Chapter 5 - Forces

Rigid Body Statics

Sum of Ext. Forces & Moments is zero \[ \sum \text{ Forces} = \sum \text{ Moments} = 0 \]

We care about statics, since a stable grasp must satisfy equilibrium.

Def: A system is stable if when perturbed from an equilibrium config, it eventually returns to the config.

Example: can suspension grasp w/ compliant fingers

Big Picture: What about a rigid

We want algorithmic, manual, and analytic methods for reasoning about forces and contacts and predicting motions.
What about a simpler analysis of grasping? Everything rigid?

Is this stable?

- $\mu = 0 \iff$ marginally stable
  counting on unmodeled physical effects to $\&$
damp motion

in classical sense

- $\mu > 0 \iff$ not stable*, since system will not
  return from all perturbations,
  but still operationally for
  many tasks.

book on table

If not level, perturbations
do not recover.
Eventually book falls off.

C-space

[diagram showing C-space with regions labeled]

3/3/04

(2)

Explain w/ 2 contacts, then 3.
Skip to page 2.1

(2.4)
Is this rectangle in equilibrium with $\mu = 0$? Possible $(x, y)$

Contact constraints

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note: $\dot{\theta}$ is arbitrary

From Physics

Equilibrium w/o friction occurs if c.g. has "locally" smallest $y$ value!

$\Delta$ c.g. height $= -L \sin(\alpha) \dot{\theta}$

$\therefore \dot{\theta} > 0$ causes c.g to move down!

Not in Equilibrium!

Geometric Interp

- Forces must intersect at a point, i.e. $\Sigma\text{Mom}=0$
- Forces must sum to zero $\Sigma\text{Force}=0$

\[
\begin{align*}
\hat{n}_1 \lambda_1 &= mg \\
\hat{n}_2 \lambda_2 &= mg
\end{align*}
\]
\[
\begin{bmatrix}
W_n^T \nu \\
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} \geq \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Note: \(\dot{\theta}\) is still arbitrary, but contacts must break.

for any motion, since \((W_n^T)^{-1}\) exists

If forcing equality
\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} = (W_n^T)^{-1} \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]


Does there exist \(\nu \in V^T f \leq 0\) ?

where \(f = \begin{bmatrix}
0 \\
-1 \\
\frac{1}{2}
\end{bmatrix}\)

\[
\begin{bmatrix}
-1 & 1 & 0 \\
1 & 1 & 0 \\
-1 & 1 & -2
\end{bmatrix} \begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} \geq \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -1 & -1/2
\end{bmatrix} \leq \begin{bmatrix}
0
\end{bmatrix}
\]

This is a Linear Program!
Solve the LP geometrically (qualitatively)!

All possible IC's cause CG to move upward.

\[ \therefore \nu = 0 \text{ is stable!} \]

Solution of LP would give rate of energy gain for each basic motion (breaking one contact)!

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This is even "more stable" than w/o friction.

ICS = 2 cones

Let $p \in \text{Int}(\Theta \text{ICS}) \cup \text{Int}(\Theta \text{ICS})$
then valid $\theta$ works against gravity.

If $p \in \partial(\Theta \text{ICS} \cup \Theta \text{ICS})$
then valid $\theta$ works against gravity & friction.
What about this?

Is it stable?

No perturbation is possible, so

Yes.

So forces don't even enter the picture here.

We will eventually discuss a force/velocity dual.

Regardless of the external force applied, the contacts can always balance.

No motion is possible.

If the object were flexible, then motion would work against body stiffness too!
Forces & Moments (Torques)

Forces cause acceleration (or not)

\[ \mathbf{f} = ma \]

Newton's eq.

3 equations

Particles of mass \( m \).

Moments cause rotation of rigid bodies

\[ \mathbf{n}_z = I_{zz} \alpha \]

Angular acceleration

moment of inertia \( \omega \) wrt \( z \)-axis

component of moment in \( z \)-direction

\[ \mathbf{n} = \mathbf{I} \alpha - \omega \times \mathbf{I} \omega \]

Euler's Eq.

\[
\mathbf{I} = \begin{bmatrix}
I_{xx} & I_{xy} & I_{xz} \\
I_{yx} & I_{yy} & I_{yz} \\
I_{zx} & I_{zy} & I_{zz}
\end{bmatrix}
\]

Positive Definite
Forces produce moments about lines and points:

About origin: \( \eta = r \times f \)  

About other point: \( \eta = (r-p) \times f \)

Moment about line:

\[ \eta_l = \hat{\ell} \cdot \rho \times f \] scalar

where \( \rho \) is diff between two points, one on each line, \( \hat{\ell} \) is unit vector along \( l \).
Forces: internal & external

internal - act between particles of a body
external - due to external effects such as gravity, wind

Total Force & Moment (resolved)

\[ F = \sum \text{of all external forces} = \sum f_i \]
\[ N = \sum \text{of all external moments} = \sum r_i \times f_i \]

Equivalent Systems of Forces

If \( F_1 = F_2 \) & \( N_1 = N_2 \), then the two systems of forces are said to be equivalent.
If there exists a single force, \( f \) such that \( f = F \) and \( r \times f = N \) (for a single \( r \)), then \( f \) is the resultant force of the system.

Line of Action - line determines moment.

\( f \) applied to a pt is completely characterized by the direction of the force. i.e. \( f = ma \)

But applied to rigid body,

Point of app. is not important! Line of action matters!
Resultant Force when 2 Lines of Action Intersect

If 2 force lines are in 1 plane, then this is equivalent to two forces acting on a single point!

Change of Reference Point

\[ \sum f_i = F_Q = F_R \]

\[ \sum r_i \times f_i = N_R \]

\[ \sum q_i \times f_i = N_Q \]

Suppose we have \( N_R \) and we want \( N_Q \)

\[ N_Q = N_R + (R - Q) \times F \]
A couple

A system of forces \( \in \) \( F = 0 \)

Note \( N \) is indep of ref pt.

Since \( N_q = N_k + (R-Q)x_F \)

A couple is a \boxed{\text{Pure Moment}}

One can always construct a system of 2 forces equivalent to a moment!

Note that the couple can be moved rigidly w/o changing moment, \( N_q \).
Equivalence Theorems

5.1 For any point Q, any system of forces is equiv. to a single force thru Q, plus a couple.

5.2 Every system of forces is equiv to just 2 forces.

5.3 A system consisting of a single nonzero force plus a couple in the same plane, has a resultant = an equiv. force.

Translate force until \( r \times F_p = N_p \).
Thm. 5.4

Birnbaum's Theorem

Every system of forces is equivalent to a single force, plus a couple w/ moment parallel to force.

This is the analog to Chavel's theorem.

DEFINITION 5.2 Wrench

A wrench is a screw plus a scalar magnitude representing force along the screw axis and moment about the screw axis.

\[
\text{pitch is } \frac{||F||}{||M||} = \frac{n}{f}
\]

\[
W = ||F|| \hat{q}
\]

\[
W_0 = ||F||q_0 + ||M||\hat{q} = ||F||q_0 + ||M||\hat{q}
\]

where \( q_0 = r \times \hat{q} \)
Wrench\
\[ w = f \] (3x1)
\[ w_0 = rf + n \] (3x1)

or
\[ w = f \]
\[ w_0 = n_0 \]

\[ [f, n_0] + (\omega, n_0) = 0 \]

\[ \text{moment of force about origin, plus the moment in the wrench} \]

Reciprocal of wrench & differential twist
\[ (\omega, n_0) \ast (f, n_0) = f \cdot n_0 + n_0 \cdot \omega = \text{power} \]

This is instantaneous power

\[ \text{explain w/ example} \]

Repelling if power > 0
Contrary if power < 0

Put these together to get a wrench & twist