Probs_Chap5_Lavalle

Tuesday, April 28, 2009
2:38 PM

(5.2) If ρ is a metric, is ρ^2 a metric?

It is trivial to see that ρ^2 satisfies 3 of the 4 required properties: non negativity, symmetry, and reflexivity. It does not satisfy <u>triangle inequality</u>.

ρ is a metric implies: $\rho(a,b) + \rho(b,c) \geq \rho(a,c)$ (1)

$$\Rightarrow \rho^2(a,b) + \rho^2(b,c) + 2\rho(a,b)\rho(b,c) \geq \rho^2(a,c) \quad (2)$$

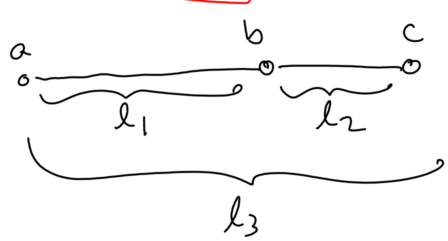
Suppose ρ^2 is a metric. Then we must have:

$$\rho^2(a,b) + \rho^2(b,c) \geq \rho^2(a,c) \quad . \quad (3)$$

Since $\rho \geq 0$, $2\rho(a,b)\rho(b,c) \geq 0$. When $2\rho(a,b)\rho(b,c)$ is strictly positive, then (2) could be satisfied when (3) is not. $\therefore$ (3) is not always satisfied,

$\boxed{\therefore \rho^2 \text{ is not a metric.}}$

Counter-Example:



using L_2 norm: $\ell_1^2 + \ell_2^2 < \ell_3^2$

(5.3). Is $\rho(x,x') = \begin{cases} 1 & \text{if } x \neq x' \\ 0 & \text{if } x = x' \end{cases}$ a metric?

Requirements:

Non neg:	$\rho(x,x') \geq 0$	✓ Yes
Reflex:	$\rho(x,x) = 0$	✓ Yes
Symmetry:	$\rho(x,x') = \rho(x',x)$	✓ Yes
Tri Ineq.:	$\rho(x,x') + \rho(x',x'') \geq \rho(x,x'')$	✓ Yes

for Triangle Ineq. there are 3 cases:

① x, x', x'' are distinct
② all are equal
③ two are distinct

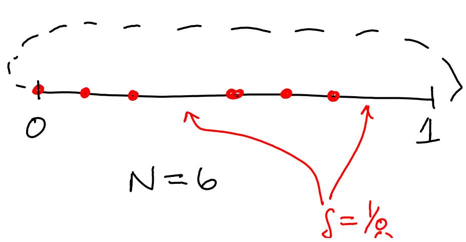
Triangle Ineq. is satisfied in all cases.

Yes, ρ is a metric

(5.6) Determine dispersion δ of a van der Corput sequence on the circle. Let P denote a set of samples. Then $\delta(P)$ is:

$$\delta(P) = \sup_{x \in X} \left\{ \min_{p \in P} \{ \rho(x,p) \} \right\}$$

$\delta(P)$ on S^1 is ½ the largest clear interval



$\delta = \left(\frac{1}{2}\right)^{k+1}$

$\delta = \left(\frac{1}{2}\right)^{k+1}$ N=6 ✓ $\delta = 1/8$

where $k = \lfloor \log_2(N) \rfloor$ ← floor function

and $N = |P|$ = # of samples

(5.9) Determine formula for dispersion δ after i samples of a van der Corput sequence in base p on $[0,1]/\sim$.

Create table of dispersion for base 3 and observe the critical aspects to derive formula.

Let L = length of longest interval.

L drops by 1/3 for each i

L drops by $1/3^2$ after each 3 pts

L drops by $1/3^3$ after each 3^2 pts

i	$\alpha(i)$	length longest interval
1	0.000	3/3
2	0.100	2/3
3	0.200	1/3 = 3/9
4	0.010	3/9
5	0.110	3/9
6	0.210	2/9
7	0.020	2/9
8	0.120	2/9
9	0.220	3/27
10–17	⋮	3/27
18–26	⋮	2/27
27–53	⋮	3/81
54–80	⋮	2/81

$$\boxed{\delta = \frac{1}{2p^x}\left[1 - \frac{\left(\lfloor \frac{i}{p^x} \rfloor - 1\right)}{p}\right] \quad \text{where } x = \lfloor \log_p(i) \rfloor}$$

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5.13 Show that for any set of points in $[0,1]^n$ a range space $\mathcal{R}$ can be designed so that the discrepancy is as close to 1 as desired

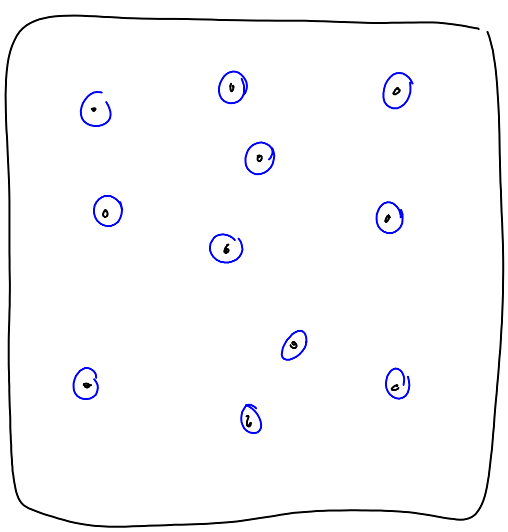
$$D(P,\mathcal{R}) = \sup_{R \in \mathcal{R}} \left\{ \frac{|P \cap R|}{N} - \frac{\mu(R)}{\mu(X)} \right\}$$

where $N = |P| = \#$ samples

$\mu(\cdot)$ is the measure of a space (Area, volume ...)

R is any one of a collection of subsets of X

Example Soln:



P is a set of points

R is a set of open discs $\ni$

$P \cap R = P$

This gives first term of D expression $= 1$

Now make $\mu(R)/\mu(X)$ go to zero

Since $\mu(X) =$ constant > 0, as long as $\mu(R) \to 0$
then we have the required result.

Letting the radius of the circles of R go to zero gives $\mu(R) \longrightarrow 0$

q.e.d.