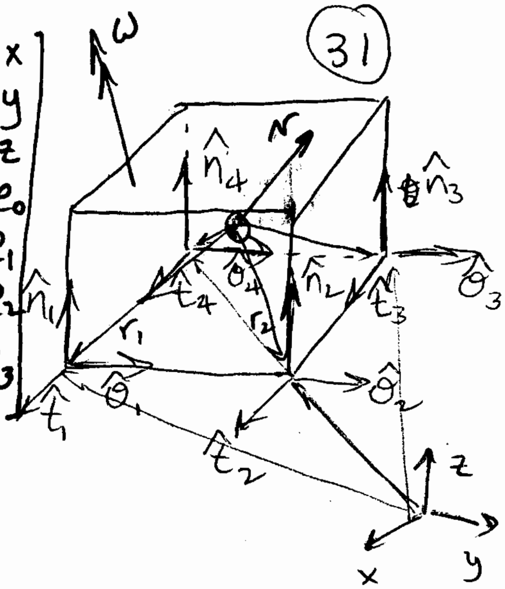
Then write two eqs that force $p_3(q_2)$ & $p_4(q_2)$ to lie on the line $\mathcal{L}_1$

Spatial Case

$\Theta(q) \Rightarrow 5$ equations

An Example: Box on Floor

$$\nu = \begin{bmatrix} N_x \\ N_y \\ N_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad q = \begin{bmatrix} x \\ y \\ z \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}$$



Assume Body-Fixed Frame is principal axes

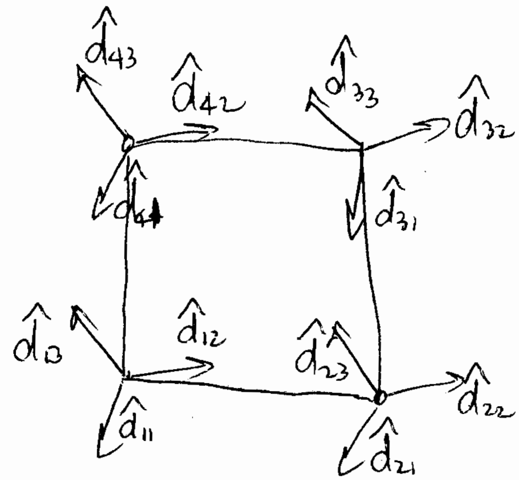
$$^{B}J = \mathrm{diag}(1, 2, 3)$$

$$m = 1$$

r's from c.g. to contact point.

$$^{N}W_n = \begin{bmatrix} ^{N}\hat{n}_1 & \cdots & ^{N}\hat{n}_4 \\ ^{N}r_1 \times {}^{N}\hat{n}_1 & \cdots & ^{N}r_4 \times {}^{N}\hat{n}_4 \end{bmatrix}$$

$$\cancel{^{N}W_f = \begin{bmatrix} ^{N}W_t & -W_t \end{bmatrix}}$$



$$^{N}W_f = \begin{bmatrix} ^{N}\hat{d}_{11} & \hat{d}_{12} & \hat{d}_{13} & \hat{d}_{21} & \cdots \\ ^{N}r_{11} \times {}^{N}\hat{d}_{11} & r_1 \times \hat{d}_{12} & r_1 \times \hat{d}_{13} & r_2 \times \hat{d}_{21} & \cdots \end{bmatrix}$$

$$M = \left[\begin{array}{ccc|c} m & & 0 & \\ & m & & 0 \\ 0 & & m & \\ \hline & 0 & & ^{N}I \end{array}\right] \qquad p_{ext} = \left[\begin{array}{c} 0 \\ 0 \\ -mg \\ \hline h(^{N}\omega \times {}^{N}I\,{}^{N}\omega) \end{array}\right]$$

$$\cancel{W_E =} \quad \Theta' = e_0^2 + e_1^2 + e_2^2 + e_3^2 - 1 = 0 \qquad [0 \quad 0 \quad 0 \quad 2e_0 \quad 2e_1 \quad 2e_2 \quad 2$$

$$W_E^T = \frac{\partial \Theta}{\partial q} G(q) \qquad W_E = G^T(q)\left(\frac{\partial \Theta}{\partial q}\right)^T \quad \text{where} \left(\frac{\partial \Theta}{\partial q}\right)^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2e_0 \\ 2e_1 \\ 2e_2 \\ 2e_3 \end{bmatrix}$$

$$U = \begin{bmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \mu_3 & \\ 0 & & & \mu_4 \end{bmatrix}_{(n_c \times n_c)}$$

$$E^T = \begin{bmatrix} 1\ 1\ 1 & & & \\ & 1\ 1\ 1 & & \\ & & 1\ 1\ 1 & \\ & & & 1\ 1\ 1 \end{bmatrix}_{(n_c \times (n_c \cdot n_f))}$$

$$\frac{\partial \Theta}{\partial t} = 0$$

$\psi_n^l / h \Leftrightarrow \psi_n^l$ = gap at current time

$$\psi_n^l = \begin{bmatrix} 0.001 \\ -0.0014 \\ -0.010 \\ -0.0004 \end{bmatrix}$$

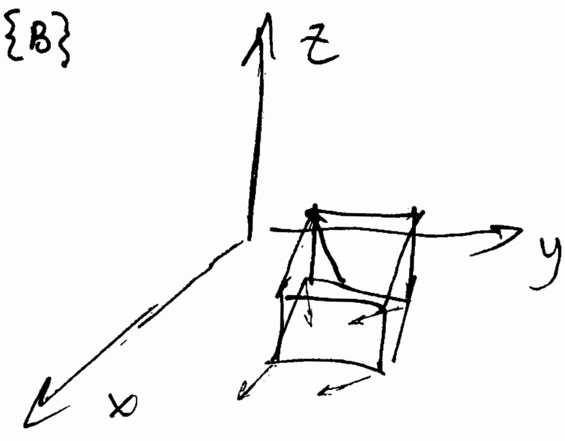
e.g.

Assume ~~unit cube.~~ Look at nonpenetration for one corner.

$$\psi_n = \left[\begin{array}{c|c} {}^N_B R(q) & \begin{matrix} q_1 \\ q_2 \\ q_3 \end{matrix} \\ \hline 0\,0\,0 & 1 \end{array}\right] \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ \hline 1 \end{bmatrix} \geq 0$$

corner in $\{B\}$

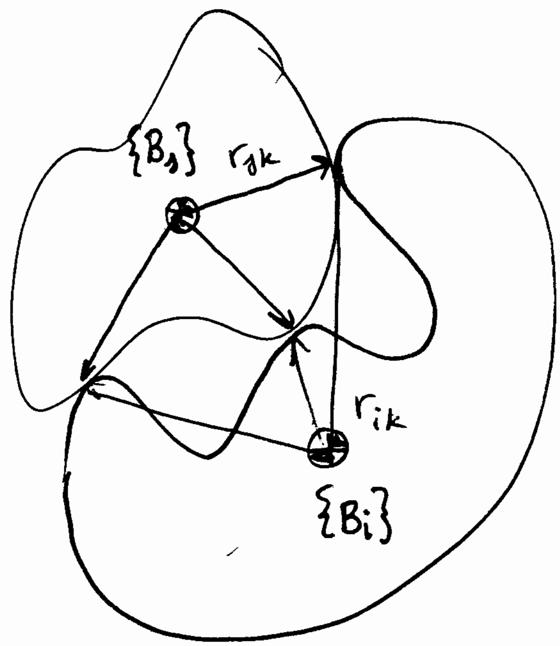
It is ok to Keep all 8 constraints active at all times.



More Bodies

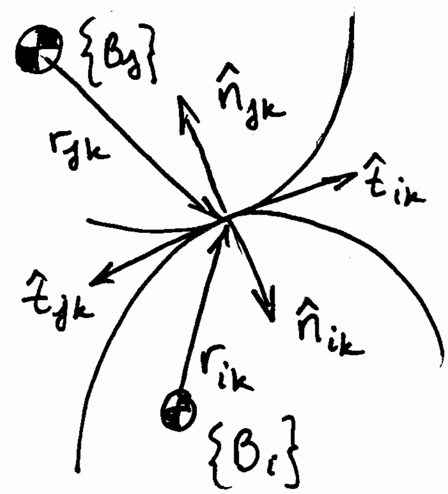
$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_{nb} \end{bmatrix} \quad \nu = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_{nb} \end{bmatrix} \quad p_{ext} = \begin{bmatrix} p_{1,ext} \\ p_{2,ext} \\ \vdots \\ p_{nb,ext} \end{bmatrix}$$

$(7nb \times 1)$ $\quad$ $(6n_b \times 1)$ $\quad$ $(6nb \times 1)$



$$M = \mathrm{diag}(M_1, \ldots, M_{nb})$$

$$\underset{(6n_b \times n_c)}{W_n} = \begin{array}{c} \left[\begin{array}{ccccc} & & \vdots & & \\ \cdots & & W_{ik} & & \cdots \\ & & \vdots & & \\ \cdots & & W_{jk} & & \cdots \\ & & \vdots & & \end{array}\right] \begin{array}{l} \\ \}\text{ body } i \\ \\ \}\text{ body } j \\ \\ \end{array} \\ 1\ 2 \cdots\ k \cdots\cdots n_c \\ \longleftarrow \text{contacts} \longrightarrow \end{array} \quad (6nb \times n_c)$$

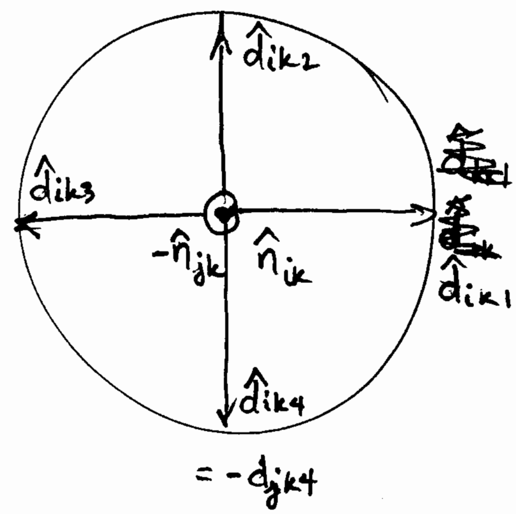


where ${}^N W_{ik} = \begin{bmatrix} {}^N\hat{n}_{ik} \\ {}^N r_{ik} \times {}^N\hat{n}_{ik} \end{bmatrix} \Rightarrow$

$${}^N W_{jk} = \begin{bmatrix} -{}^N\hat{n}_{ik} \\ {}^N r_{jk} \times (-{}^N\hat{n}_{ik}) \end{bmatrix} = \begin{bmatrix} {}^N\hat{n}_{jk} \\ {}^N r_{jk} \times {}^N\hat{n}_{jk} \end{bmatrix}$$

$$
\begin{array}{cc}
 & \text{Contact } k \\
W_f = & \begin{bmatrix} & & \\ & W_{ik,f} & \\ & & \\ & W_{jk,f} & \\ & & \end{bmatrix}
\end{array}
\begin{array}{l} \\ \text{Body } i \\ \\ \text{Body } j \\ \\ \end{array}
$$

Contact k



$$
W_{ik,f} = \begin{bmatrix} \hat{d}_{ik,1} & \cdots & \hat{d}_{ik,n_f} \\ r_{ik} \times \hat{d}_{ik,1} & \cdots & r_{ik} \times \hat{d}_{ik,n_f} \end{bmatrix}
$$

Similar for $W_{jk,f}$.

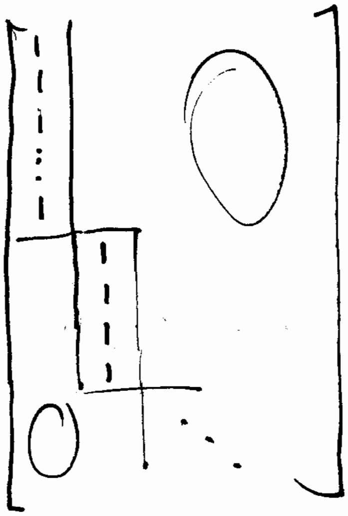
Convenient to let ${}^N\hat{d}_{ik,1} = -{}^N\hat{d}_{jk,1}$, etc.

Then it is easy to use one set of contact ~~force~~ impulse parameters $p_{kn}, p_{kf1}, p_{kf2}, \ldots$

$$
\Theta(q) = \begin{bmatrix} e_{10}^2 + e_{11}^2 + e_{12}^2 + e_{13}^2 - 1 \\ e_{20}^2 + e_{21}^2 + e_{22}^2 + e_{23}^2 - 1 \\ \vdots \\ \end{bmatrix}
$$

$$
\frac{\partial \Theta}{\partial q} = \begin{bmatrix} 0 \ 0 \ 0 \ 2e_{10} \ 2e_{11} \ 2e_{12} \ 2e_{13} & & \\ 0 \ 0 \ 0 & 0 \ 0 \ 0 \ 2e_{20} \ 2e_{21} \ 2e_{22} \ 2e_{23} & \\ & & \ddots \\ 0 & & \end{bmatrix}
$$

$E =$ $(n_c \times (n_c \cdot n_f))$



$\leftarrow$ contacts $\rightarrow$

$$p_f = \begin{bmatrix} \vdots \\ p_{kf1} \\ p_{kf2} \\ \vdots \\ p_{(k+1)f1} \\ p_{(k+1)f2} \\ p_{(k+1)f3} \\ \vdots \end{bmatrix} \updownarrow \text{contacts}$$

Final Size of LCP $\quad (7n_b + n_c(2+n_f))$

Could eliminate $7n_b$ variables to make problem smaller, but then the LCP matrix becomes dense and solver converges more slowly.