Expansive Spaces: Analysis of Sample-Based Planning

Goal:
Characterize the complexity of a space (C_{free}) w.r.t. path planning.

i.e. How hard is it to create a plan in C_{free}?

i.e. How many sample points are needed to make plans for multiple queries?

What can be proven that would guide the design of motion planning algorithms?

Coverage & Connectivity

Coverage is “good” if any pt. in C_{free} can be connected to the graph by a traj. that is easy to design.
(not necessarily a straight line in C-space).

Not good since
C_free has one component, but G has two
Add green nodes &
cells. Now G gives
good coverage & has
only one component.

“Expansiveness” - attempt to describe how easy/hard
it is to build a good graph.

Def: Visibility of a point:
the fraction of C_free that
can be seen from q.

Def: ε-goodness of a space:
is the lowest value of
visibility fraction of any

Examples:
C_free in above right fig
is 0.5-good.
point in the space.

\( C_{\text{free}} \rightarrow \text{is about} \quad 0.01 \)-good

**Def:** \( \beta \)-Lookout of a subset \( S \) of \( C_{\text{free}} \)

is the subset \( L \) of \( S \) \( \exists \) every \( q \in L \) can see at least \( \beta \) (fraction) of \( C_{\text{free}} \backslash S \).

Every pt in \( L \) sees at least 50% of \( C_{\text{free}} \backslash S \). Pts on \( \partial L \) see more than 50% of \( C_{\text{free}} \backslash S \).

Change \( S \) and try again.

\( q \) is in a 0.75-lookout

(see all of left side portion of \( C_{\text{free}} \backslash S \). \( S \) does not block view.)
Let $\alpha = \frac{\text{Volume}(L)}{\text{Volume}(S)}$

**Def:** $(\varepsilon, \alpha, \beta)$-Expansiveness

$C_{\text{free}}$ is $(\varepsilon, \alpha, \beta)$-expansive if $C_{\text{free}}$ is $\varepsilon$-good and if for each subset $S$ of $C_{\text{free}}$, its $\beta$-lookout contains at least $\alpha$ (fraction) of $S$.

Here $\alpha = 0.08$.
$\beta = 0.50$
$\varepsilon = 0.50$

Is there $S \subset C_{\text{free}}$ such that its 0.5-lookout has $\alpha < 0.08$?

Very difficult to answer in closed form.

Let $S = \text{old } S \setminus \text{old } L$. 
Let $S = \text{old } S \setminus \text{old } L$.

Now the 0.5-lookout is a set like the blue one.

Here it appears that $\alpha < 0.08$, so

Space is not

$(0.5, 0.08, 0.5)$-expansive

If we keep $\beta$ constant, then $\alpha$ must be less than 0.08.

$\epsilon$ is fixed by the space

**Thm** A topological graph $G$ with $\frac{16 \ln(Y \beta)}{\epsilon \alpha} + \frac{\beta}{\epsilon \alpha}$ uniformly sampled points has the correct $C_{\text{free}}$ connectivity with probability $\geq 1 - \beta$
Driving $\tau \to 0 \implies \# \text{pts} \to \infty$

Making $\epsilon, \alpha, \beta$ large $\implies$ fewer points needed.

- For general problem settings there is no efficient way to approximate $\epsilon, \alpha,$ and $\beta$.

- Computing $\epsilon, \alpha, \beta$ would be at least as hard as solving the hardest path finding problem in the space.