

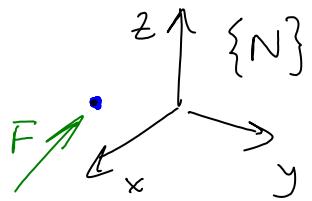
Discrete Dynamics

Saturday, February 09, 2008

11:08 AM

$$\text{Let } u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad v = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

Begin w/ Particle



No rotational dynamics!

$$F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

Newton's Law

$$\sum \text{Forces} = \frac{d}{dt}(mv) = \dot{m}v + m\ddot{v}$$

Assume mass is fixed

$$\sum \text{Forces} = m\ddot{v} = F$$

Also need kinematic differential equation

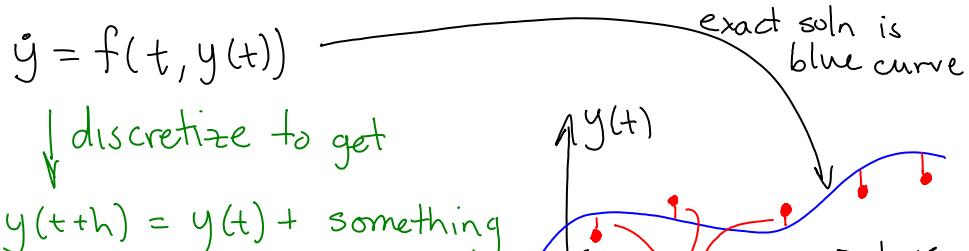
$$\boxed{\begin{aligned} \dot{v} &= \frac{1}{m} F \\ \ddot{v} &= V v \end{aligned}} \quad (\text{V} = I_{(3 \times 3)} \text{ in this case})$$

Approximate solution via time-stepping method

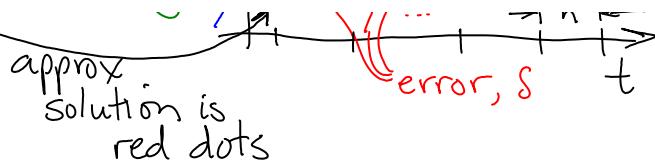
$$\text{Let } t_e = hl, \quad u^e \triangleq u(t_e), \quad v^e \triangleq v(t_e)$$

where h is a small time interval

Goal of time stepping method is to produce a point-wise approximation to the solution of a differential equation



Properties desired



Convergence:

A time stepping method is convergent if the maximum s goes to zero as h goes to zero.

Order:

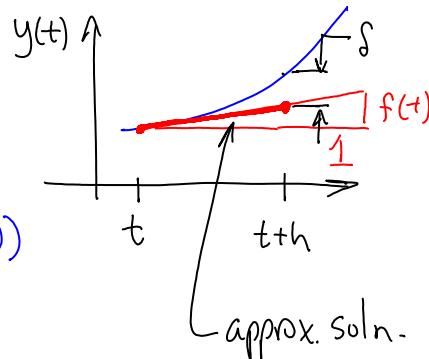
A time stepping method has order p if the error s of a single time step is $\mathcal{O}(h^{1+p})$ as h goes to zero.

Explicit Euler

$$\dot{y} \approx \frac{y(t+h) - y(t)}{h}$$

$$\Rightarrow y(t+h) = y(t) + h f(t, y(t))$$

$$\text{or } y^{l+1} = \underline{y^l + h f^l}$$



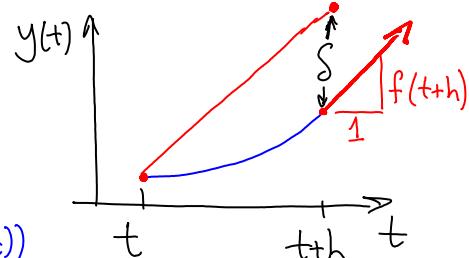
The new value is a function of known quantities!

Backward (Implicit) Euler

$$\dot{y} \approx \frac{y(t) - y(t-h)}{h}$$

$$\Rightarrow y(t) = y(t-h) + h f(t, y(t))$$

$$\text{or } y^{l+1} = \underline{y^l + h f^{l+1}}$$



The new value is a function of an unknown quantity.
Usually is nonlinear and requires iteration.

Both Euler methods are convergent and first order when $f(t)$ is smooth.

Euler's Method Applied to Particle Dynamics

$$\dot{u} \approx \frac{u^{l+1} - u^l}{h}, \quad \dot{v} \approx \frac{v^{l+1} - v^l}{h}$$

Apply to dynamic equations

$$v^{l+1} = v^l + \frac{h}{m} F^{(?)}$$

at what time do we evaluate F & v ?

$$u^{l+1} = u^l + h v^{(?)}$$

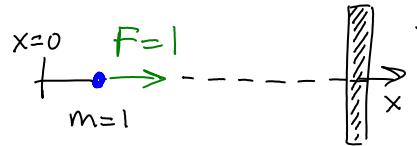
Explicit Euler (Right-hand sides known at t_k)

$$v^{l+1} = v^l + \frac{h}{m} F^l$$

$$u^{l+1} = u^l + h v^l$$

Consider 1-D example \rightarrow

$$u = x, \quad v = \dot{x}$$



$$\text{for simplicity, } h = \underline{\underline{1}} = m = F$$

wall at $x = 11$

Initial Conditions

$$x_x^0 = x^0 = 0$$

Time stepping equations

$$x_x^{l+1} = x_x^l + \underline{\underline{1}}$$

$$x^{l+1} = x^l + x_x^l$$

l	x^l	v_x^l
0	0	0
1	0	1
2	1	2
3	3	3
4	6	4
5	10	5
\vdots	\vdots	\vdots

Semi-Implicit Euler

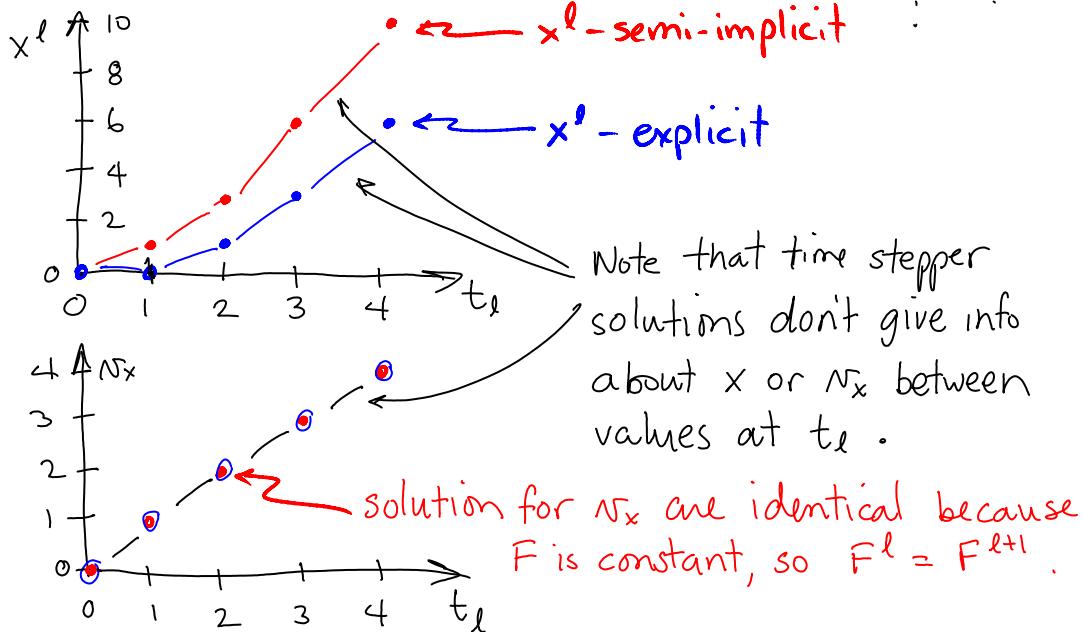
$$N_x^{l+1} = N_x^l + 1$$

$$x^{l+1} = x^l + N_x^{l+1}$$

Note: compute v^{l+1} first, then u^{l+1}

R.H.S. not completely known at t_{l+1}

l	x^l	N_x^l
0	0	0
1	1	1
2	3	2
3	6	3
4	10	4
⋮	⋮	⋮



Recall that $x = \int N_x dt + \text{const.}$

Semi-implicit scheme implicitly assumes:

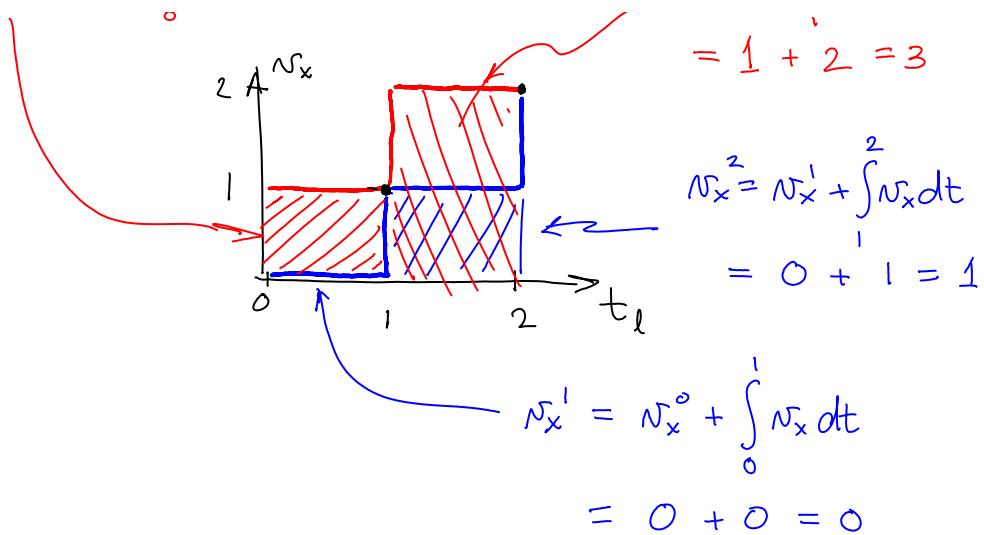
$$N_x(t) = N_x^{(l+1)} \quad \text{for } t \in (t^l, t^{l+1})$$

Explicit scheme implicitly assumes:

$$N_x(t) = N_x^{(l)} \quad \text{for } t \in (t^l, t^{l+1})$$

$$N_x^1 = N_x^0 + \int_0^1 N_x dt = 0 + 1 = 1$$

$$N_x^2 = N_x^1 + \int_1^2 N_x dt = 1 + 2 = 3$$

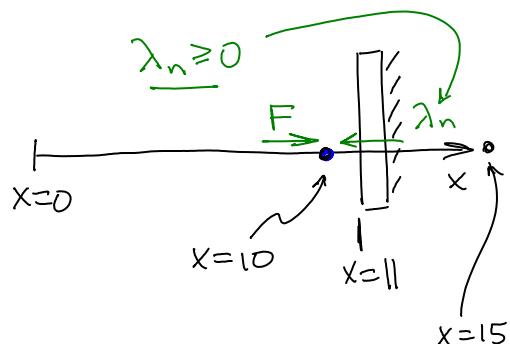


Consider Dynamics when collision is imminent.

Define distance function, $\Psi_n(u)$

In 1D example,

$$\Psi_n(x) = 11 - x$$



with no wall, x at next time step would be $\underline{15}$

We want time-stepper to anticipate collision and prevent penetration!

\therefore must "see" wall coming up and apply λ_n

Contact force modeling:

$$\text{No penetration} \Rightarrow \Psi_n \geq 0$$

$$\text{No tensile contact force} \Rightarrow \lambda_n \geq 0$$

Also:

$$\Psi_n > 0 \Rightarrow \lambda_n = 0$$

$$\lambda_n > 0 \Rightarrow \Psi_n = 0$$

Therefore, we require

$$0 \leq \lambda_n \perp \Psi_n \geq 0$$

New dynamic model

$$\dot{v} = \frac{1}{m}(F - \lambda_n)$$

$$\dot{u} = v$$

$$0 \leq \lambda_n \perp \Psi_n(u) \geq 0$$

If collision is "bouncy," then integrate with adaptive step size and then apply collision law

Newton's Hypothesis

$$N_n(t_c^+) = -N_n(t_c^-) e$$

approach velocity

coef. of restitution

time just before collision

time just after collision

Other Hypotheses : Poisson's, Strange's

energetically consistent

defined in terms of impulses

Suppose bounce is negligible (Stewart-Trinkle time-stepper)

$$v^{l+1} = v^l + \frac{h}{m}(F - \lambda_n)$$

$$u^{l+1} = u^l + h v^?$$

$l?$ $l+1?$

Note that if we use v^l , the contact force does not enter. Therefore use v^{l+1}

Now, we have another goal - make system consistent at the end of every timestep, i.e. we want all solutions to satisfy Newton's Law and nonpenetration

\therefore use λ_n^{l+1} .

$$v^{l+1} = v^l + \frac{h}{m} (F - \lambda_n^{l+1})$$

$$u^{l+1} = u^l + h v^{l+1}$$

$$0 \leq \lambda_n^{l+1} \perp \psi_n^{l+1} \geq 0$$

Apply to 1D example

$$n_x^{l+1} = n_x^l + 1 - \lambda_n^{l+1} \quad (1)$$

$$x^{l+1} = x^l + n_x^{l+1} \quad (2)$$

$$0 \leq \lambda_n^{l+1} \perp 11 - x^{l+1} \geq 0 \quad (3)$$

Convert above mixed LCP

to a pure LCP. Substitute

(1) & (2) into (3).

l	x^l	n_x^l	ψ_n^l	λ_n^l
0	0 ⁺	0	11	0
1	1 ⁺	1	10	0
2	3 ⁺	2	8	0
3	6 ⁺	3	5	0
4	10	4	1	0
5	?	?	?	?

$$0 \leq \lambda_n^{l+1} \perp 11 - (x^l + n_x^l + 1 - \lambda_n^{l+1}) \geq 0$$

$$0 \leq \lambda_n^{l+1} \perp 10 - x^l - n_x^l + \lambda_n^{l+1} \geq 0$$

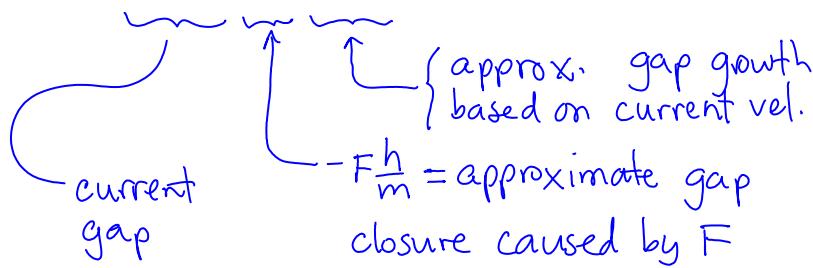
$$0 \leq \lambda_n^{l+1} \perp \lambda_n^{l+1} + 10 - x^l - n_x^l \geq 0$$

LCP of size 1.

$$0 \leq z \perp Bz + b \geq 0$$

$B=1 \Rightarrow$ unique solution exists since $B>0$

$$b = 11 - x^l - 1 - N_x^l$$



$$0 \leq \lambda_n^{l+1} \perp \lambda_n^{l+1} + b \geq 0$$

$$\uparrow \text{ If } b > 0, \lambda_n^{l+1} = 0$$

$$\text{if } b < 0, \lambda_n^{l+1} = -b$$

$$0 \leq \lambda_n^{l+1} \perp \lambda_n^{l+1} + 10 - x^l - N_x^l \geq 0$$

$$N_x^{l+1} = N_x^l + 1 - \lambda_n^{l+1}$$

$$x^{l+1} = x^l + N_x^{l+1}$$

$$l=5$$

$$0 \leq \lambda^6 \perp \lambda^6 - 4 \geq 0$$

$$\Rightarrow \lambda^6 = 4$$

$$N_x^6 = 4 + 1 - 4 = 1$$

$$x^6 = 10 + 1 = 11$$

l	x^l	N_x^l	ψ_n^l	λ_n^l
0	0 ⁺	0	11	not computed
1	1 ⁺	1	10	0
2	3 ⁺	2	8	0
3	6 ⁺	3	5	0
4	10	4	1	0
5	11	1	0	4
6	11	0	0	2
7	11	0	0	1
\vdots	\vdots	\vdots	\vdots	\vdots

Note that two time steps are required to complete the impact.

First time step prevents penetration by apply impulse to reduce velocity

Second time step brings velocity to zero

with a second impulse

Finally the steady state solution with contact leads to a contact force $\lambda_n = F$ for all time.



Consider case when wall position is a function of time.

$$\ddot{v} = \frac{1}{m}F$$

$$\dot{u} = v$$

$$0 \leq \lambda_n \perp \Psi_n(u, t) \geq 0$$

Now discretize

$$v^{l+1} = v^l + \frac{h}{m}(F - \lambda_n^{l+1})$$

$$u^{l+1} = u^l + h v^{l+1}$$

$$0 \leq \lambda_n^{l+1} \perp \Psi_n(u^{l+1}, t_{l+1}) \geq 0$$

generally nonlinear, so
use linear approximation

$$\Psi_n^{l+1} \approx \Psi_n^l + \frac{\partial \Psi_n^l}{\partial u} \Delta u + \frac{\partial \Psi_n^l}{\partial t} h$$

$$\Delta u = u^{l+1} - u^l = h v^{l+1}$$

Apply to 1D problem.

Let $\Psi_n(u, t) = x_w(t) - x$ linear in x, t
approx of Ψ_n^{l+1} will be exact.

where $x_w = l + v_w t$

wall velocity
wall position

.. v_w .. l .. $v_w t$.. $l + v_w t$.. $l + v_w t + v_w h$.. $l + v_w t + v_w h + v_w h$..

$$\begin{aligned}\Psi_n^{l+1} &= \Psi_n^l + (-1) \nabla_{\omega}^{l+1} + \nabla_w^l \nabla^{-1} \\ &= \Psi_n^l - \nabla_x^{l+1} + \nabla_w^l\end{aligned}$$

Let wall velocity be constant $\nabla_w = -1$ (moving left)

$$\Psi_n^{l+1} = \Psi_n^l - \nabla_x^{l+1} - 1$$

LCP

$$\nabla_x^{l+1} = \nabla_x^l + 1 - \lambda_n^{l+1}$$

$$x^{l+1} = x^l + \nabla_x^{l+1}$$

$$0 \leq \lambda_n^{l+1} \perp \lambda_n^{l+1} + 9 - x^l - \nabla_x^l - t^l \geq 0$$

$$\Psi_n^l = 11 - t^l - x^l$$

substitute

$$\Psi_n^{l+1} = \Psi_n^l - (\nabla_x^l + 1 - \lambda_n^{l+1}) - 1$$

Recall,

$$\Psi_n^l = 11 - t^l - x^l$$

$$\therefore \Psi_n^{l+1} = 11 - 2 - t^l - x^l - \nabla_x^l + \lambda_n^{l+1}$$

$t^l = l$	x^l	∇_x^l	Ψ_n^l	λ_n^l
0	0	0	11	not computed
1	1	1	9	0
2	3	2	6	0
3	6	3	2	0
4	7	1	0	3
5	6	-1	0	3
6	5	-1	0	1
:	:	:	:	:

steady state

Time stepping

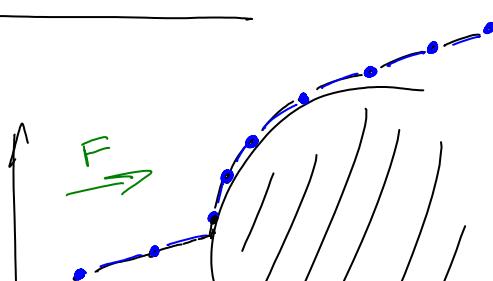
- 1.) solve for λ_n^{l+1}
- 2.) solve for ∇_x^{l+1}
- 3.) solve for x^{l+1}
- 4.) Increment time

anticipate impact
complete impact

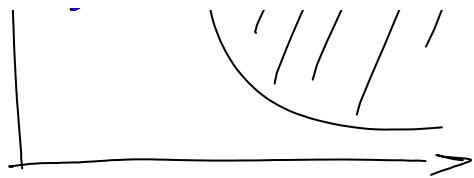
3D : Now let particle move in Space

Obstacle

$\Psi_n(u(t), t)$
↑ possibly



moving
obstacle



No difference from

previous derivation, but now Ψ_n^{l+1} will be an

$$\text{Mixed NCP} \left\{ \begin{array}{l} v^{l+1} = v^l + \frac{h}{m} (F - \lambda_n^{l+1}) \\ u^{l+1} = u^l + h v^{l+1} \\ 0 \leq \lambda_n^{l+1} \perp \Psi_n^{l+1} \geq 0 \end{array} \right.$$

generally nonlinear, even nonsmooth when using polyhedral bodies.

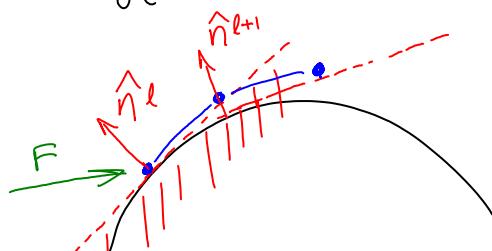
We prefer LCPs, so

approximate Ψ_n^{l+1} as above

$$\Psi_n^{l+1} \approx \Psi_n^l + \frac{\partial \Psi_n^l}{\partial u} \Big|_{t_e} v^{l+1} h + \frac{\partial \Psi_n^l}{\partial t} \Big|_{t_e} h$$

To make the complementarity problem linear,
we evaluate $\frac{\partial \Psi_n^l}{\partial u}$ and $\frac{\partial \Psi_n^l}{\partial t}$ at $t = t_e$.

Consider particle
moving on a
curved surface

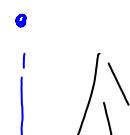


$$\frac{\partial \Psi_n^l}{\partial u} = \hat{n}$$

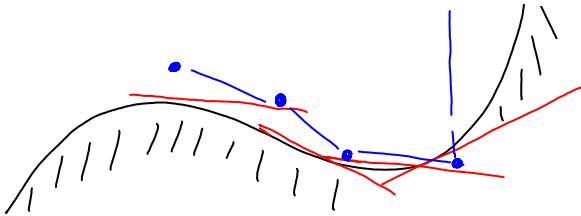
$$\Psi_n^l + h \left(\frac{\partial \Psi_n^l}{\partial u} v^{l+1} + \frac{\partial \Psi_n^l}{\partial t} \right) \geq 0$$

local approximation of
surface is planar.
Large step yields large
error.

Large time steps
can cause large

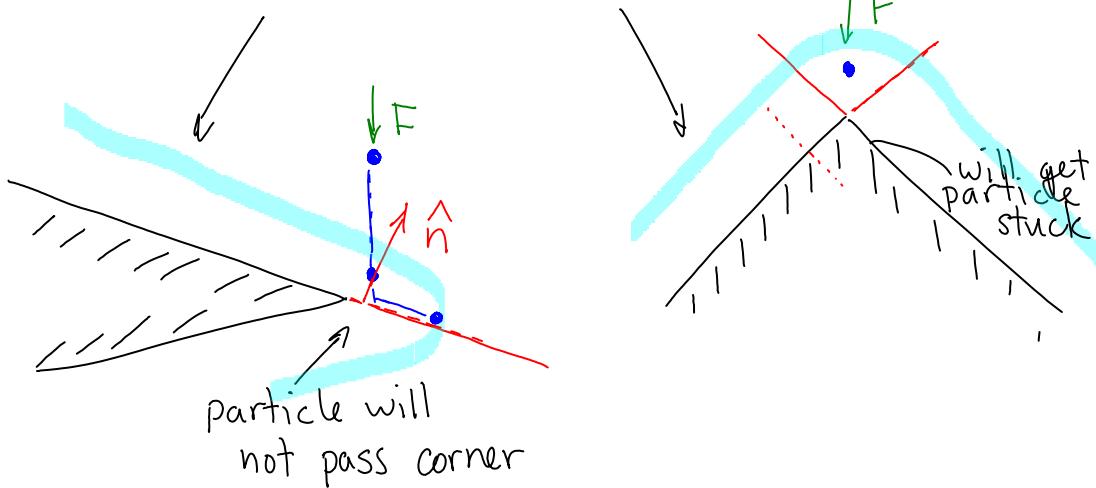


can cause large penetrations, which cause large non-physical forces/impulses which manifest as bouncing even if the coeff. of restitution is zero!



Particle near a nonsmooth surface.
Half space is not a good model

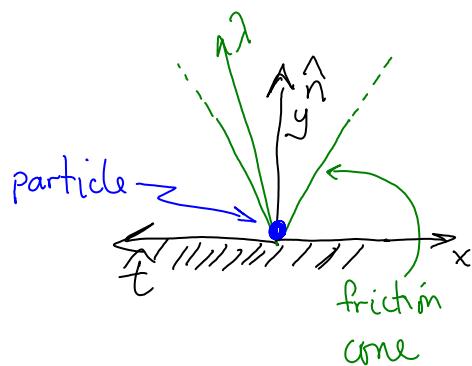
dVC uses boundary layer to select imminent contacts



Let's add Coulomb Friction

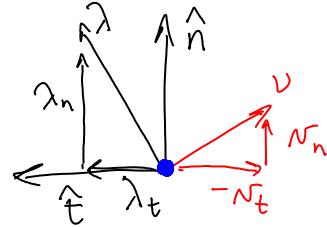
Planar Case

$$\nu = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} N_x \\ N_y \end{bmatrix}$$



Convenient to work in contact frame $\{\hat{n}, \hat{t}\}$.

$$v = \begin{bmatrix} N_n \\ N_t \end{bmatrix} = \begin{bmatrix} N_y \\ -N_x \end{bmatrix}$$



Rules for Coulomb's Law:

$$\text{If } N_n > 0, \text{ then } \lambda = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $N_n = 0$, then :

$$\lambda_t = -\mu \lambda_n \quad \text{if } N_t > 0$$

$$\lambda_t = \mu \lambda_n \quad \text{if } N_t < 0$$

$$-\mu \lambda_n \leq \lambda_t \leq \mu \lambda_n \quad \text{if } N_t = 0$$

For convenience, divide λ_t into positive and negative parts (standard math programming trick)

$$\lambda_t = \lambda_{f_1} - \lambda_{f_2}$$

$$\text{s.t. } \lambda_{f_1}, \lambda_{f_2} \geq 0$$

$$\lambda_{f_1} \perp \lambda_{f_2}$$

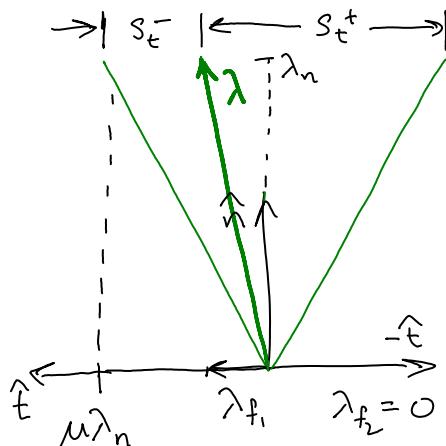
Introduce 2 nonnegative slack variables, s_t^-, s_t^+

s_t^+ = distance to right edge of frict. cone

s_t^- = distance to left edge of frict cone

$$\therefore s_t^+ + s_t^- = 2\mu \lambda_n$$

Physical implications :



Physical implications :

$s_t^+ = 0 \Rightarrow$ sliding to left

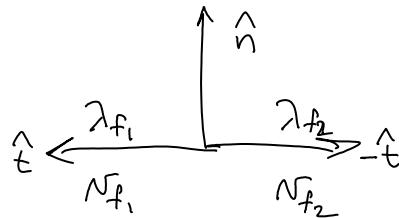
$s_t^- = 0 \Rightarrow$ sliding to right

Similarly, write N_t as a sum of pos. & neg. parts

$$N_t = N_{f_1} - N_{f_2}$$

$$\text{s.t. } N_{f_1}, N_{f_2} \geq 0$$

$$N_{f_1} \perp N_{f_2}$$



Write friction model as complementarity conditions

$$0 \leq \mu \lambda_n + \lambda_t \perp N_{f_1} \geq 0$$

$$0 \leq \mu \lambda_n - \lambda_t \perp N_{f_2} \geq 0$$

Four cases :

① Sticking

$$\begin{aligned} \mu \lambda_n + \lambda_t > 0 &\Rightarrow N_{f_1} = 0 \\ \mu \lambda_n - \lambda_t > 0 &\Rightarrow N_{f_2} = 0 \end{aligned} \} \Rightarrow N_t = 0$$

② Slide Right

$$\mu \lambda_n + \lambda_t > 0 \Rightarrow N_{f_1} = 0$$

$$N_{f_2} > 0 \Rightarrow \lambda_t = \mu \lambda_n$$

③ Slide Left

$$N_{f_1} > 0 \Rightarrow \lambda_t = -\mu \lambda_n$$

$$\mu \lambda_n - \lambda_t > 0 \Rightarrow N_{f_2} = 0$$

④ Degenerate Sliding

$$\left. \begin{array}{l} N_{f_1} > 0 \Rightarrow \lambda_t = -\mu \lambda_n \\ N_{f_2} > 0 \Rightarrow \lambda_t = \mu \lambda_n \end{array} \right\} \Rightarrow \mu = 0 \text{ or } \lambda_n = 0$$

Alternative formulation (that extends to 3D).

$$0 \leq \lambda_f \perp G_f^T v + s \geq 0$$

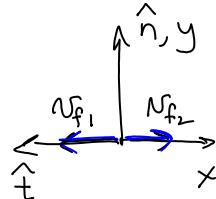
$$0 \leq s \perp \mu \lambda_n - 1^T \lambda_f \geq 0$$

where $\lambda_f = \begin{bmatrix} \lambda_{f_1} \\ \lambda_{f_2} \end{bmatrix}$, $1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $G_f = \begin{bmatrix} \hat{t} & -\hat{t} \end{bmatrix}$

$G_f^T v$ = tangential velocity components in the f_1 and f_2 directions (in $\{\mathcal{N}\}$). no moment components for a particle

For the particle example, we have:

$$G_f^T v = \begin{bmatrix} \hat{t} \text{ expressed in } \{\mathcal{N}\} \\ -\hat{t} \text{ expressed in } \{\mathcal{N}\} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} N_{f_1} \\ N_{f_2} \end{bmatrix}$$



The complementarity conditions are:

$$0 \leq \lambda_{f_1} \perp -\dot{x} + s \geq 0 \quad (1)$$

$$0 \leq \lambda_{f_2} \perp \dot{x} + s \geq 0 \quad (2)$$

$$0 \leq s \perp \mu \lambda_n - \lambda_{f_1} - \lambda_{f_2} \geq 0 \quad (3)$$

How many cases are there to consider?

$$\underline{\underline{2^3 = 8}}$$

Consider all 8 cases systematically:

Case 1 Sticking

eqs. $\lambda_c > 0 \Rightarrow -\dot{x} + s = 0$

case #	Signs on left	Sign on Right
1	+ +	0 0

eqs.

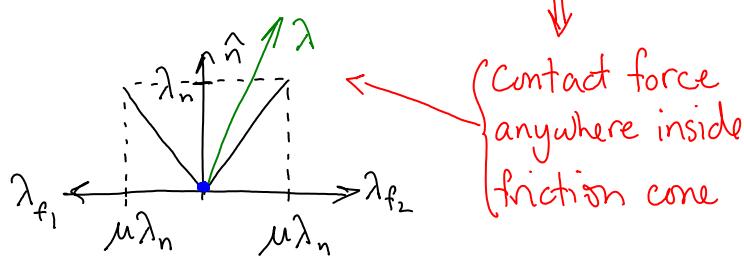
$$(1) \lambda_{f_1} > 0 \Rightarrow -\dot{x} + s = 0$$

$$(2) \lambda_{f_2} > 0 \Rightarrow \dot{x} + s = 0$$

$$(3) s > 0 \Rightarrow \mu \lambda_n - \lambda_{f_1} - \lambda_{f_2} = 0$$

$$\begin{cases} -\dot{x} = -s \\ \dot{x} = -s \end{cases} \Rightarrow \dot{x} = s = 0$$

$$\begin{cases} \lambda_{f_1} > 0 \\ \lambda_{f_2} > 0 \\ \mu \lambda_n = \lambda_{f_1} + \lambda_{f_2} \end{cases} \Rightarrow \begin{cases} 0 < \lambda_{f_1} < \mu \lambda_n \\ 0 < \lambda_{f_2} < \mu \lambda_n \end{cases}$$



	+	0
1	+	0
	+	0
2	+	0
	0	+
3	+	0
	0	+
4	+	0
	0	+
5	0	+
	+	0
6	0	+
	+	0
7	0	+
	0	+
8	0	+
	0	+
	0	+

Case 2: [Sticking] same idea, but with $s=0$

$$(1) \lambda_{f_1} > 0 \Rightarrow -\dot{x} = -s = 0$$

$$(2) \lambda_{f_2} > 0 \Rightarrow \dot{x} = -s = 0$$

$$(3) s = 0 \Leftarrow \mu \lambda_n > \lambda_{f_1} + \lambda_{f_2}$$

Sticking
 $N_t = N_{f_1} - N_{f_2} = 0$

$$\begin{cases} 0 < \lambda_{f_1} < \mu \lambda_n \\ 0 < \lambda_{f_2} < \mu \lambda_n \end{cases}$$

Only diff. from Case 2
 is that $s = 0 = N_{f_1} = N_{f_2}$
 λ constraints are same

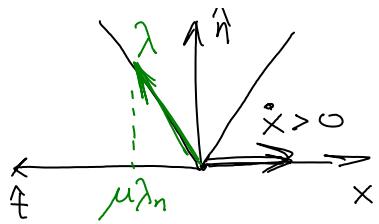
Case 3: [Right Sliding]

$$(1) \lambda_{f_1} > 0 \Rightarrow -\dot{x} = -s$$

$$\Rightarrow \dot{x} = s > 0$$

$$(2) \lambda_{f_2} = 0 \Leftarrow \dot{x} > -s$$

$$(3) s > 0 \Rightarrow \mu \lambda_n - \lambda_{f_1} - \lambda_{f_2} = 0 \Rightarrow \lambda_{f_1} = \mu \lambda_n$$



Cases 4, 5, 6, 7, 8 are left
as an exercise to the student

~~~~~  
Extensions to RIGID  
BODIES 2D & 3D