1. A planar object is grasped with two hard fingers. The Coefficient of friction at both contact points is 0.5.

   a. Show graphically that the grasp to the right has frictional form closure.

   b. Assume contact 1 is fixed. Highlight the edges of the object where the second contact could be placed such that the grasp has frictional form closure.

   c. Suppose the robot is
not perfectly accurate, so when attempting to place finger \(1\) as shown, one can only guarantee that it is placed in the contact region shown. Assume also that finger \(2\) is perfectly accurate. Sketch the region of finger \(2\) placements that would yield frictional form closure for every placement of finger \(1\) inside the region shown.

2. The hand in the planar system to the right makes two contacts with the object. Contact \(1\) is modeled as a hard finger (point w/ friction) contact.
The other as a point w/o friction.

a. Determine $G$ & $J$

$$
J = \begin{bmatrix}
0 & 0 & 0 \\
-1 & 5 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad G = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & -5 \\
1 & 0 & 0
\end{bmatrix}
$$

If you use $\{N\}$ with origin at center of gravity and axes oriented like so $\uparrow$, then $G^T = \begin{bmatrix}
0.6 & -0.8 & 0.8 \\
0.8 & -0.6 & 0.6 \\
0 & 0 & -5
\end{bmatrix}$

b. If the contact points on the fingers could move arbitrarily, could they be chosen to cause any desired $\nu \in \mathbb{R}^3$?

Yes, because $\text{Rank}(G) = 3$, $G$ provides a "1-to-1" and "onto" mapping between $\nu_{cc}$ & $\nu$, i.e. for every $\nu \in \mathbb{R}^3 \exists$ exactly one $\nu_{cc} \in \mathbb{R}^6$ such that $G^T \nu_{cc} = \nu$.
c. Does this grasp have form closure?
   No. There must be at least 4 contacts for form cl. in the plane.

d. Does the grasp have force closure.
   No. \( \text{Rank}(G) = 3 \), \( \text{Rank}(GJ) = 1 \).
   Another necessary condition is at least 4 distinct friction cone edges.
   The grasp has only 3: \( s_1, s_2, s_3 \).

e. Complete the picture below, i.e. identify the dimensions of the various subspaces.
\[ \text{Rank}(J) = 1 \implies \text{Dim}(R(J)) = 1 = \text{Dim}(R(J^T)) \]

f. Identify an element of \( R(J) \) and interpret it in physical terms specifically applied to the hand. You can explain in terms of velocities or forces, which ever you are more comfortable with.

\[ R(J) = \text{column space of } J = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \alpha_1 + \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \end{bmatrix} \alpha_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \alpha_3 \]

arbitrary scalar

The elements of \( R(J) \) are:

\[ Jq = \begin{bmatrix} N_{1n} \\ N_{2n} \\ N_{1t} \\ N_{2t} \end{bmatrix} = \begin{bmatrix} 0 \\ -\alpha_1 + 5\alpha_2 \\ 0 \end{bmatrix} \]

\[ \therefore \text{each element of } R(J) \]

has \( N_{1n} = N_{2n} = 0 \), \( N_{1t} \) arbitrary

An element in \( R(J) \) is motion in the contact constraint directions that can be accomplished by the hand.
So, the hand can only cause contact point 0 to move in the ± $t_1$ direction.

Force interp: The hand can control $\lambda_{t1}$, but not $\lambda_{1n}$ or $\lambda_{2n}$.

More analysis (not required for this problem).

$$\mathbf{N}(J) = \begin{bmatrix} 1 & 0 \\ 5 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{N}(J^T) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3. A frictionless particle moves toward a corner. Let $m = h = 1$. 

$$\mathbf{v}^l = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \mathbf{P}_{ext} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{u}^l = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$$
Let \( m = h = 1 \).

\[
\psi_{2n} = \frac{1}{\sqrt{4}} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}
\]

\[
u = \psi_n = \begin{bmatrix} x \\ y \end{bmatrix}
\]

a. Set up the time-stepping LCP including both \( \psi_n \) and \( \psi_{2n} \).

No friction; the eqns become

\[
\frac{\partial \psi_n}{\partial t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
G_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Mixed LCP

\[
\begin{bmatrix} \psi_n \\ \rho_n^{\text{lin}} \end{bmatrix} = \begin{bmatrix} M & -G_n^T \\ G_n & 0 \end{bmatrix} \begin{bmatrix} \nu^{\text{lin}} \\ \rho_n^{\text{lin}} \end{bmatrix} + \begin{bmatrix} -Mv^l - \text{ext} \\ \frac{\psi_n}{h} + \frac{\partial \psi_n}{\partial t} \end{bmatrix}
\]

\[0 \leq \rho_n^{\text{lin}} + \rho_n^{\text{lin}} \geq 0\]

Configuration update

\[\nu^{l+1} = \nu^l + h\nu^{l+1}\]

Convert mixed LCP to pure LCP

\[0 = M\nu^{l+1} - G_n\rho_n^{l+1} - Mv^l \Rightarrow \nu^{l+1} = \nu^l + \rho_n^{l+1}\]

\[\rho_n^{l+1} = G_n^T\nu^{l+1} + \frac{\psi_n^l}{h} \Rightarrow \rho_n^{l+1} = \nu_n^{l+1} + \frac{\psi_n^l}{h} = 1\]
Here is the pure LCP including both contacts.

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} \leq
\begin{bmatrix}
P_{n}^{l+1} \\
P_{2n}^{l+1}
\end{bmatrix} - \begin{bmatrix}
P_{n}^{l+1} \\
P_{2n}^{l+1}
\end{bmatrix} + \begin{bmatrix}
N_{x}^{l} \\
N_{y}^{l}
\end{bmatrix} + \begin{bmatrix}
\psi_{in}^{l} \\
\psi_{2n}^{l}
\end{bmatrix} \leq \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

b. Temporarily ignore \( \psi_{2n} \). Determine \( u^{l+1} \) and \( p_{n}^{l+1} \).

\[
0 \leq P_{n}^{l+1} - P_{n}^{l+1} + N_{x}^{l} + \psi_{n}^{l} \geq 0
\]

\[
0 \leq p_{n}^{l+1} - 1 + \frac{1}{4} \geq 0 \quad \Rightarrow \quad p_{n}^{l+1} = \frac{3}{4}
\]

State updates: \( u^{l+1} = u^{l} + p_{n}^{l+1} \)

\[
\begin{bmatrix}
N_{x}^{l+1} \\
N_{y}^{l+1}
\end{bmatrix} = \begin{bmatrix}
-1 \\
-1
\end{bmatrix} + \begin{bmatrix}
\frac{3}{4} \\
0
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{4} \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
x^{l+1} \\
y^{l+1}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{4} \\
\frac{1}{2}
\end{bmatrix} + \begin{bmatrix}
-\frac{1}{4} \\
-1
\end{bmatrix} = \begin{bmatrix}
0 \\
-\frac{1}{2}
\end{bmatrix}
\]

Apparently ignoring \( \psi_{2n} \) was a bad idea.

c. For the next time step include both constraints.

Compute \( p_{n}^{l+2} \) and \( u^{l+2} \).
\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = \begin{bmatrix}
p_{x,n+2} \\
p_{y,n+2}
\end{bmatrix} - \begin{bmatrix}
p_{x,n} \\
p_{y,n}
\end{bmatrix} + \begin{bmatrix}
x_{x,n+1} \\
y_{y,n+1}
\end{bmatrix} + \begin{bmatrix}
x_{1,n} \\
y_{1,n}
\end{bmatrix} + \begin{bmatrix}
-\frac{1}{4} \\
-1
\end{bmatrix} + \begin{bmatrix}
0 \\
-\frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
p_{x,n+2} \\
p_{y,n+2}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{4} \\
\frac{3}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_{x,n+2} \\
y_{y,n+2}
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{4} \\
-1
\end{bmatrix} + \begin{bmatrix}
\frac{1}{4} \\
\frac{3}{2}
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{1}{2}
\end{bmatrix}
\]

notice that the particle "bounced" of the horizontal obstacle, just fast enough to eliminate the penetration at the end of the time step.

d. Assume that Coulom friction with coefficients \( \mu_1 \) & \( \mu_2 \) act between the particle and the two constraint surfaces. Define \( E, U, G_f \) for this problem.

\[
U = \begin{bmatrix}
\mu_1 & 0 \\
0 & \mu_2
\end{bmatrix} \begin{bmatrix}
\hat{t}_i \\
\hat{t}_j
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
0 & -1
\end{bmatrix}
\]
\[ u = \begin{bmatrix} 0 & \mu \end{bmatrix} \]

\[ E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ G_f^T = \begin{bmatrix} \hat{t}_1^T \\ -\hat{t}_1^T \\ \hat{t}_2^T \\ -\hat{t}_2^T \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix} \]