Non uniqueness in Rigid Body Dynamics

Painleve's Paradox - inconsistent force/acceleration analysis of rigid body contact

Newton-Euler Equation

\[ M \ddot{u} = G \mathbf{n} \mathbf{t} + G \mathbf{t} \mathbf{a} + \mathbf{g}_{\text{ext}} \]

\( m = \text{mass} \)
\( J = \text{mom. of inertia about c.g.} \)

Let \((x, y)\) be position of center of mass
\[ u = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \]

\[ v = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \]

Gap Function, \( \Psi_n \)

Define quantities:
\[ d = \|r\| \]
\[ \mathbf{N}_n, \mathbf{N}_t = \text{normal and tang. components of relative velocity at contact} \]

\( n, a = \text{normal and tang. comms of} \)
\( a_n, a_t = \text{normal and tang. comp. of relative acceleration} \)

\[
\begin{align*}
\dot{\psi}_n &= g a p = y - d \sin \theta \\
\dot{\psi}_n &= a_n = \dot{y} - d \dot{\theta} \cos \theta \\
\ddot{\psi}_n &= \ddot{a}_n = \ddot{y} + d \dot{\theta}^2 \sin \theta - d \dot{\theta} \dot{\theta} \cos \theta
\end{align*}
\]

\[
\begin{align*}
\dot{\psi}_n &= a_n = \dot{y} - d \dot{\theta} \cos \theta = \nu T_{\psi} G_{n} \\
\ddot{\psi}_n &= a_n = \ddot{y} + d \dot{\theta}^2 \sin \theta - d \dot{\theta} \dot{\theta} \cos \theta = \nu T_{\psi} G_{n} + G_{n} \nu
\end{align*}
\]

\[
G_{n} = \frac{\partial \psi_{n}}{\partial \nu} \nu = \begin{bmatrix} 0 & 1 & -d \cos \theta \end{bmatrix}
\]

\[
\nu T_{\psi} = \begin{bmatrix} 0 & 0 & d \dot{\theta} \sin \theta \end{bmatrix}
\]

We can also define a tangential displacement function \( \psi_t(u) \), i.e. position of contact on object relative to initial contact point on horizontal support.

\[
\psi_t(u) = -x + d \cos(\theta)
\]

\[
\frac{d\psi_t}{du} = \nu_t = -\nu_x - d \dot{\theta} \sin \theta = G_{u} \nu
\]

\[
\frac{d^2\psi_t}{du^2} = a = -\nu_x - d \dot{\theta} \dot{\theta} \sin \theta - d \dot{\theta}^2 \cos \theta
\]
\[
\frac{d\psi}{dt^2} = a_t = -\dot{\kappa} - d\dot{\omega}_x \sin(\Theta) - d\dot{\omega}_z \cos(\Theta)
\]
\[
= \mathbf{G}^T_t \mathbf{V} + \mathbf{G}^T_t \dot{\mathbf{V}}
\]

\[
\frac{\partial \psi}{\partial \mathbf{u}} \mathbf{V} = \mathbf{G}^T_t = \begin{bmatrix}
-1 & 0 & -d\sin(\Theta)
\end{bmatrix}
\]

\[
\mathbf{G}^T_t = \begin{bmatrix}
0 & 0 & -d\omega_z \cos(\Theta)
\end{bmatrix}
\]

**Newton Euler Equations.**

\[
M \ddot{\mathbf{u}} = \mathbf{G}_n \lambda_n + \mathbf{G}_t \lambda_t + \mathbf{g}_{ext}
\]

\[M = \text{diag} (m, m, J)\]

\[
\mathbf{G}_t = \begin{bmatrix}
-1 \\
0 \\
-d\sin(\Theta)
\end{bmatrix}
\]

\[
\mathbf{g}_{ext} = \begin{bmatrix}
0 \\
-mg \\
0
\end{bmatrix}
\]

**Coulomb's Friction Law**

Assume contact exists.
$\psi_n = 0$

**Cases**

<table>
<thead>
<tr>
<th>$\psi_n$</th>
<th>$n_n$</th>
<th>$a_n$</th>
<th>$n_t$</th>
<th>$a_t$</th>
<th>Law</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$=0$</td>
<td>$&gt;0$</td>
<td></td>
<td></td>
<td>$\lambda_n = \lambda_t = 0$</td>
</tr>
<tr>
<td>$SL$</td>
<td>$=0$</td>
<td>$=0$</td>
<td>$&gt;0$</td>
<td></td>
<td>$\lambda_t = -\mu \lambda_n$</td>
</tr>
<tr>
<td>$SR$</td>
<td>$=0$</td>
<td>$=0$</td>
<td>$&lt;0$</td>
<td></td>
<td>$\lambda_t = \mu \lambda_n$</td>
</tr>
<tr>
<td>$S-SL$</td>
<td>$=0$</td>
<td>$=0$</td>
<td>$&lt;0$</td>
<td>$&gt;0$</td>
<td>$\lambda_t = -\mu \lambda_n$</td>
</tr>
<tr>
<td>$S-SR$</td>
<td>$=0$</td>
<td>$=0$</td>
<td>$&lt;0$</td>
<td>$&lt;0$</td>
<td>$\lambda_t = \mu \lambda_n$</td>
</tr>
</tbody>
</table>

**Velocity-Level Model**

**Acceleration-Level**

- **B** - Contact Break
- **SL** - Slide Left
- **SR** - Slide Right
- **S-SL** - Stick → Slide Left Transition
- **S-SR** - Stick → Slide Right Transition

**Painleve Paradox arises in accel-level analysis**

Assume sliding to left

$\psi_n = n_n = 0$,
\[ N_t > 0 \implies \lambda_t = -\mu \lambda_n \]

Solve for \( a_n, \lambda_n \)

\[ \dot{\lambda} = M^{-1} (G_n \lambda_n + G_{t} \lambda_t + g_{ext}) \]
\[ 0 \leq a_n \perp \lambda_n \geq 0 \]

Substitute

\[ \dot{\lambda} = M^{-1} (G_n \lambda_n + g_{ext}) \]

Define \( a_n \) from \( N_n = G_n^\top \lambda_n \)

\[ a_n = G_n^\top \dot{\lambda} + \dot{G}_n^\top \lambda \]
\[ = G_n^\top M^{-1} (G_n - \mu G_t) \lambda_n + G_n^\top M^{-1} g_{ext} + \dot{G}_n^\top \lambda \]

Substitute expressions for \( G_n, G_t, M, g_{ext}, G_n \)

\[ a_n = A \lambda_n + b \]

LCP

\[ 0 \leq a_n \perp \lambda_n \geq 0 \]

where

\[ A = \frac{1}{m} + \frac{d^2}{J} \cos \theta (\cos \theta - \mu \sin \theta) \]
\[ b = d \omega_z^2 \sin \theta - g \]

Case 1: 0 or 1 Solution

Assume body is initially translating \(\Rightarrow \omega_z = 0\)

\[ \therefore b = -q \implies b < 0 \]
A can be negative!

\[ A > 0 \]
1 soln.

\[ A < 0 \]
0 soln.

Geometric Interpretation

When is \( A > 0 \)?

when \( \cos \theta > \mu \sin \theta \)

Determined by cg location

Case 1:
\[
\begin{align*}
\cos \theta &< 0 \\
\sin \theta &> 0
\end{align*}
\]

Case 2:
\[
\begin{align*}
\cos \theta &> 0 \\
\sin \theta &> 0
\end{align*}
\]
and \( \cos \theta > \mu \sin \theta \)

Nonexistence only when cg in right half
Case 2: 0,1,2 Solutions

\[ \sin \theta > 0 \quad \text{and} \quad \omega_z \neq 0 \]

More generally, \( b > 0 \)

i.e. \( d\omega_z \sin \theta > g > 0 \)

\[ \frac{d\omega_z}{\lambda_n} \sin \theta - g = b \]

Same arguments apply to sign of \( A \).

Two solution case, \( b < 0 \)

Maintain sliding contact
\[ a_n = 0, \quad \lambda_n > 0 \]

Contact separation
\[ a_n > 0, \quad \lambda_n = 0 \]

Some values that make \( A < 0 \).
\[ d = 1 \]
\[ m = 1 \]
\[ J = \frac{1}{10} \]
\[ \mu = 1 \]
\[ \theta = 1.1 \]
\[ g = 1 \]

If object is slender rod, then

\[ J = \frac{1}{3} m d^2 \]

\[ \therefore \text{ For these numbers, mass is more concentrated near } cg \text{ than for a slender rod.} \]

\[ A = \frac{1}{m} + \frac{d^2}{J} \cos \theta (\cos \theta - \mu \sin \theta) \]

Substitute

\[ A = -0.985 \]

Resolution of Painleve's Paradox via Time Stepping

Let \( h = 1 \), \( N_x = -2 \)

\[ \mathbf{P}_{ext} = h \mathbf{g}_{ext} = \begin{bmatrix} 0 \\ -mgh \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \end{bmatrix} \]
\((Y^k)^T = [-2 0 0]\)

Simulation results for one time step with different \(\mu\) solved with pathleap in Matlab.

<table>
<thead>
<tr>
<th>(\mu)</th>
<th>(\rho^1_i)</th>
<th>(p_{f_i}^1)</th>
<th>(p_{h_i}^1)</th>
<th>(S_{h_i}^1)</th>
<th>(N_{x}^{h_i})</th>
<th>(N_{y}^{h_i})</th>
<th>(W_{x}^{h_i})</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0.33</td>
<td>0</td>
<td>0</td>
<td>3.32</td>
<td>-2.0</td>
<td>-0.67</td>
<td>-1.48</td>
</tr>
<tr>
<td>0.3</td>
<td>0.54</td>
<td>0</td>
<td>0.16</td>
<td>2.73</td>
<td>-1.84</td>
<td>-0.46</td>
<td>-1.00</td>
</tr>
<tr>
<td>0.5</td>
<td>0.96</td>
<td>0</td>
<td>0.48</td>
<td>1.59</td>
<td>-1.52</td>
<td>-0.04</td>
<td>-0.08</td>
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<tr>
<td>0.6</td>
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<td>0.92</td>
<td>1.07</td>
<td>0.54</td>
<td>1.21</td>
<td></td>
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<tr>
<td>1</td>
<td>1.54</td>
<td>0.31</td>
<td>1.23</td>
<td>0</td>
<td>-1.08</td>
<td>0.54</td>
<td>1.21</td>
</tr>
<tr>
<td>&gt;1</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

\(\mu^* = \text{critical } \mu = 0.6\)

\(p_{f_2} - p_{f_1} = 0.92 \quad \forall \ \mu > \mu^*\)

Physical Interpretation of Solution

for \(\mu > \mu^* = 0.6\)

A solution exists with \(p_n > 0\), so it appears contact will be maintained.

Impulse passes just right of c.g. causing counterclockwise rotation.
Sliding speed is zero. Solution is much like pole vault.

The previous was solved using Matlab and "pathlicp" obtained from cpenet.org. pathlicp solves standard LCPs only Convert mixed LCP to standard as follows

\[ \nu^{l+1} = \nu^l + M^{-1} (G_n p_n^{l+1} + G_f p_f^{l+1} + P_{ext}) \]

Substitute

\[ 0 \leq p_n^{l+1} \perp G_n^T \nu^{l+1} \geq 0 \]
\[ 0 \leq p_f^{l+1} \perp G_f^T \nu^{l+1} + E_s s^{l+1} \geq 0 \]
\[ 0 \leq s^{l+1} \perp U p_n^{l+1} - E^T p_f^{l+1} \geq 0 \]

(Where \( \frac{\partial \psi_n}{\partial t}, \frac{\partial \psi_f}{\partial t}, \psi_n \) were assumed = 0)
The standard LCP is:

\[
0 \preceq \begin{bmatrix}
  p_n^{\text{let}} \\
  p_f^{\text{let}} \\
  s^{\text{let}}
\end{bmatrix} \prec \begin{bmatrix}
  B_{nn} & B_{nf} & 0 \\
  B_{fn} & B_{ff} & E \\
  U & -E^T & 0
\end{bmatrix}\begin{bmatrix}
  p_n^{\text{let}} \\
  p_f^{\text{let}} \\
  s^{\text{let}}
\end{bmatrix} + \begin{bmatrix}
  b_n \\
  b_f \\
  0
\end{bmatrix} \succeq 0
\]

\[
B_{nn} = G_n^T M^{-1} G_n \quad B_{nf} = G_n^T M^{-1} G_f
\]

\[
B_{fn} = G_f^T M^{-1} G_n \quad B_{ff} = G_f^T M^{-1} G_f
\]

\[
b_n = \quad G_n^T (\nu^l + M^{-1} p_{\text{ext}})
\]

\[
b_f = \quad G_f^T (\nu^l + M^{-1} p_{\text{ext}})
\]