1. Two bodies in the plane touch at a contact point with Coulomb friction.

Let \( \lambda = \begin{bmatrix} \lambda_n \\ \lambda_t \end{bmatrix} \) be the contact force applied to body 2 by body 1.

The relative velocity of the contact point of body 2 wrt. body 1 is \( \begin{bmatrix} \dot{N}_n \\ \dot{N}_t \end{bmatrix} \).

Assume \( N_n = 0, \mu > 0, \lambda_n > 0 \).

Let \( N_t \) be represented by the difference of its positive and negative parts, i.e.,

\[
N_t = N_{f_1} - N_{f_2}, \quad N_{f_1} \geq 0, \quad N_{f_2} \geq 0
\]

I claim that the following pair of linear complementarity conditions model planar Coulomb friction:

\[
0 \leq \mu \lambda_n + \lambda_t \perp N_{f_1} \geq 0
\]

\[
0 \leq \mu \lambda_n - \lambda_t \perp N_{f_2} \geq 0
\]
Demonstrate that I am right or wrong.

The model must allow the contact force to be anywhere inside the friction cone when \( N_{f_1} = N_{f_2} = 0 \). If \( N_{f_1} > 0 \) \& \( N_{f_2} = 0 \), then \( \lambda_t \) must equal \(- \mu \lambda_n\). If \( N_{f_2} > 0 \) and \( N_{f_1} = 0 \), then \( \lambda_t \) must equal \( \mu \lambda_n \).

**Case 1**: \( N_{f_1} = N_{f_2} = 0 \) \( \Rightarrow \) \( \mu \lambda_n + \lambda_t \geq 0 \) \& \( \mu \lambda_n - \lambda_t \geq 0 \). These define the friction cone.

**Case 2**: \( N_{f_1} = 0 \), \( \mu \lambda_n - \lambda_t = 0 \) \( \Rightarrow \) \( N_{f_2} \geq 0 \), \( \mu \lambda_n + \lambda_t \geq 0 \). \( \lambda_t = \mu \lambda_n \). \( \lambda_t \) is to the left which is opposite sliding direction.

\( \lambda_t = \mu \lambda_n \) satisfies so all is good.

**Case 3**: \( N_{f_2} = 0 \), \( \mu \lambda_n + \lambda_t = 0 \). Analysis same as Case 2, but with signs flipped.

**Case 4**: \( \mu \lambda_n + \lambda_t = 0 \) \& \( \mu \lambda_n - \lambda_t = 0 \) \( \Rightarrow \) \( N_{f_1}, N_{f_2} \geq 0 \) sticking or sliding in
$\lambda_t = \lambda_n = 0$

Degenerate case.

Cases 1, 2, 3, properly model friction, case 4 does not hurt anything, just defines a rare case.

The claim is true!

\[ \text{A hand with two fingers is grasping an object with contact points in a hole.} \]

\[ \text{a. Construct } G \text{ and } J \text{ (If the assumed order of } \nu_{cc} \text{ is } \nu_{cc} = [N_1, N_1, N_2, N_2]^T \]
\[
G = \begin{bmatrix}
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}
\]

is \( \nu_{cc} = [\nu_1 \nu_{1+} \nu_{2n} \nu_{2b}]^T \)

then a possible basis of \( \mathcal{N}(G) \) is \([1 \ 0 \ 0 \ 0]^T\)

\[
J = \begin{bmatrix}
-1 & -1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

b. What are the dimensions of the four subspaces of \( G \) and the four of \( J \)?

\[
\dim(\mathcal{R}(J)) = \dim(\mathcal{R}(J^T)) = 4
\]
\[
\dim(\mathcal{N}(J)) = \dim(\mathcal{N}(J^T)) = 0
\]
\[
\dim(\mathcal{R}(G)) = \dim(\mathcal{R}(G^T)) = 3
\]
\[
\dim(\mathcal{N}(G)) = 1 \quad \dim(\mathcal{N}(G^T)) = 0
\]

c. Show that the grasp has frictional form closure for any \( \mu > 0 \).

Since \( \hat{\mathbf{n}}_1 \neq -\hat{\mathbf{n}}_2 \) are colinear, then the line segment joining contact points 1 & 2 will always
be in the negative friction cones. q.e.d.

d: Show that the grasp has force closure.

In addition to frictional form closure, we must have \( \mathcal{N}(G) \cap \mathcal{N}(J^T) = \emptyset \).

Since \( \mathcal{N}(J^T) = 0 \), q.e.d.

e: You may permanently lock a single joint. Can you choose one which will cause the grasp to lose force closure? [No]. All possible \( J \):

\[
J_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad J_2 = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad J_3 = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad J_4 = \begin{bmatrix}
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
N(J_1^T) = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad N(J_2^T) = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad N(J_3^T) = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \quad N(J_4^T) = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\[
\mathcal{N}(J_i^T) \cap \mathcal{N}(G) = \mathcal{N}(J_i^T) \cap \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \text{ for all } i
\]
f. You may permanently lock two joints. Can you choose two which will cause the grasp to lose force closure?

Yes] Joints 1 & 3. \( \Rightarrow \) \( J = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \)

\[ N(J^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad N(G) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

\( N(G) \cap N(J^T) \neq 0 \) \( \Rightarrow \) \( N(J^T)_\alpha \Leftrightarrow N(G)_\alpha \)

\( \therefore \) Force closure is lost by locking joints 1 & 3.

Remember, force closure does not require \( \text{Rank}(GJ) = n \). It requires \( \text{Rank}(G) = n \), frictional form closure, and

\( N(G) \cap N(J^T) = 0 \).

\( \psi_{2n}^\theta = -1 \)

\( \psi_{2n} = 1 \)

3. A particle moving in the plane is near a corner.
the plane is near a corner. \{N\} \times \hat{e}_1

Assume mass = 1, h = 1, \mu_1 = \mu_2 = 0
\nN_x = 0, \quad N_y = -2, \quad F_y = 1

(a) Set up the time-stepping LCP
taking both edges into account.

We need the big matrix & vector that define the LCP.

\[
\begin{bmatrix}
M & -G_n & -G_f & 0 \\
G_n^T & 0 & 0 & 0 \\
G_f^T & 0 & -E \cdot U & 0 \\
0 & U & -E^T & 0
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
-1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}, \quad U = \begin{bmatrix}
\mu_1 & 0 \\
0 & \mu_2
\end{bmatrix}
\]

\[
G_n = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad G_f = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-Mu^0 - p_{ext} \\
\psi_n^0 + \partial \psi_n^h / \partial t \\
\partial \psi_f / \partial t \\
0
\end{bmatrix}
\]

\[
V^0 = \begin{bmatrix}
0 \\
-2
\end{bmatrix}, \quad p_{ext} = \begin{bmatrix}
0
\end{bmatrix}
\]

\[
\psi_n^0 = \begin{bmatrix}
1 \\
-1/2
\end{bmatrix}, \quad \partial \psi_n / \partial t = 0, \quad \partial \psi_f / \partial t = 0
\]
b. Solve for $u^{l+1}$, $u^{t+1}$, $p^{t+1}$

\[
\begin{bmatrix}
0 \\
\rho_{n}^{l+1}
\end{bmatrix} =
\begin{bmatrix}
M - G_n \\
G_n^T
\end{bmatrix}
\begin{bmatrix}
u^{l+1} \\
p^{t+1}_n
\end{bmatrix}
+ \begin{bmatrix}
-Mu^e - \text{Pext} \Omega \\
\psi_n \phi + \frac{\partial \psi}{\partial u}
\end{bmatrix}
\]

\[0 \leq \rho_n^{l+1} \geq 0\]

\[u^{l+1} = u^l + G_n \rho_n^{l+1} + \text{Pext} \Rightarrow \begin{bmatrix}
\n_x^{l+1} \\
_y^{l+1}
\end{bmatrix} = \begin{bmatrix}
0 \\
-2
\end{bmatrix} + \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\rho_n^{l+1} \\
p^{t+1}_n
\end{bmatrix}
+ \begin{bmatrix}
0
\end{bmatrix}
\]

\[\rho_n^{l+1} = G_n^T u^{l+1} + \frac{\psi_n^l}{\phi^l} \geq 0 \Rightarrow \]

\[\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
-2
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\rho_n^{l+1} \\
p^{t+1}_n
\end{bmatrix}
+ \begin{bmatrix}
0
\end{bmatrix}
\begin{bmatrix}
0 \n
\end{bmatrix}
+ \begin{bmatrix}
-\frac{1}{2}
\end{bmatrix} \geq 0
\]

\[0 \leq \begin{bmatrix}
\rho_n^{l+1} \\
p^{t+1}_n
\end{bmatrix}
- \begin{bmatrix}
\rho_n^{l+1} \\
p^{t+1}_n
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-2
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
-\frac{1}{2}
\end{bmatrix} \geq 0
\]

**Solution is unique** ...

\[
\begin{bmatrix}
\rho_n^{l+1} \\
p^{t+1}_n
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{1}{2}
\end{bmatrix}
\]

\[u^{l+1} = \begin{bmatrix}
\n_x^{l+1} \\
_y^{l+1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
-2
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\rho_n^{l+1} \\
p^{t+1}_n
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{2} \\
-1
\end{bmatrix} = u^{t+1}
\]

\[u^{l+1} = \begin{bmatrix}
x^l \\
y^l
\end{bmatrix}
+ \sqrt{u^{t+1}} \kappa \Rightarrow \begin{bmatrix}
x^l \\
y^l
\end{bmatrix}
= \begin{bmatrix}
x^l + \frac{1}{2} \\
y^l - 1
\end{bmatrix}
\]
\[ u^{l+1} = \begin{bmatrix} x^l \\ y^l \end{bmatrix} + \sqrt{u^{l+1}} \frac{k}{I_{2x2}} \Rightarrow u^{l+1} = \begin{bmatrix} x^l + \frac{y^l}{2} \\ y^l - 1 \end{bmatrix} \]

Particle moves to corner.

\[ u^l \rightarrow_{\frac{1}{2}} \rightarrow_{1} u^{l+1} \]

**c.** If you did part b. correctly, then \( N_{x}^{l+1} > 0 \).

Since \( g_{app} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) \& \( N_{x}^l = 0 \),

What caused \( N_{x}^{l+1} \) to change?

The particle was in violation of the extended vertical wall constraint.