On Dynamic Multi-Rigid-Body Contact Problems with Coulomb Friction *

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September 28, 1995

Abstract. This paper is a summary of a comprehensive study of the problem of predicting the accelerations of a set of rigid, three-dimensional bodies in contact in the presence of Coulomb friction. We begin with a brief introduction of this problem and its governing equations. This is followed by the introduction of complementarity formulations for the contact problem under two friction laws: Coulomb's Law of quadratic friction and an approximated pyramid law. Existence and uniqueness results for the complementarity problems are presented. Algorithms for solving these problems are proposed and their convergence properties are discussed. Computational results are presented and conclusions are drawn.

Key Words. Multi-rigid-body contact problem, Coulomb friction, complementarity problem, quasi-variational inequality, Lemke's algorithm, interior point algorithm.

1 Introduction

The dynamic multi-rigid-body frictional contact problem is concerned with predicting the accelerations and contact forces of several rigid bodies in contact under friction. For our purposes, the dynamic system is assumed to consist of a number of passive, three-dimensional, bodies referred to as *objects* that move in response to external forces and forces arising from contacts with a number of active (or actuated) bodies referred to as *manipulator links*. Multi-rigid-body contact problems crop up in many engineering applications in which it would be desirable to ignore deformations of the bodies. Thus despite some deficiencies, the multi-rigid-body contact problem is quite important. Indeed, models of multi-rigid body contact have recently received great interest in the virtual reality

*Any findings, conclusions, or recommendations expressed herein are those of the authors and do not necessarily reflect the views of the funding agencies.

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(VR), graphics, and robotics research communities. The VR community is motivated by the desire to enhance the believability of the scenes in which the VR user is submersed. Currently, systems cannot handle situations involving the loss of contact and multiple contacts between objects [17], but progress is being made toward this goal by Baraff [1], who has followed Lötstedt's lead in the application of complementarity theory and pivoting methods [15]. The robotics and graphics communities are interested in the inverse problem. Specifically, roboticists and animators want to discover input trajectories that accomplish a task or action described at a very high level [14, 16]. However, most current systems require too much interaction and input from the human user to be effective. Although some simple robot task planners and animators' tools have been successfully developed that require very little user interaction [3, 6, 25], their enhancements have been significantly hampered by a lack of understanding of rigid body contact mechanics.

The governing equations and constraints of the multi-rigid-body frictional contact problem are the Newton-Euler equations of motion, unilateral and bilateral kinematic consrtaints, a friction law, contact conditions, and the maximum work principle. The bilateral (equality) constraints are associated with permanent joint connections. In contrast, the unilateral (inequality) constraints disallow interpenetration of the bodies at their contact points while the contacts persist. The contact conditions stipulate complementary relations between the normal forces and accelerations, while the maximum work principle implies that the tangential forces oppose the tangential accelerations.

A common approach to solving the multi-rigid-body contact problem, as seen in the DADS package developed by Haug [7], has been based on a formulation of the problem as a system of differential algebraic equations (DAEs). In such an approach, the contact interactions (*e.g.* rolling, sliding, and breaking) are assumed known, so that all unilateral constraints can be replaced by bilateral constraints. The DAEs are integrated over time while contact forces and velocities are monitored to determine when to change the assumed contact interactions. While the DAE approach seems to work well in practice, its basic assumption that *one can always deduce the impending contact interactions* is fundamentally flawed for rigid body systems. Even the simple example of a rod in one-point contact (initially rolling) with a tabletop can have multiple solutions for its impending motion (one with continued rolling, another with the initiation of sliding). This makes choosing the "correct" set of kinematic constraints impossible in some situations.

The present paper reports analytical and computational results obtained in our study of the three-dimensional multi-rigid-body contact problem subject to quadratic Coulomb friction laws and their variations. This paper is a continuation of the previous one [19] in which various complementarity formulations were given for this contact problem and a main existence result was established under the assumption that all contacts are initially rolling. Using the formulations in the previous paper and assuming small friction coefficients and full row rank of the "system Jacobian matrix," we establish herein existence and uniqueness results for the contact problem allowing initial sliding and rolling; we show that for a friction pyramid model and under the same assumptions, Lemke's algorithm [5, Algorithm 4.4.5] will *always* compute a solution of the multi-rigid-body contact model; we also describe a feasible interior-point method [12] for solving the same model. Finally, we report computational results from our MATLAB implementation of these two methods.

2 The Multi-Rigid-Body Contact Model

In this section, we present the fundamental mathematical model for the three-dimensional multi-rigid-body frictional contact problem. During an integration step of the overall continuous-time problem, the model must be formulated and solved to determine the accelerations of the bodies. The accelerations are then used to update the velocities and positions. After the update, the bodies would be tested for collisions. Upon the detection of a collision, an impulse model would be applied. Then the process would repeat. In this paper, we will assume that an impulse model has already been applied if necessary. We will study only the formulation of the equations of motion and the applicable solution techniques for a given time step. The time instant for which the equations of motion are formulated will be referred to as the *current time*. The other assumptions that will apply for the remainder of this paper are:

1. The bodies are rigid.
2. The normal direction at each contact is well-defined.
3. Dry friction exists at each contact point.
4. Each manipulator joint has one degree of freedom.
5. The manipulator has no closed loops formed by the links and joints. Loops involving (unilateral) contacts are allowed.
6. All links are connected (at least indirectly) to ground.

Note that the fourth assumption is not limiting, because joints with more than one degree of freedom can be modeled as a set of one degree-of-freedom joints. The reason for the fifth assumption is to avoid having to consider load distribution in the manipulator joints. This assumption could be removed by applying known techniques. Assumption 6 is applicable to typical Earth-bound systems, (*e.g.*, manufacturing systems). Removing this assumption would fundamentally alter the formulation of the model. Notice that there are no restrictions on the shapes of the bodies.

2.1 The Newton-Euler equations

We begin by numbering the objects from 1 to n_{obj} and manipulator links from 1 to n_{man}. All grounded links are considered to be a single manipulator link. When two bodies i and k (with $i \neq k$) are in contact, we label the contact point as j and consider this point j as associated uniquely with the pair (i, k). Let n_c denote the total number of contact points at the current time. Each contact point j of bodies i and k defines the origin of the contact frame C_j. Let $\hat{\boldsymbol{n}}_j$ denote the contact normal. The other two axes of frame $C_j, \hat{\boldsymbol{t}}_j$ and $\hat{\boldsymbol{o}}_j$, span the contact tangent plane Further, let $\boldsymbol{c}_{i,j}$ denote the contact force acting on body i through contact j and expressed in frame C_j. The three components of $\boldsymbol{c}_{i,j}$ are denoted $(c_{i,j})_n$ (the normal component), $(c_{i,j})_t$ and $(c_{i,j})_o$ (the two tangential components). Next, define $\boldsymbol{W}_{i,j}$ to be the (6×3) wrench matrix that transforms contact force $\boldsymbol{c}_{i,j}$ into the equivalent wrench (generalized force) in the frame B_i. The wrench matrices contain all geometric information relevant to contact j (definitions can be found in [24]).

Assuming that object i is acted upon by an external generalized force vector $\boldsymbol{g}_{\text{obj},i}$ of length 6, and a set of contact forces, the Newton-Euler equation governing the motion of object i is

$$\sum_{j \in \mathcal{B}_i} \boldsymbol{W}_{i,j}\boldsymbol{c}_{i,j} + \boldsymbol{g}_{\text{obj},i} + \boldsymbol{h}_{\text{obj},i} = \boldsymbol{M}_{\text{obj},i}\ddot{\boldsymbol{q}}_i; \quad \in \quad R^6 \tag{1}$$

where $\mathcal{B}_i$ is the index set of contact points that involve body i, $\boldsymbol{h}_{\text{obj},i}$ is the vector of velocity product terms, $\ddot{\boldsymbol{q}}_i$ is the generalized acceleration of object i, and $\boldsymbol{M}_{\text{obj},i}$ is the mass matrix of object i.

The Newton-Euler equations of all the objects can be combined and cast in matrix form as:

$$\boldsymbol{W}_n\boldsymbol{c}_n + \boldsymbol{W}_t\boldsymbol{c}_t + \boldsymbol{W}_o\boldsymbol{c}_o + \boldsymbol{g}_{\text{obj}} + \boldsymbol{h}_{\text{obj}} = \boldsymbol{M}_{\text{obj}}\ddot{\boldsymbol{q}}; \quad \in \quad R^{6n_{obj}} \tag{2}$$

where the elements of the vector $\boldsymbol{c}_n \in R^{n_c}$, are the intensities of the normal wrenches at the contacts, and $\boldsymbol{W}_n$ is the matrix that transforms the normal wrenches to the appropriate body-fixed coordinate frames. The vectors $\boldsymbol{c}_t$ and $\boldsymbol{c}_o$ and the matrices $\boldsymbol{W}_t$ and $\boldsymbol{W}_o$ are analogous. The vectors $\boldsymbol{g}_{\text{obj}}$ and $\boldsymbol{h}_{\text{obj}}$ are the generalized external and velocity product forces, respectively. The generalized acceleration of the passive bodies $\ddot{\boldsymbol{q}}$ and the block diagonal mass matrix $\boldsymbol{M}_{\text{obj}}$ are composed of the acceleration vectors and mass matrices of the individual passive bodies.

The development of the dynamic equations of the manipulator parallels that of equation (2). Let n_θ be the number of joints in the manipulator and let $\boldsymbol{\tau}$ be the n_θ-vector of joint efforts [1]. Summing the effects of all the contacts on the manipulator yields:

$$\boldsymbol{\tau} - (\boldsymbol{J}_n^T \boldsymbol{c}_n + \boldsymbol{J}_t^T \boldsymbol{c}_t + \boldsymbol{J}_o^T \boldsymbol{c}_o + \boldsymbol{g}_{\text{man}} + \boldsymbol{h}_{\text{man}}) = \boldsymbol{M}_{\text{man}} \ddot{\boldsymbol{\theta}}; \quad \in \quad R^{n_\theta} \tag{3}$$

where $\boldsymbol{J}_n^T$, $\boldsymbol{J}_t^T$, and $\boldsymbol{J}_o^T$ map the wrench intensities into the joint axis directions.

2.2 Kinematic constraints

According to our convention, a contact point j uniquely determines the two bodies i and k that touch at this point, and vice versa. Throughout the rest of the paper, we shall refer only to the contact points and not to the bodies in contact; in particular, the pair (i, j) which refers to body i in contact with another body $k > i$ at contact point j is abbreviated as j (with i omitted). We will use the notation $j\alpha$ to mean contact point j along the direction $\alpha \in \{n, t, o\}$.

At the current time, there are $n_c = n_{\mathcal{R}} + n_{\mathcal{S}}$ contacts, where $n_{\mathcal{R}}$ and $n_{\mathcal{S}}$ are the number of rolling and sliding contacts, and $\mathcal{R}$ and $\mathcal{S}$ are the index sets corresponding to the rolling and sliding contacts, respectively. Let $\boldsymbol{v}_j = [v_{jn}\ v_{jt}\ v_{jo}]^T$ and $\boldsymbol{a}_j = [a_{jn}\ a_{jt}\ a_{jo}]^T$, for $j = 1, \ldots, n_c$, be the relative linear velocity and acceleration vectors, expressed in frame C_j of bodies i and k at contact point j. Next, partition $\boldsymbol{v}_n$, $\boldsymbol{v}_t$, and $\boldsymbol{v}_o$ and $\boldsymbol{a}_n$, $\boldsymbol{a}_t$, and $\boldsymbol{a}_o$ into their n, t, and o component vectors, $\boldsymbol{v}_n$, $\boldsymbol{v}_t$, $\boldsymbol{v}_o$, $\boldsymbol{a}_n$, $\boldsymbol{a}_t$, and $\boldsymbol{a}_o$, each of length n_θ. The vectors $\boldsymbol{v}_\alpha$ and $\boldsymbol{a}_\alpha$, for $\alpha \in \{n, t, o\}$, can be expressed in the following convenient form:

$$\begin{aligned} \boldsymbol{v}_\alpha &= \boldsymbol{W}_\alpha^T \dot{\boldsymbol{q}} - \boldsymbol{J}_\alpha \dot{\boldsymbol{\theta}}, \quad \alpha \in \{n, t, o\}, \\ \boldsymbol{a}_\alpha &= \boldsymbol{W}_\alpha^T \ddot{\boldsymbol{q}} - \boldsymbol{J}_\alpha \ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{W}}_\alpha^T \dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}_\alpha \dot{\boldsymbol{\theta}}, \quad \alpha \in \{n, t, o\}, \end{aligned} \tag{4}$$

By definition, the normal component of relative velocity at a contact must be zero and rolling is indicated by both the t and o components being zero also. To prevent interpenetration, the normal component of relative acceleration at must be nonnegative. If it is zero, the contact will be maintained. Otherwise, the contact will break. Specifically, we have: $j \in \mathcal{R}$ if $v_{jt}^2 + v_{jo}^2 = 0$ and $j \in \mathcal{S}$ otherwise. The following equations and inequalities apply at the current time:

$$\boldsymbol{v}_n = \boldsymbol{0} \tag{5}$$

$$v_{jt} = v_{jo} = 0, \quad \text{for all } j \in \mathcal{R} \tag{6}$$

$$\boldsymbol{a}_n \geq \boldsymbol{0}. \tag{7}$$

Notice that equations (5) and (6) constrain the initial conditions of the problem at the current time, but inequality (7) is a constraint on the unknown accelerations of the bodies.

[1] By "joint effort" we mean the force in the direction of motion of a prismatic joint or the moment in the direction of motion of a revolute joint.

2.3 Friction constraints

The remaining constraints enforce certain friction laws. Two will be considered: first, Coulomb's Law which requires each contact force to lie within a quadratic cone, and second, an approximation of Coulomb's Law in which the quadratic cone at each rolling contact is replaced by a four-sided pyramid.

2.3.1 Coulomb's friction law

Coulomb's Law stipulates that the j-th contact force $\boldsymbol{c}_j$ lies within or on the boundary of its corresponding friction cone represented as follows:

$$c_{jt}^2 + c_{jo}^2 \leq \mu_j^2 c_{jn}^2, \quad \text{for } j = 1, \ldots, n_c \tag{8}$$

where μ_j is the coefficient of friction at the j-th contact point. Since the contact forces must be nontensile, we have:

$$c_{jn} \geq 0, \quad \text{for } j = 1, \ldots, n_c. \tag{9}$$

While a contact is sliding, the contact force must lie on the boundary of the friction cone with its friction component directly opposite the relative sliding velocity:

$$\mu_j c_{jn} v_{j\alpha} + c_{j\alpha}\sqrt{v_{jt}^2 + v_{jo}^2} = 0, \quad \text{for } \alpha = t, o \text{ and all } j \in \mathcal{S}. \tag{10}$$

While a contact is rolling, the contact force may have any direction and magnitude, provided it lies in the cone defined by (8) and (9). However, since a rolling contact may convert to sliding, we have:

$$\mu_j c_{jn} a_{j\alpha} + c_{j\alpha}\sqrt{a_{jt}^2 + a_{jo}^2} = 0, \quad \text{for } \alpha = t, o \text{ and all } j \in \mathcal{R}. \tag{11}$$

Note that at an initially rolling contact, either the tangential acceleration components (a_{jt}, a_{jo}) are zero, indicating that the rolling persists, or the contact force (c_{jt}, c_{jo}) lies on the boundary of the friction cone (*i.e.*, (8) holds as an equation). In the latter case, the rolling contact begins to slide and the friction force opposes the sliding motion.

Mathematically, the equation in (10) differs in one significant way from that in (11). Namely, the former is linear in the unknowns, c_{jn}, c_{jt}, and c_{jo}, while the latter is not, because both a_{jt} and a_{jo} are unknown. This observation motivates us to approximate the quadratic friction cones by polygons (similar to related situations in deformable-body contact problems; see *e.g.* [13, 8, 10]); one such approximation is the pyramid model described in the next subsection.

Three-Dimensional Multi-Rigid-Body Contact Problem with Coulomb Friction Law: *Given $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\boldsymbol{g}_{obj}$, $\boldsymbol{h}_{obj}$, $\boldsymbol{M}_{obj}$, $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, $\boldsymbol{\tau}$, $\boldsymbol{g}_{man}$, $\boldsymbol{h}_{man}$, $\boldsymbol{M}_{man}$, and the $\boldsymbol{W}$ and $\boldsymbol{J}$ matrices, with $\dot{\boldsymbol{q}}$ and $\dot{\boldsymbol{\theta}}$ satisfying equations (5) and (6); determine $\ddot{\boldsymbol{q}}$, $\ddot{\boldsymbol{\theta}}$, $\boldsymbol{c}_n$, $\boldsymbol{c}_t$, $\boldsymbol{c}_o$, $\boldsymbol{a}_n$, $\boldsymbol{a}_t$, $\boldsymbol{a}_o$ satisfying equations (2–4), (7–11), and the complementarity condition:*

$$(\boldsymbol{c}_n)^T \boldsymbol{a}_n = 0. \tag{12}$$

The last equation (12) is the main contact condition. It enforces the fact that if a contact force is compressive, then the relative acceleration of the bodies in the normal direction at that contact must be zero (*i.e.*, the contact is maintained). Similarly, if the normal component of the relative acceleration at a contact is positive (*i.e.*, the contact is breaking), then the normal component of the contact force, must be zero; thus the entire contact force must be zero also, by (8).

A solution to the above problem naturally yields four possible types of contact transitions:

(i) rolling $\rightarrow$ rolling:

$$c_{jn} \geq 0 \qquad a_{jn} = a_{jt} = a_{jo} = 0; \quad \text{for all } j \in \mathcal{R};$$

(ii) rolling $\rightarrow$ sliding:

$$c_{jn} \geq 0 \qquad a_{jn} = 0 \qquad a_{jt}^2 + a_{jo}^2 \neq 0, \quad \text{for all } j \in \mathcal{R}; \tag{13}$$

$$c_{jt} = -\frac{\mu_j a_{jt}}{\sqrt{a_{jt}^2 + a_{jo}^2}} c_{jn} \qquad c_{jo} = -\frac{\mu_j a_{jo}}{\sqrt{a_{jt}^2 + a_{jo}^2}} c_{jn}; \quad \text{for all } j \in \mathcal{R}; \tag{14}$$

(iii) sliding $\rightarrow$ sliding:

$$c_{jn} \geq 0, \qquad a_{jn} = 0; \quad \text{for all } j \in \mathcal{S};$$

(iv) rolling or sliding $\rightarrow$ breaking:

$$a_{jn} \geq 0 \qquad c_{jn} = 0 \qquad (\Rightarrow c_{jt} = c_{jo} = 0); \quad \text{for } j = 1, \ldots, n_c.$$

2.4 The friction pyramid law

One reason for approximating the friction cone by a pyramid, thereby replacing the nonlinear constraints (8) and (11) by linear ones, is to facilitate the analysis and algorithmic development. Figure 1 shows a friction cone and its approximation as a four-sided friction pyramid. The interior and boundary of the pyramid can be expressed as a system of *linear* inequalities:

$$\mu_j c_{jn} - c_{j\alpha} \geq 0, \quad \mu_j c_{jn} + c_{j\alpha} \geq 0, \quad \text{for } \alpha = t, o \text{ and all } j \in \mathcal{R}. \tag{15}$$

Along with these friction constraints, we also replace the rolling contact conditions (11) by the following conditions:

$$\mu_j c_{jn} a_{j\alpha} + c_{j\alpha} |a_{j\alpha}| = 0, \quad \text{for } \alpha = t, o \text{ and all } j \in \mathcal{R}. \tag{16}$$

Three-Dimensional Multi-Rigid-Body Contact Problem with Friction Pyramid Law: *Same as the Coulomb law problem except that conditions (8) and (11) are replaced by (15) and (16).*

Accordingly, the rolling-to-sliding contact interaction constraints (13) and (14) now become

(ii) rolling $\rightarrow$ sliding:

$$c_{jn} \geq 0 \qquad a_{jn} = 0 \qquad a_{jt} \neq 0 \text{ OR } a_{jo} \neq 0, \quad \text{for all } j \in \mathcal{R}. \tag{17}$$

$$\left.\begin{array}{l} c_{jt} = \begin{cases} -\frac{\mu_j a_{jt}}{|a_{jt}|} c_{jn} & \text{if } a_{jt} \neq 0 \\ \in [-\mu_j c_{jn}, \mu_j c_{jn}] & \text{otherwise} \end{cases} \\ c_{jo} = \begin{cases} -\frac{\mu_j a_{jo}}{|a_{jo}|} c_{jn} & \text{if } a_{jo} \neq 0 \\ \in [-\mu_j c_{jn}, \mu_j c_{jn}] & \text{otherwise} \end{cases} \end{array}\right\} \quad \text{for all } j \in \mathcal{R}. \tag{18}$$

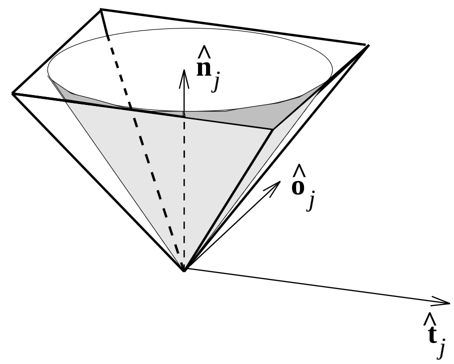

Figure 1: Friction Cone and Pyramid

A deficiency with the friction pyramid law is that if the model predicts a given rolling contact begins to slide (*i.e.*, the solution of the model satisfies equation (17)), then according to equation (16), the friction force will generally not exactly oppose the direction of sliding. Practically speaking, the contact force always lies along one of the edges of the pyramid. Despite this deficiency, the friction pyramid law has the important benefit of leading to a computationally simpler model. In addition, other polyhedral approximations of the quadratic friction cone can also be used; all of which lead to linear complementarity problems as opposed to the nonlinear complementarity problem obtained from the quadratic model.

2.5 Variations of the model

The same techniques to be described below can be applied to two specific variations of the multi-rigid-body contact model. First, the input, $\boldsymbol{\tau}$, and output, $\ddot{\boldsymbol{\theta}}$ can be interchanged. That is, one can assume that the joint effort vector, $\boldsymbol{\tau}$, is unknown, while the joint acceleration vector, $\ddot{\boldsymbol{\theta}}$ is known. More generally, let $\boldsymbol{\tau}_\alpha$ be the subset of the elements of $\boldsymbol{\tau}$ that are known and $\boldsymbol{\tau}_{\bar{\alpha}}$ denote the unknown elements. The corresponding subsets of elements of $\ddot{\boldsymbol{\theta}}$, $\ddot{\boldsymbol{\theta}}_\alpha$ and $\ddot{\boldsymbol{\theta}}_{\bar{\alpha}}$, must be, respectively, unknown and known. Second, the contact friction model can be extended to what is referred to as a "soft finger" model [22]. This model assumes that in addition to three components of force, a contact can also transmit a moment in the direction of the contact normal. More details of this extension can be found in the paper [20].

There are two interesting variations that change the form of the basic model. The first is the quasistatic, multi-rigid-body contact problem which is formed by setting the terms $\boldsymbol{M}_{\text{obj}}\ddot{\boldsymbol{q}}$ and $\boldsymbol{M}_{\text{man}}\ddot{\boldsymbol{\theta}}$ to zero. The quasistatic assumption fundamentally alters the model in ways discussed in [21]. The other variation arises when relaxing the rigid body assumption. In this case, the unknown contact forces are replaced by functions of the configuration and velocity of the system, which relate forces to deformations; see [8, 9, 10, 11, 13].

3 Complementarity Formulations

The mixed nonlinear complementarity problem (mixed NCP) is defined as follows: given two functions $\boldsymbol{f}$ and $\boldsymbol{g}$ from the $(n+m)$-dimensional Euclidean space into itself, find an $(n+m)$-dimensional vector $(\boldsymbol{u}, \boldsymbol{v})$ such that

$$\boldsymbol{u} \geq \mathbf{0}, \quad \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{v}) \geq \mathbf{0}, \quad \boldsymbol{u}^T \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{v}) = 0,$$
$$\boldsymbol{v} \text{ unrestricted in sign}, \quad \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \mathbf{0}.$$

If $m = 0$ and $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{r} + \boldsymbol{M}\boldsymbol{z}$, where $\boldsymbol{r}$ is a given n-vector and $\boldsymbol{M}$ is a given $n \times n$ matrix, then this problem becomes the standard linear complementarity problem, which we denote as LCP $(\boldsymbol{r}, \boldsymbol{M})$. A comprehensive study of the latter problem is contained in the book [5]. In what follows, we show that the contact problem with Coulomb friction can be formulated as a mixed NCP, whereas that with pyramid friction can be formulated as an LCP.

Both formulations begin by eliminating the accelerations, $\ddot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{\theta}}$, as follows:

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}_{\text{obj}}^{-1} \left[\boldsymbol{W}_n \boldsymbol{c}_n + \boldsymbol{W}_t \boldsymbol{c}_t + \boldsymbol{W}_o \boldsymbol{c}_o + \boldsymbol{g}_{\text{obj}} + \boldsymbol{h}_{\text{obj}} \right] \tag{19}$$

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{M}_{\text{man}}^{-1} \left[\boldsymbol{\tau} - \boldsymbol{J}_n^T \boldsymbol{c}_n - \boldsymbol{J}_t^T \boldsymbol{c}_t - \boldsymbol{J}_o^T \boldsymbol{c}_o - \boldsymbol{g}_{\text{man}} - \boldsymbol{h}_{\text{man}} \right]. \tag{20}$$

Note that because $\boldsymbol{M}_{\text{obj}}$ and $\boldsymbol{M}_{\text{man}}$ are positive definite and symmetric, their inverses exist. Substituting these expressions into equation (4) defining $\boldsymbol{a}_\alpha$, we obtain

$$\begin{bmatrix} \boldsymbol{a}_n \\ \boldsymbol{a}_t \\ \boldsymbol{a}_o \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{c}_n \\ \boldsymbol{c}_t \\ \boldsymbol{c}_o \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_n \\ \boldsymbol{b}_t \\ \boldsymbol{b}_o \end{bmatrix} \tag{21}$$

where the matrix $\boldsymbol{A}$ is symmetric positive semidefinite and given by

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{nn} & \boldsymbol{A}_{nt} & \boldsymbol{A}_{no} \\ \boldsymbol{A}_{tn} & \boldsymbol{A}_{tt} & \boldsymbol{A}_{to} \\ \boldsymbol{A}_{on} & \boldsymbol{A}_{ot} & \boldsymbol{A}_{oo} \end{bmatrix} \equiv \mathcal{J}^T \mathcal{M} \mathcal{J}$$

with the system inertia and constraint Jacobian matrices, $\mathcal{M}$ and $\mathcal{J}$, defined as follows:

$$\mathcal{M} \equiv \begin{bmatrix} \boldsymbol{M}_{\text{obj}}^{-1} & 0 \\ 0 & \boldsymbol{M}_{\text{man}}^{-1} \end{bmatrix} \qquad \mathcal{J} \equiv \begin{bmatrix} \boldsymbol{W}_n & \boldsymbol{W}_t & \boldsymbol{W}_o \\ \boldsymbol{J}_n^T & \boldsymbol{J}_t^T & \boldsymbol{J}_o^T \end{bmatrix} \tag{22}$$

and

$$\begin{bmatrix} \boldsymbol{b}_n \\ \boldsymbol{b}_t \\ \boldsymbol{b}_o \end{bmatrix} \equiv \dot{\mathcal{J}}^T \begin{bmatrix} \dot{\boldsymbol{q}} \\ -\dot{\boldsymbol{\theta}} \end{bmatrix} + \mathcal{J}^T \mathcal{M} \begin{bmatrix} \boldsymbol{g}_{\text{obj}} + \boldsymbol{h}_{\text{obj}} \\ \boldsymbol{g}_{\text{man}} + \boldsymbol{h}_{\text{man}} - \boldsymbol{\tau} \end{bmatrix}. \tag{23}$$

Next, we can solve the equations (10) for the friction force components, c_{jt} and c_{jo}, to eliminate those corresponding to the sliding contacts. Since there are no additional restrictions on the tangential accelerations, a_{jt} and a_{jo}, for $j \in \mathcal{S}$, the equations in (21) that define these components can be dropped from the model without affecting its solvability. Let the vectors $\boldsymbol{a}_{\mathcal{S}n}$ and $\boldsymbol{a}_{\mathcal{R}n}$ be the partitions of $\boldsymbol{a}_n$ corresponding to the sliding and rolling contacts respectively (*i.e.*, the elements of

$\boldsymbol{a}_{\mathcal{S}n}$ are a_{jn} for $j \in \mathcal{S}$, and $\boldsymbol{a}_{\mathcal{R}n}$ is defined similarly). Next define $\boldsymbol{a}_{\mathcal{S}t}$, $\boldsymbol{a}_{\mathcal{R}t}$, $\boldsymbol{a}_{\mathcal{S}o}$, $\boldsymbol{a}_{\mathcal{R}o}$, $\boldsymbol{c}_{\mathcal{S}n}$, $\boldsymbol{c}_{\mathcal{R}n}$, $\boldsymbol{c}_{\mathcal{S}t}$, $\boldsymbol{c}_{\mathcal{R}t}$, $\boldsymbol{c}_{\mathcal{S}o}$, and $\boldsymbol{c}_{\mathcal{R}o}$ analogously. Eliminating $\boldsymbol{c}_{\mathcal{S}t}$ and $\boldsymbol{c}_{\mathcal{S}o}$, and removing the equations defining $\boldsymbol{a}_{\mathcal{S}t}$ and $\boldsymbol{a}_{\mathcal{S}o}$ result in:

$$\begin{bmatrix} \boldsymbol{a}_{\mathcal{S}n} \\ \boldsymbol{a}_{\mathcal{R}n} \\ \boldsymbol{a}_{\mathcal{R}t} \\ \boldsymbol{a}_{\mathcal{R}o} \end{bmatrix} = \tilde{\boldsymbol{A}} \begin{bmatrix} \boldsymbol{c}_{\mathcal{S}n} \\ \boldsymbol{c}_{\mathcal{R}n} \\ \boldsymbol{c}_{\mathcal{R}t} \\ \boldsymbol{c}_{\mathcal{R}o} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_{\mathcal{S}n} \\ \boldsymbol{b}_{\mathcal{R}n} \\ \boldsymbol{b}_{\mathcal{R}t} \\ \boldsymbol{b}_{\mathcal{R}o} \end{bmatrix} \tag{24}$$

where

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} (\tilde{\boldsymbol{A}}_{nn})_{\mathcal{S}\mathcal{S}} & (\boldsymbol{A}_{nn})_{\mathcal{S}\mathcal{R}} & (\boldsymbol{A}_{nt})_{\mathcal{S}\mathcal{R}} & (\boldsymbol{A}_{no})_{\mathcal{S}\mathcal{R}} \\ (\tilde{\boldsymbol{A}}_{nn})_{\mathcal{R}\mathcal{S}} & (\boldsymbol{A}_{nn})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{nt})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{no})_{\mathcal{R}\mathcal{R}} \\ (\tilde{\boldsymbol{A}}_{tn})_{\mathcal{R}\mathcal{S}} & (\boldsymbol{A}_{tn})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{tt})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{to})_{\mathcal{R}\mathcal{R}} \\ (\tilde{\boldsymbol{A}}_{on})_{\mathcal{R}\mathcal{S}} & (\boldsymbol{A}_{on})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{ot})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{oo})_{\mathcal{R}\mathcal{R}} \end{bmatrix}. \tag{25}$$

Here

$$\begin{bmatrix} (\tilde{\boldsymbol{A}}_{nn})_{\mathcal{S}\mathcal{S}} \\ (\tilde{\boldsymbol{A}}_{nn})_{\mathcal{R}\mathcal{S}} \\ (\tilde{\boldsymbol{A}}_{tn})_{\mathcal{R}\mathcal{S}} \\ (\tilde{\boldsymbol{A}}_{on})_{\mathcal{R}\mathcal{S}} \end{bmatrix} \equiv \begin{bmatrix} (\boldsymbol{A}_{nn})_{\mathcal{S}\mathcal{S}} \\ (\boldsymbol{A}_{nn})_{\mathcal{R}\mathcal{S}} \\ (\boldsymbol{A}_{tn})_{\mathcal{R}\mathcal{S}} \\ (\boldsymbol{A}_{on})_{\mathcal{R}\mathcal{S}} \end{bmatrix} - \begin{bmatrix} (\boldsymbol{A}_{nt})_{\mathcal{S}\mathcal{S}} \\ (\boldsymbol{A}_{nt})_{\mathcal{R}\mathcal{S}} \\ (\boldsymbol{A}_{tt})_{\mathcal{R}\mathcal{S}} \\ (\boldsymbol{A}_{ot})_{\mathcal{R}\mathcal{S}} \end{bmatrix} \boldsymbol{V}_{\mathcal{S}t} - \begin{bmatrix} (\boldsymbol{A}_{no})_{\mathcal{S}\mathcal{S}} \\ (\boldsymbol{A}_{no})_{\mathcal{R}\mathcal{S}} \\ (\boldsymbol{A}_{to})_{\mathcal{R}\mathcal{S}} \\ (\boldsymbol{A}_{oo})_{\mathcal{R}\mathcal{S}} \end{bmatrix} \boldsymbol{V}_{\mathcal{S}o} \tag{26}$$

where $\boldsymbol{V}_{\mathcal{S}t}$ and $\boldsymbol{V}_{\mathcal{S}o}$ are, respectively, the diagonal matrices with diagonal entries given by

$$\frac{\mu_j v_{jt}}{\sqrt{v_{jt}^2 + v_{jo}^2}} \text{ and } \frac{\mu_j v_{jo}}{\sqrt{v_{jt}^2 + v_{jo}^2}}, \qquad \text{for all } j \in \mathcal{S}$$

and in general, for an $N \times N$ matrix $\boldsymbol{M}$, if α and β are subsets of $\{1, \ldots, N\}, \boldsymbol{M}_{\alpha\beta}$ denotes the submatrix of $\boldsymbol{M}$ consisting of rows and columns indexed by α and β respectively.

3.1 Formulation with Coulomb friction law

To complete the formulation of the contact problem as a mixed NCP, we introduce a slack variable s_j to write (8) as an equation:

$$s_j = \mu_j^2 c_{jn}^2 - c_{jt}^2 - c_{jo}^2 \geq 0, \quad j \in \mathcal{R}; \tag{27}$$

we also rewrite (11) equivalently as: for $j \in \mathcal{R}$,

$$\begin{cases} \mu_j c_{jn} a_{j\alpha} + c_{j\alpha}\lambda_j = 0, & \alpha = t, o \\ \lambda_j \geq 0, \quad \lambda_j s_j = 0. \end{cases} \tag{28}$$

Along with the following conditions on the normal components:

$$(\boldsymbol{a}_n, \boldsymbol{c}_n) \geq 0, \quad (\boldsymbol{a}_n)^T \boldsymbol{c}_n = 0, \tag{29}$$

we obtain the equivalent formulation of the three-dimensional multi-rigid-body contact problem with Coulomb friction as the mixed NCP defined by equations (24), (27)–(29).

3.2 Formulation with friction pyramid law

The formulation of the three-dimensional multi-rigid-body contact problem with the friction pyramid law is similar and is obtained by the introduction of four nonnegative slack variables, $s^+_{jt}, s^-_{jt}, s^+_{jo}$, and s^-_{jo}, at each rolling contact:

$$\left.\begin{array}{ll} s^+_{jt} \equiv \mu_j c_{jn} + c_{jt} & s^-_{jt} \equiv \mu_j c_{jn} - c_{jt} \\ s^+_{jo} \equiv \mu_j c_{jn} + c_{jo} & s^-_{jo} \equiv \mu_j c_{jn} - c_{jo} \end{array}\right\} \quad \text{for all } j \in \mathcal{R}. \tag{30}$$

Letting $a^{\pm}_{j\alpha}$ be the nonnegative and nonpositive part of the tangential contact acceleration $a_{j\alpha}$ respectively, we may write for $\alpha = t, o$,

$$a_{j\alpha} = a^+_{j\alpha} - a^-_{j\alpha} \quad \text{and} \quad |a_{j\alpha}| = a^+_{j\alpha} + a^-_{j\alpha}.$$

The contact conditions (16) can be seen to be equivalent to the following complementarity relations:

$$s^+_{jt} a^+_{jt} = s^-_{jt} a^-_{jt} = s^+_{jo} a^+_{jo} = s^-_{jo} a^-_{jo} = 0. \tag{31}$$

We may solve for the variables $\boldsymbol{c}_{\mathcal{R}t}$ and $\boldsymbol{c}_{\mathcal{R}o}$ from the equations in (30) and substitute into equation (21); this yields

$$\begin{bmatrix} \boldsymbol{a}_{\mathcal{S}n} \\ \boldsymbol{a}_{\mathcal{R}n} \\ \boldsymbol{a}^+_{\mathcal{R}t} \\ \boldsymbol{a}^+_{\mathcal{R}o} \\ \boldsymbol{s}^-_{\mathcal{R}t} \\ \boldsymbol{s}^-_{\mathcal{R}o} \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} \boldsymbol{c}_{\mathcal{S}n} \\ \boldsymbol{c}_{\mathcal{R}n} \\ \boldsymbol{s}^+_{\mathcal{R}t} \\ \boldsymbol{s}^+_{\mathcal{R}o} \\ \boldsymbol{a}^-_{\mathcal{R}t} \\ \boldsymbol{a}^-_{\mathcal{R}o} \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_{\mathcal{S}n} \\ \boldsymbol{b}_{\mathcal{R}n} \\ \boldsymbol{b}_{\mathcal{R}t} \\ \boldsymbol{b}_{\mathcal{R}o} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{32}$$

where

$$\boldsymbol{M} \equiv \begin{bmatrix} (\boldsymbol{M}_{nn})_{\mathcal{S}\mathcal{S}} & (\boldsymbol{M}_{nn})_{\mathcal{S}\mathcal{R}} & (\boldsymbol{A}_{nt})_{\mathcal{S}\mathcal{R}} & (\boldsymbol{A}_{no})_{\mathcal{S}\mathcal{R}} & \boldsymbol{0} & \boldsymbol{0} \\ (\boldsymbol{M}_{nn})_{\mathcal{R}\mathcal{S}} & (\boldsymbol{M}_{nn})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{nt})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{no})_{\mathcal{R}\mathcal{R}} & \boldsymbol{0} & \boldsymbol{0} \\ (\boldsymbol{M}_{tn})_{\mathcal{R}\mathcal{S}} & (\boldsymbol{M}_{tn})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{tt})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{to})_{\mathcal{R}\mathcal{R}} & \boldsymbol{I} & \boldsymbol{0} \\ (\boldsymbol{M}_{on})_{\mathcal{R}\mathcal{S}} & (\boldsymbol{M}_{on})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{ot})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{oo})_{\mathcal{R}\mathcal{R}} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & 2\boldsymbol{U}_{\mathcal{R}} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 2\boldsymbol{U}_{\mathcal{R}} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \tag{33}$$

with $\boldsymbol{U}_{\mathcal{R}}$ being the diagonal matrix with diagonal entries μ_j, for $j \in \mathcal{R}$, and

$$\begin{bmatrix} (\boldsymbol{M}_{nn})_{\mathcal{SS}} & (\boldsymbol{M}_{nn})_{\mathcal{SR}} \\ (\boldsymbol{M}_{nn})_{\mathcal{RS}} & (\boldsymbol{M}_{nn})_{\mathcal{RR}} \\ (\boldsymbol{M}_{tn})_{\mathcal{RS}} & (\boldsymbol{M}_{tn})_{\mathcal{RR}} \\ (\boldsymbol{M}_{on})_{\mathcal{RS}} & (\boldsymbol{M}_{on})_{\mathcal{RR}} \end{bmatrix} \equiv$$

$$\begin{bmatrix} (\boldsymbol{A}_{nn})_{\mathcal{SS}} & (\boldsymbol{A}_{nn})_{\mathcal{SR}} \\ (\boldsymbol{A}_{nn})_{\mathcal{RS}} & (\boldsymbol{A}_{nn})_{\mathcal{RR}} \\ (\boldsymbol{A}_{tn})_{\mathcal{RS}} & (\boldsymbol{A}_{tn})_{\mathcal{RR}} \\ (\boldsymbol{A}_{on})_{\mathcal{RS}} & (\boldsymbol{A}_{on})_{\mathcal{RR}} \end{bmatrix} - \begin{bmatrix} (\boldsymbol{A}_{nt})_{\mathcal{SS}} & (\boldsymbol{A}_{nt})_{\mathcal{SR}} \\ (\boldsymbol{A}_{nt})_{\mathcal{RS}} & (\boldsymbol{A}_{nt})_{\mathcal{RR}} \\ (\boldsymbol{A}_{tt})_{\mathcal{RS}} & (\boldsymbol{A}_{tt})_{\mathcal{RR}} \\ (\boldsymbol{A}_{ot})_{\mathcal{RS}} & (\boldsymbol{A}_{ot})_{\mathcal{RR}} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_{\mathcal{S}t} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_{\mathcal{R}} \end{bmatrix} -$$

$$\begin{bmatrix} (\boldsymbol{A}_{no})_{\mathcal{SS}} & (\boldsymbol{A}_{no})_{\mathcal{SR}} \\ (\boldsymbol{A}_{no})_{\mathcal{RS}} & (\boldsymbol{A}_{no})_{\mathcal{RR}} \\ (\boldsymbol{A}_{to})_{\mathcal{RS}} & (\boldsymbol{A}_{to})_{\mathcal{RR}} \\ (\boldsymbol{A}_{oo})_{\mathcal{RS}} & (\boldsymbol{A}_{oo})_{\mathcal{RR}} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_{\mathcal{S}o} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_{\mathcal{R}} \end{bmatrix}.$$

The three-dimensional multi-rigid-body contact problem with pyramid friction is now equivalent to the LCP defined by the equation (32) and the following conditions:

$$\boldsymbol{a}_n, \boldsymbol{a}^+_{\mathcal{R}t}, \boldsymbol{a}^-_{\mathcal{R}t}, \boldsymbol{a}^+_{\mathcal{R}o}, \boldsymbol{a}^-_{\mathcal{R}o}, \boldsymbol{c}_n, \boldsymbol{s}^+_{\mathcal{R}t}, \boldsymbol{s}^-_{\mathcal{R}t}, \boldsymbol{s}^+_{\mathcal{R}o}, \boldsymbol{s}^-_{\mathcal{R}o} \geq \boldsymbol{0} \tag{34}$$

$$(\boldsymbol{a}_n)^T \boldsymbol{c}_n = (\boldsymbol{a}^+_{\mathcal{R}t})^T(\boldsymbol{c}^+_{\mathcal{R}t}) = (\boldsymbol{a}^-_{\mathcal{R}t})^T(\boldsymbol{c}^-_{\mathcal{R}t}) = (\boldsymbol{a}^+_{\mathcal{R}o})^T(\boldsymbol{c}^+_{\mathcal{R}o}) = (\boldsymbol{a}^-_{\mathcal{R}o})^T(\boldsymbol{c}^-_{\mathcal{R}o}) = 0. \tag{35}$$

4 Existence and Uniqueness of Solution

In this section, we present a main theorem which summarizes the solution existence and uniqueness results for the contact problems. Let $\mathcal{F}$ be the subset of the Euclidean space R^{3n_c} consisting of triples $(\boldsymbol{c}_n, \boldsymbol{c}_t, \boldsymbol{c}_o)$ satisfying the nonnegativity condition (9) and friction constraints: (8) for the cone model and (15) for the pyramid model. Further let $\mathcal{N}$ be the null space of the system Jacobian matrix $\mathcal{J}$ (defined in equation (22)). Let $\mathcal{F}_{\mathcal{R}} \equiv \mathcal{F} \cap \mathcal{N}$.

The following theorem contains three conclusions. The first conclusion refers to the case where all contacts are initially rolling; *i.e.*, $\mathcal{S} = \emptyset$. In this case, the cone $\mathcal{F}_{\mathcal{R}}$ and the body velocities $\dot{\boldsymbol{q}}$ and $\dot{\boldsymbol{\theta}}$ play an important role in providing a sufficient condition for the existence of a solution. The second and third conclusion allow sliding contacts but assume that $\mathcal{J}$ has full column rank. In this case, a solution exists if the friction coefficients corresponding to the sliding contacts are "sufficiently small"; furthermore such a solution is unique if in addition the friction coefficients corresponding to the rolling contacts are also small.

Theorem 1 *The following statements hold for the contact problems.*

(a) If $\mathcal{S} = \emptyset$ and if

$$\begin{bmatrix} \dot{\boldsymbol{q}} \\ -\dot{\boldsymbol{\theta}} \end{bmatrix}^T \dot{\mathcal{J}} \begin{bmatrix} \boldsymbol{c}_n \\ \boldsymbol{c}_t \\ \boldsymbol{c}_o \end{bmatrix} \geq 0, \quad \text{for all } (\boldsymbol{c}_n, \boldsymbol{c}_t, \boldsymbol{c}_o) \in \mathcal{F}_{\mathcal{R}}, \tag{36}$$

then the contact problem with either the friction cone law or the friction pyramid law has a solution.

(b) Suppose $\mathcal{J}$ has full column rank. There exists a positive scalar $\bar{\mu}$ such that if $\mu_j \in [0, \bar{\mu}]$ for all $j \in \mathcal{S}$, the three-dimensional multi-rigid-body contact problem with either Coulomb friction cone law or the pyramid law has a solution.

(c) Suppose $\mathcal{J}$ has full column rank. There exists a positive scalar $\bar{\mu}$ such that if $\mu_j \in [0, \bar{\mu}]$ for all $j = 1, \ldots, n_c$, the solution in part (b) is unique.

The condition (36) is trivially satisfied when the columns of the system Jacobian matrix are linearly independent, since then its null space becomes the origin. Thus, our existence condition (when specialized to the planar case) is less restrictive than previous results obtained by Lötstedt[15] and Baraff [2]. As noted in [19], when the constant vector, $\boldsymbol{b}$, defined in equation (23), lies in the column space of $\mathcal{J}^T$, then (36) must hold and the existence of a solution follows (assuming $\mathcal{S} = \emptyset$). In particular, for the assembly stability testing problem which has $\dot{\boldsymbol{q}}$ and $\dot{\boldsymbol{\theta}}$ both equal to zero, a solution exists.

Lötstedt [15] showed that the planar problem with both rolling and sliding contacts has a unique solution if every matrix in a family of matrices (closely related to $\tilde{\boldsymbol{A}}$ defined in equation (25)) belongs to the class of **P**-matrices. The second and third conclusion of Theorem 1 extend Lötstedt's results to the 3-dimensional case. In essence, the condition that the friction coefficients be small is needed to ensure that $\tilde{\boldsymbol{A}}$ is a **P**-matrix. By an elementary linear algebraic argument, it is therefore possible to derive an estimate for the largest upper bound for $\bar{\mu}$; however, such an estimate is typically very conservative and can be expected to be much smaller than one would expect to encounter in real systems; see [19, Appendix I] for the derivation of such an estimate.

We note one important difference between the assumptions in (b) and (c). Namely, in (b) only those friction coefficients at the sliding contacts are assumed small; whereas in (c), all friction coefficients are assumed small.

Sketch of Theorem 1's proof. The proof of statement (a) can be found in [19]. In the following proof of (b), we will focus on the friction cone problem and give only the essential ideas. The omitted details can be found in the technical report [23].

Since $\mathcal{J}$ has full column rank, the matrix $\boldsymbol{A}$ is symmetric positive definite. The matrix $\tilde{\boldsymbol{A}}$, being a modification of $\boldsymbol{A}$ involving the friction coefficients μ_j, $j \in \mathcal{S}$ (see (25) and (26)), is positive definite (but not symmetric), provided that these coefficients are sufficiently small. This implies that the principal submatrix

$$\begin{bmatrix} (\tilde{\boldsymbol{A}}_{nn})_{\mathcal{SS}} & (\boldsymbol{A}_{nn})_{\mathcal{SR}} \\ (\tilde{\boldsymbol{A}}_{nn})_{\mathcal{RS}} & (\boldsymbol{A}_{nn})_{\mathcal{RR}} \end{bmatrix}$$

is **P**. By LCP results [5], it follows that for each fixed but arbitrary pair $(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o})$, there exists a unique pair $(\boldsymbol{a}_n, \boldsymbol{c}_n)$ satisfying

$$\left[\begin{array}{c} \boldsymbol{a}_{\mathcal{S}n} \\ \boldsymbol{a}_{\mathcal{R}n} \end{array}\right] = \left[\begin{array}{cc} (\tilde{\boldsymbol{A}}_{nn})_{\mathcal{S}\mathcal{S}} & (\boldsymbol{A}_{nn})_{\mathcal{S}\mathcal{R}} \\ (\tilde{\boldsymbol{A}}_{nn})_{\mathcal{R}\mathcal{S}} & (\boldsymbol{A}_{nn})_{\mathcal{R}\mathcal{R}} \end{array}\right] \left[\begin{array}{c} \boldsymbol{c}_{\mathcal{S}n} \\ \boldsymbol{c}_{\mathcal{R}n} \end{array}\right] + \left[\begin{array}{cc} (\boldsymbol{A}_{nt})_{\mathcal{S}\mathcal{R}} & (\boldsymbol{A}_{no})_{\mathcal{S}\mathcal{R}} \\ (\boldsymbol{A}_{nt})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{no})_{\mathcal{R}\mathcal{R}} \end{array}\right] \left[\begin{array}{c} \boldsymbol{c}_{\mathcal{R}t} \\ \boldsymbol{c}_{\mathcal{R}o} \end{array}\right] + \left[\begin{array}{c} \boldsymbol{b}_{\mathcal{S}n} \\ \boldsymbol{b}_{\mathcal{R}n} \end{array}\right] \tag{37}$$

$$(\boldsymbol{a}_n, \boldsymbol{c}_n) \geq \mathbf{0} \qquad (\boldsymbol{a}_n)^T \boldsymbol{c}_n = 0;$$

moreover $(\boldsymbol{a}_n, \boldsymbol{c}_n)$ is Lipschitz continuous as a function of $(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o})$. Let $(\boldsymbol{a}_n(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}), \boldsymbol{c}_n(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}))$ denote the unique solution function of the LCP (37). Also define

$$\boldsymbol{F}(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}) \equiv \left[\begin{array}{cccc} (\tilde{\boldsymbol{A}}_{tn})_{\mathcal{R}\mathcal{S}} & (\boldsymbol{A}_{tn})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{tt})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{to})_{\mathcal{R}\mathcal{R}} \\ (\tilde{\boldsymbol{A}}_{on})_{\mathcal{R}\mathcal{S}} & (\boldsymbol{A}_{on})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{ot})_{\mathcal{R}\mathcal{R}} & (\boldsymbol{A}_{oo})_{\mathcal{R}\mathcal{R}} \end{array}\right] \left[\begin{array}{c} \boldsymbol{c}_{\mathcal{S}n}(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}) \\ \boldsymbol{c}_{\mathcal{R}n}(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}) \\ \boldsymbol{c}_{\mathcal{R}t} \\ \boldsymbol{c}_{\mathcal{R}o} \end{array}\right] + \left[\begin{array}{c} \boldsymbol{b}_{\mathcal{R}t} \\ \boldsymbol{b}_{\mathcal{R}o} \end{array}\right].$$

Let $N \equiv 2|\mathcal{R}|$. The function $\boldsymbol{F}$ maps the Euclidean space R^N, which contains the pair of vectors $(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o})$, into itself; $\boldsymbol{F}$ is a piecewise linear map, hence Lipschitz continuous; moreover, $\boldsymbol{F}$ is strongly monotone.

We define a set-valued map $\boldsymbol{K} : R^N \to R^N$ as follows. For a given pair $(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}) \in R^N$, let the set $\boldsymbol{K}(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o})$ consist of all vectors $(\boldsymbol{c}'_{\mathcal{R}t}, \boldsymbol{c}'_{\mathcal{R}o}) \in R^N$ such that for all $j \in \mathcal{R}$,

$$(c'_{jt})^2 + (c'_{jo})^2 \leq \mu_j^2 (c_{jn}(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}))^2.$$

The pair $(\boldsymbol{K}, \boldsymbol{F})$ defines a quasi-variational inequality problem which is to find a pair of vectors $(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}) \in R^N$ with two properties: (a) $(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}) \in \boldsymbol{K}(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o})$; and (b) for all vectors $(\boldsymbol{c}'_{\mathcal{R}t}, \boldsymbol{c}'_{\mathcal{R}o}) \in \boldsymbol{K}(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o})$,

$$\boldsymbol{F}(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o})^T \left[\begin{array}{c} \boldsymbol{c}'_{\mathcal{R}t} - \boldsymbol{c}_{\mathcal{R}t} \\ \boldsymbol{c}'_{\mathcal{R}o} - \boldsymbol{c}_{\mathcal{R}o} \end{array}\right] \geq 0.$$

It is easy to show that if $(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o})$ solves the QVI $(\boldsymbol{K}, \boldsymbol{F})$, then

$$(\bar{\boldsymbol{a}}_n, \bar{\boldsymbol{c}}_n, \bar{\boldsymbol{a}}_{\mathcal{R}t}, \bar{\boldsymbol{a}}_{\mathcal{R}o}, \boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o})$$

solves the friction cone problem, where

$$\left[\begin{array}{c} \bar{\boldsymbol{a}}_n \\ \bar{\boldsymbol{c}}_n \end{array}\right] \equiv \left[\begin{array}{c} \boldsymbol{a}_n(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}) \\ \boldsymbol{c}_n(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}) \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} \bar{\boldsymbol{a}}_{\mathcal{R}t} \\ \bar{\boldsymbol{a}}_{\mathcal{R}o} \end{array}\right] \equiv \boldsymbol{F}(\boldsymbol{c}_{\mathcal{R}t}, \boldsymbol{c}_{\mathcal{R}o}).$$

The proof of (b) can now be completed by invoking an existence result [4] for the quasi-variational inequality problem.

The proof of (c) is based on an extension of the argument given by Lötstedt for the planar case [15]. We refer the reader to [23] for details. Q.E.D.

In general, solutions to the two contact models could be quite distinct. The next result, which requires none of the assumptions in Theorem 1, gives a simple condition for a solution of the friction pyramid model to be a solution of the cone model. The extent to which this observation can be put to use will be investigated separately.

Proposition 1 *If a solution to the friction pyramid model satisfies the cone constraints (8), then it must be a solution of the friction cone model.*

At first glance, this result seems trivial. Nevertheless a closer look at the two models suggests that they differ not only in the quadratic (8) versus polyhedral (15) friction constraints, but the resulting contact conditions, (11) versus (16), are also different. A simple proof of Proposition 1 can be found in [23].

5 Solution Methods for Friction Pyramid Model

The formulation of the friction pyramid model as the LCP defined by (32)–(35) permits the numerical solution of this model by a host of algorithms as described in [5]. The main concern with the application of these algorithms is whether their convergence criteria are satisfied by the data of this particular LCP. In our study, we have focused on two algorithms: one classical and the other contemporary. The former is Lemke's almost complementarity pivot algorithm; the latter is a feasible interior point algorithm.

We refer the reader to [5] for a comprehensive treatment of Lemke's algorithm and its convergence theory. The following is the main convergence result of this algorithm applied to the LCP in question.

Theorem 2 *Suppose that the system Jacobian matrix $\mathcal{J}$ has full column rank. There exists a positive scalar $\bar{\mu}$ such that if $\mu_j \in [0, \bar{\mu}]$ for all $j = 1, \ldots, n_c$, Lemke's algorithm, under a standard nondegeneracy assumption, will in a finite number of pivots successfully compute a solution to the LCP (32)–(35).*

Proof. One can easily verify that the matrix $\boldsymbol{M}$ given by (33) is copositive. Moreover the constant vector $\boldsymbol{r}$ in the equation (32) satisfies the following implication:

$$\left.\begin{aligned} \boldsymbol{x} &\geq \boldsymbol{0} \\ \boldsymbol{M}\boldsymbol{x} &\geq \boldsymbol{0} \\ \boldsymbol{x}^T \boldsymbol{M}\boldsymbol{x} &= 0 \end{aligned}\right\} \Rightarrow \boldsymbol{r}^T \boldsymbol{x} \geq 0.$$

The desired conclusion now follows from a known LCP result [5, Theorem 4.4.13]. Q.E.D.

The family of interior point methods is a recent entry into the field of the LCP. Fueled by their great success for solving linear programs, these methods have received much attention in the mathematical programming literature. For this reason, we have chosen a feasible interior point (FIP) method as a candidate algorithm for solving the LCP arising from the multi-rigid-body problem with friction pyramids. As we shall see in the next section, the interior point method provides a viable alternative to Lemke's method.

Since the engineering community is probably not well acquainted with the interior point method, we give a brief summary of the method in the Appendix. What follows is the main convergence result of this method for solving the friction pyramid model.

Theorem 3 *Suppose that the system Jacobian matrix, $\mathcal{J}$ has full column rank. There exists a positive scalar $\bar{\mu}$ such that if $\mu_j \in (0, \bar{\mu}]$ for all $j = 1, \ldots, n_c$, the feasible interior point method, as described in the Appendix and applied to the LCP defined by (32–35), generates a sequence of iterates*

$$\left(\boldsymbol{a}_n^\nu, \boldsymbol{c}_n^\nu, \boldsymbol{a}_{\mathcal{R}t}^{\nu,\pm}, \boldsymbol{a}_{\mathcal{R}o}^{\nu,\pm}, \boldsymbol{s}_{\mathcal{R}t}^{\nu,\pm}, \boldsymbol{s}_{\mathcal{R}o}^{\nu,\pm}\right), \quad \nu = 0, 1, 2, \ldots,$$

each of which is strictly feasible to this LCP (i.e., equation (32) is satisfied and the inequalities in (34) are strictly satisfied) and

$$\lim_{\nu\to\infty}(\boldsymbol{a}_n^\nu)^T\boldsymbol{c}_n^\nu = \lim_{\nu\to\infty}(\boldsymbol{a}_{\mathcal{R}t}^{\nu,\pm})^T\boldsymbol{s}_{\mathcal{R}t}^{\nu,\pm} = \lim_{\nu\to\infty}(\boldsymbol{a}_{\mathcal{R}o}^{\nu,\pm})^T\boldsymbol{s}_{\mathcal{R}o}^{\nu,\pm} = 0.$$

Moreover, every accumulation point of the generated sequence solves the LCP; finally, this LCP must have a solution.

Unlike Theorem 2 which permits zero coefficients of friction at the rolling contacts, the positivity of the friction coefficients at the rolling contacts is essential for the validity of Theorem 3.

6 Numerical Results

In this section, we report our computational experience gained through applying Lemke's algorithm and our FIP algorithm (detailed in the Appendix) to some multi-rigid contact problems with friction pyramid law. The algorithms were implemented in MATLAB (Lemke's algorithm was provided by Michael Ferris), while our data generation code was written in C. Both algorithms were run on the same 328 data sets. These were generated by randomly generating 35 (physically meaningful) problems of varying sizes with varying numbers of passive bodies, contact geometries, and initial conditions. The dimensions of the corresponding $\boldsymbol{M}$ matrices in the LCPs ranged from 2 to 170. Each problem was used to produce about 10 data sets by varying the coefficient of friction, which for convenience, was assumed to be equal at all contact points. While the termination criterion for Lemke's algorithm was standard, our FIP algorithm was terminated if the total complementary gap fell below 10^{-6}. We set upper limits of 100 and 300 iterations in Lemke's and the FIP algorithms, respectively.

Table 1 summarizes our numerical results. However, due to the large number of data sets, only representative results are included in the table. The column headings in the table are defined as follows: "size" is the dimension of the $\boldsymbol{M}$ matrix appearing in the LCP formulation, "# data sets" is the number of data sets of a given size that were attempted, "frac. solved" is the fraction of the data sets solved by either Lemke's or our FIP algorithm (as indicated by the column headings), "ave. # starts" is the average number of times Lemke's algorithm was run (using new covering vectors) for the data sets which it eventually solved, "ave. # iter." is the average number of iterations required for a solved data set. The latter number for Lemke's algorithm includes the iterations performed during failed runs on a given data set provided that it was eventually solved by Lemke's algorithm after a later restart. Overall, the two algorithms solved just under 70% (229 out of 328) of the data sets. For the problems that were not solved, it is possible that they do not have solutions to begin with.

		Lemke's algorithm			FIP algorithm	
size	# data sets	frac. solved	ave. # restarts	ave. # iter.	frac. solved	ave. # iter.
2	20	1.00	1.0	2.0	0.50	8.4
10	60	0.92	1.3	8.9	0.35	10.4
16	10	1.00	1.4	20.0	1.00	17.2
30	26	0.62	1.7	44.1	0.42	19.6
50	20	0.35	2.0	107.4	0.30	61.1
90	23	0.00	–	–	0.35	30.4
120	10	0.10	1.0	< 100	0.70	< 300
150	6	0.00	–	–	0.50	46.3
170	10	0.00	–	–	0.00	–

Table 1: Summary of numerical results

Much more computational testing has been carried out and reported in [23].

7 Summary and Conclusions

In this paper, we have introduced various mathematical formulations for the three-dimensional multi-rigid-body contact problem with Coulomb friction. Existence and uniqueness of a solution to two models of the problem are presented under a full column rank assumption on the system Jacobian matrix and a smallness assumption on the coefficients of friction. Two algorithms have been established to compute a solution to the friction pyramid model under the same assumptions. Although the required assumptions for the theoretical results are difficult to verify, the numerical results we have obtained suggest that the algorithms are fairly effective for solving some randomly generated three-dimensional multi-rigid-body contact problems under the friction pyramid law.

Our study is imperfect. Theoretically, the results are not strong enough to handle the case of arbitrary friction coefficients and/or column rank deficiency of the system Jacobian matrix; numerically, we do not know if the two algorithms are capable of solving all friction pyramid problems which possess solutions. In spite of these deficiencies, we believe that we have made an important contribution toward the understanding of the three-dimensional multi-rigid-body contact problem with Coulomb friction. Theoretically, we have provided the most comprehensive results for this problem.

There remains much more to be accomplished. Trying to improve the theoretical results is clearly something worth looking into. The search for alternative algorithms for solving the friction pyramid model would be equally useful. In this regard, the NE/SQP method [18] and the infeasible interior-point method in [26] are possible candidates for solving the friction cone model.

Finally, there is practical side to this study. Our motivation was the hope that our results would be useful in the design and development of two classes of systems: simulation systems and planning systems. Both types of systems would be extremely useful: the former helping us to better understand existing mechanical systems, and the latter enabling the automation of a plethora of contact tasks in both hazardous and safe environments. The primary impediment to the achievement of our practical objectives is the fact that the multi-rigid-body models discussed here do not always have unique solutions, and the sufficient conditions that we have developed are conservative. What is needed are algorithms to always and efficiently compute a solution if one exists, or determine that either no solution or multiple solutions exist. If such algorithms were available, one could develop intelligent strategies to deal with the nonuniqueness of the model in both simulation and planning applications. We are currently studying these and related issues.

Acknowledgements

The authors would like to express their appreciation to Professor Michael Ferris at the University of Wisconsin at Madison for supplying an implementation of Lemke's algorithm written in MATLAB.

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8 Appendix: Some Details of The Interior Point Method

We present some details of an interior point method for solving the general LCP $(\boldsymbol{r}, \boldsymbol{M})$ satisfying certain assumptions. We refer the reader to the monograph [12] for some background results and detailed historical discussion of the family of interior point methods for solving the LCP.

Consider an LCP $(\boldsymbol{r}, \boldsymbol{M})$ where the vector $\boldsymbol{r} \in R^n$ and matrix $\boldsymbol{M} \in R^{n \times n}$. Let

$$\mathcal{F} \equiv \{(\boldsymbol{w}, \boldsymbol{z}) \in R^{2n}_+ \ : \ \boldsymbol{w} = \boldsymbol{r} + \boldsymbol{M}\boldsymbol{z}\}$$

denote the feasible region of this problem. We write

$$\mathcal{F}_+ \equiv \mathcal{F} \cap R^{2n}_{++}$$

where $R^{2n}_{++} \equiv \{(\boldsymbol{w}, \boldsymbol{z}) \in R^{2n} : (\boldsymbol{w}, \boldsymbol{z}) > \boldsymbol{0}\}$. We postulate the following assumptions on the pair $(\boldsymbol{r}, \boldsymbol{M})$:

(a) $\boldsymbol{M}$ is a $\mathbf{P}_0$-matrix;

(b) the LCP $(\boldsymbol{r}, \boldsymbol{M})$ is strictly feasible; that is, $\mathcal{F}_+ \neq \emptyset$;

(c) for any scalars $\alpha > \beta > 0$, the level set $L(\alpha, \beta) \equiv \{(\boldsymbol{w}, \boldsymbol{z}) \in \mathcal{F}_+ \ : \ \alpha \boldsymbol{e} \geq \boldsymbol{w} \circ \boldsymbol{z} \geq \beta \boldsymbol{e}\}$ is bounded, where $\boldsymbol{e}$ denotes the n-vector of all ones and $\boldsymbol{u} \circ \boldsymbol{v}$ denotes the Hadamard product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, that is, $\boldsymbol{u} \circ \boldsymbol{v}$ is the vector whose i-th component is equal to $u_i v_i$ for all i.

Except for condition (c) which is a weakening of some standard assumptions for the family of interior point methods for solving the LCP, the setting herein is the same as in [12]. The interior point method for solving the LCP $(\boldsymbol{r}, \boldsymbol{M})$ satisfying the conditions (a), (b), and (c) starts at a strictly feasible pair $(\boldsymbol{w}^0, \boldsymbol{z}^0) \in \mathcal{F}_+$. In general, given a pair of vectors $(\boldsymbol{w}^k, \boldsymbol{z}^k) \in \mathcal{F}_+$, the method consists of two major computational steps. Let $\beta_k \in [0, 1)$ be a given scalar, called the *centering parameter*. This parameter plays an extremely important role in the practical success of the interior point method; we refer to [12] for a full discussion of this role. Let

$$\xi_k \equiv \frac{(\boldsymbol{w}^k)^T \boldsymbol{z}^k}{n}.$$

We solve the following system of linear equations to compute the direction vector $(d\boldsymbol{w}^k, d\boldsymbol{z}^k)$:

$$\begin{bmatrix} -\boldsymbol{I} & \boldsymbol{M} \\ \boldsymbol{Z}^k & \boldsymbol{W}^k \end{bmatrix} \begin{bmatrix} d\boldsymbol{w}^k \\ d\boldsymbol{z}^k \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{w}^k \circ \boldsymbol{z}^k - \beta_k \xi_k \boldsymbol{e} \end{bmatrix} \tag{38}$$

where $\boldsymbol{W}^k = \text{diag}(\boldsymbol{w}^k)$ and $\boldsymbol{Z}^k = \text{diag}(\boldsymbol{z}^k)$. By the $\mathbf{P}_0$-property of $\boldsymbol{M}$ and the positivity of $(\boldsymbol{w}^k, \boldsymbol{z}^k)$, it can be shown that the matrix on the left side of (38) is nonsingular; thus $(d\boldsymbol{w}^k, d\boldsymbol{z}^k)$ is well defined. Next, we need to introduce the merit function for the method, which is defined as

$$\psi(\boldsymbol{w}, \boldsymbol{z}) \equiv (n + \zeta) \log \boldsymbol{w}^T \boldsymbol{z} - \sum_{i=1}^{n} \log w_i z_i, \quad \text{for } (\boldsymbol{w}, \boldsymbol{z}) > \boldsymbol{0},$$

where ζ is a positive constant. Again, we refer to [12] for the motivation and derivation of this function. The vector $(d\boldsymbol{w}^k, d\boldsymbol{z}^k)$ computed above turns out to be a descent direction for the function ψ at the current iterate $(\boldsymbol{w}^k, \boldsymbol{z}^k)$. Hence, a line search can be executed on this function starting at this iterate and moving along the generated direction. This search is a standard routine in a nonlinear programming algorithm. It requires two fixed constants, $\rho, \sigma \in (0, 1)$, where ρ controls the step size and σ controls the decrease of ψ.

The following is a detailed description of the interior point method for solving the LCP $(\boldsymbol{r}, \boldsymbol{M})$ satisfying conditions (a), (b), and (c).

The interior point method

Step 0. (Initialization) Let $\varepsilon, \sigma, \rho, \eta \in (0,1)$ be given scalars. Let $\zeta > 0$, $(\boldsymbol{w}^0, \boldsymbol{z}^0) \in \mathcal{F}_+$, and $\beta_0 \in [0,1)$ be arbitrary. Set $k = 0$.

Step 1. (Computing search direction) Solve the system of linear equations (38) to obtain $(d\boldsymbol{w}^k, d\boldsymbol{z}^k)$.

Step 2. (Computing step size) Calculate

$$\bar{\lambda}_k \equiv \min \left\{ 1, \left\{ -\frac{w_i^k}{dw_i^k} : dw_i^k < 0 \right\}, \left\{ -\frac{z_i^k}{dz_i^k} : dz_i^k < 0 \right\} \right\}$$

and let $\lambda'_k \equiv \eta \bar{\lambda}_k$. Let m_k be the smallest nonnegative integer m such that

$$\psi(\boldsymbol{w}^k + \rho^m \lambda'_k d\boldsymbol{w}^k, \boldsymbol{z}^k + \rho^m \lambda'_k d\boldsymbol{z}^k) - \psi(\boldsymbol{w}^k, \boldsymbol{z}^k) \leq -\sigma(1-\beta_k)\zeta\rho^m \lambda'_k. \tag{39}$$

Set $\lambda_k \equiv \rho^{m_k} \lambda'_k$.

Step 3. (Update and termination check) Set $\lambda_k \equiv \rho^{m_k} \lambda'_k$ and

$$(\boldsymbol{w}^{k+1}, \boldsymbol{z}^{k+1}) \equiv (\boldsymbol{w}^k + \lambda_k d\boldsymbol{w}^k, \boldsymbol{z}^k + \lambda_k d\boldsymbol{z}^k).$$

If $(\boldsymbol{w}^{k+1})^T \boldsymbol{z}^{k+1} \leq \varepsilon$, terminate; the pair $(\boldsymbol{w}^{k+1}, \boldsymbol{z}^{k+1})$ is a desired approximate solution of the LCP $(\boldsymbol{r}, \boldsymbol{M})$. Otherwise, choose $\beta_{k+1} \in [0, \beta_k]$; replace k by $k+1$ and return to Step 1.

We give some further explanation to Step 2. The scalar $\bar{\lambda}_k$ is the largest step size $\lambda \in (0,1]$ for which $(\boldsymbol{w}^k + \lambda d\boldsymbol{w}^k, \boldsymbol{z}^k + \lambda d\boldsymbol{z}^k)$ is nonnegative; thus by scaling $\bar{\lambda}_k$ by the factor $\eta \in (0,1)$, we are ensured that $(\boldsymbol{w}^k + \lambda d\boldsymbol{w}^k, \boldsymbol{z}^k + \lambda d\boldsymbol{z}^k)$ is positive for all $\lambda \in [0, \lambda'_k]$. The integer m_k can be determined by starting with $m = 0$ and increasing m by 1 each time the inequality (39) fails to hold; it can be shown that in a finite number of trials, the desired integer m_k can be obtained.

The tolerance ε is used in practical implementation to check the successful termination of the method. In theory, an infinite sequence of iterates $\{(\boldsymbol{w}^k, \boldsymbol{z}^k)\}$ is generated by the method. The following result summarizes the main properties of this sequence and shows that the LCP $(\boldsymbol{r}, \boldsymbol{M})$ must have a solution. Due to the weakened assumption (c), this result is not a direct consequence of the theory in [12]; we refer the reader to [23] for details of the omitted proof.

Theorem 4 *Under assumptions (a), (b), and (c), the above method generates a well-defined sequence of iterates* $\{(\boldsymbol{w}^k, \boldsymbol{z}^k)\}$ *having the following properties:*

(i) $\{(\boldsymbol{w}^k, \boldsymbol{z}^k)\} \subset \mathcal{F}_+$;

(ii) $\{\psi(\boldsymbol{w}^k, \boldsymbol{z}^k)\}$ *is strictly decreasing;*

(iii) $\lim_{k\to\infty} (\boldsymbol{w}^k)^T \boldsymbol{z}^k = 0$;

(iv) every accumulation point of the sequence $\{(\boldsymbol{w}^k, \boldsymbol{z}^k)\}$, *if it exists, is a solution of the LCP* $(\boldsymbol{r}, \boldsymbol{M})$.

Moreover, the LCP $(\boldsymbol{r}, \boldsymbol{M})$ *must have a solution.*