Solution to Important Background Knowledge Problems

Tuesday, January 20, 2009
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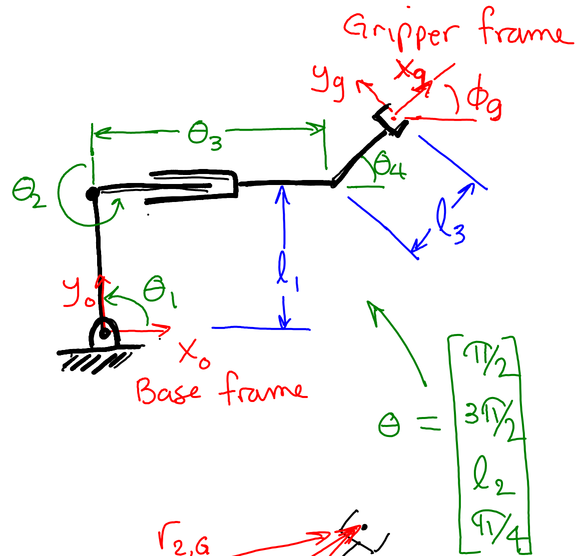
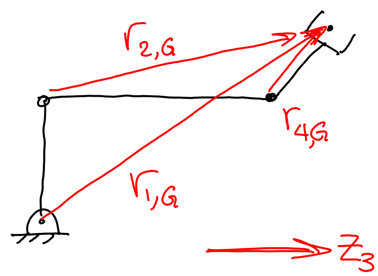
Want ${}^{0}J_G$ & ${}^{4}J_G$ of this planar RRPR manipulator.



Let $r_{i,G}$ denote vector from joint axis to gripper origin $i = 1,2,3,4$



Let $\hat{z}_i$ denote unit vector in direction of i^{th} joint axis.

We need velocity components of gripper origin induced by joint velocities

$$\begin{bmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{\phi}_G \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix}$$

Recall that velocity of tip of a rotating vector is

$$v = \omega \times r$$

For above RRPR manipulator, we have

$$ {}^{0}r_{1,G} = \begin{bmatrix} l_1 c_1 + \theta_3 c_{1,2} + l_3 c_{1,2,4} \\ l_1 s_1 + \theta_3 s_{1,2} + l_3 s_{1,2,4} \\ 0 \end{bmatrix} \qquad {}^{0}z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

For joint 1 we have gripper velocity induced:

$$ \begin{bmatrix} -({}^{0}r_{1G})_y \\ ({}^{0}r_{1G})_x \\ 1 \end{bmatrix} \dot{\theta}_1 $$

where $({}^{0}r_{1G})_x$ & $({}^{0}r_{1G})_y$ are the x- & y-comps. of ${}^{0}r_{1,G}$.

Joint 3 is prismatic, so no angular velocity is induced.

$$ z_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $$

$\therefore$ velocity induced is:

$$ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_3 $$

So, the full Jacobian, ${}^{0}J_G$ is

$$\begin{bmatrix} {}^0\dot{x}_G \\ {}^0\dot{y}_G \\ {}^0\dot{\phi}_G \end{bmatrix} = \underbrace{\begin{bmatrix} -({}^0r_{1G})_y & -({}^0r_{2G})_y & 1 & -({}^0r_{4G})_y \\ ({}^0r_{1G})_x & ({}^0r_{2G})_x & 0 & ({}^0r_{4G})_x \\ 1 & 1 & 0 & 1 \end{bmatrix}}_{{}^0J_G} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix}$$

where

$${}^0r_{2G} = \begin{bmatrix} \theta_3 C_{1,2} + l_3 C_{1,2,4} \\ \theta_3 S_{1,2} + l_3 S_{1,2,4} \\ 0 \end{bmatrix} \qquad {}^0\hat{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^0r_{4G} = \begin{bmatrix} l_3 C_{1,2,4} \\ l_3 S_{1,2,4} \\ 0 \end{bmatrix} \qquad {}^0\hat{z}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now change frames to get ${}^4J_G$.

$${}^0\nu_G = {}^0J_G\,\dot{\theta}$$

${}^0\nu_G$ is a vector whose frame of expression we wish to change.

To make this clearer, assume ν_G is in 3D form:

i.e. $${}^0\nu_G = \begin{bmatrix} {}^0v_G \\ {}^0\omega_G \end{bmatrix} \qquad {}^4\nu_G = \begin{bmatrix} {}^4v_G \\ {}^4\omega_G \end{bmatrix}$$

Now change frame of ${}^0v_G$ and ${}^0\omega_G$

$${}^4v_G = {}^4_0R\,{}^0v_G \qquad {}^4\omega_G = {}^4_0R\,{}^0\omega_G$$

$$\therefore \quad {}^4\mathcal{V}_G = \underbrace{\begin{bmatrix} {}^4_0R & 0 \\ 0 & {}^4_0R \end{bmatrix}}_{6\times 6} {}^0\mathcal{V}_G = {}^4_0\bar{R}\,{}^0\mathcal{V}_G$$

$${}^4_0R = \begin{bmatrix} c_{124} & -s_{124} & 0 \\ s_{124} & c_{124} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now go back to 2D. We don't care about z translational velocity or x, y angular velocity

$${}^4\begin{bmatrix} \dot{x} \\ \dot{y} \\ \cancel{\dot{z}} \\ \cancel{\omega_x} \\ \cancel{\omega_y} \\ \omega_z \end{bmatrix}_G = \begin{bmatrix} {}^4_0R & \\ & {}^4_0R \end{bmatrix} {}^0\begin{bmatrix} \dot{x} \\ \dot{y} \\ \cancel{\dot{z}} \\ \cancel{\omega_x} \\ \cancel{\omega_y} \\ \omega_z \end{bmatrix}_G$$

remove row3 and columns 3,4,5.

So for this planar example, after removing the row3 & columns

$${}^4_0\bar{R} \Longrightarrow {}^4_0R$$
3D $\qquad$ 2D $\qquad$ (Not always true!)

$${}^4v_G = {}^4_0R\,{}^0v_G = {}^4_0R\,{}^0J_G\,\dot{\theta} = {}^4J_G\,\dot{\theta}$$

$$\Rightarrow \ {}^{4}J_G = {}^{4}_{0}R \ {}^{0}J_G$$

In general, transform a Jacobian by

$$\boxed{{}^{i}J_F = {}^{i}_{k}\bar{R} \ {}^{k}J_F}$$

6x6

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2. Given $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

a.) sketch the row space of A

b.) what is the null space of A

c.) find all solutions to $Ax = y$ (you'll need matrices representing the row space and null space of A.

a.)

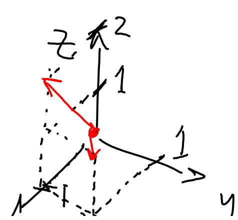
Row space of $\underset{(m \times n)}{A}$ = span of rows of A

Let $\mathcal{R}(A)$ denote the range of A (a.k.a. column space of A)

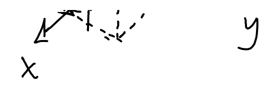
$$\mathcal{R}(A) = \{ y \mid y = Ax, \ x \in \mathbb{R}^n \}$$

equivalently, $\mathcal{R}(A)$ is the vector space formed by all linear combinations of the column of A.

For this example,



$\mathcal{R}(A)$ is the plane

containing the points 

$(1,0,2)$, $(1,1,1)$, and $(0,0,0)$

b.) Find the null space of A, $\mathcal{N}(A)$

$$\mathcal{N}(A) = \{ x \mid Ax = 0 \}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_3$$

$$= -\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_3 = -\begin{bmatrix} 2 \\ -1 \end{bmatrix} x_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_3$$

Now we can write x as a fcn of x_3

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} x_3$$

$$\therefore \mathcal{N}(A) = \left\{ x \mid x = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \alpha, \ \alpha \in \mathbb{R}^1 \right\}$$

Note: $$\begin{array}{r} \text{Dimension of } \mathcal{R}(A^T) = 2 \\ + \ \text{Dim}(\mathcal{N}(A)) = 1 \\ \hline \#\text{ columns}(A) = 3 \end{array}$$

Generally...

$$\boxed{\text{For every } A, \ \text{Dim}(\mathcal{R}(A^T)) + \text{Dim}(\mathcal{N}(A)) = \#\text{cols}(A)}$$

Note also that every vector in $\mathcal{R}(A^T)$ is orthogonal to every vector in $\mathcal{N}(A)$

For this example, show this...

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\in \mathcal{R}(A^T)} \cdot \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \alpha}_{\in \mathcal{N}(A)}$$

$$[x_1 \ x_2] \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\Downarrow} \alpha = \ ?$$

$$[x_1 \ x_2] \begin{bmatrix} 0 \\ 0 \end{bmatrix} \alpha = 0 \quad \therefore \text{ orthogonal}$$

c) Find all solutions of $Ax = y$

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Note that A^{-1} does not exist, so $x = A^{-1}y$ fails

Assuming no knowledge of four fundamental subspaces, this can be worked out exactly as the $\mathcal{N}(A)$ was worked out.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{\text{call this } B} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_3 \right)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B \begin{bmatrix} 1 \\ 2 \end{bmatrix} - B \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_3 \quad , \quad x_3 \in \mathbb{R}^1$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_3$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} x_3}$$

null space component

row space component

Since $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

and since $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

changing x_3 does not violate the equation

$\underline{Ax = y}$

3. a). $p^N = T_B^N \, p^B$

where $T_B^N = \begin{bmatrix} R_B^N & p^N_{Borg} \\ 0 \; 0 \; 0 & 1 \end{bmatrix}_{(4\times4)}$

b.) $v^N = R_B^N \, v^B$