

Dynamic Multi-Rigid-Body Systems with Concurrent Distributed Contacts*

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Abstract. Consider a system of bodies with multiple concurrent contacts. The multi-rigid-body contact problem is to predict the accelerations of the bodies and the normal and friction loads acting at the contacts. This paper presents theoretical results for the multi-rigid-body contact problem under the assumptions that one or more contacts occur over locally planar, finite regions and friction forces are consistent with the maximum work inequality. We present an existence and uniqueness result for this problem under some mild assumptions on the system inputs.

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1 Introduction

Multi-body dynamic systems are ubiquitous in our society: motors, engines, and the automation devices used to build portions of these machines are common examples. Where possible, machine designers use joints that provide bilateral kinematic constraints between the connected bodies (*e.g.*, pin joints). In some situations, however, design constraints dictate the use of “joints” which provide only unilateral kinematic constraint (*e.g.*, a cam and follower). In the domain of automated manufacturing, robots are used to position and orient parts for “presentation” to other robots and automated devices for further processing. Under normal operations, robots can position and orient only those parts that are light enough and small enough to be grasped securely and lifted.

However, with a solid understanding of contact mechanics, a robot can use pushing operations to reliably position and orient objects that are too heavy to lift and too large to grasp securely [7].

Dynamic analyses of rigid body systems are based on the Newton-Euler equations, the kinematic constraints, and a friction model of the contacts. These equations and inequalities are typically formulated as a system of differential-algebraic equations [4]. However, their formulation as a system of *equations* requires prior knowledge of the impending contact states (*i.e.*, rolling, sliding, or breaking). Once formulated, the equations will have a unique solution if the system Jacobian matrix has full row rank. If there are “too many” contacts, the contact forces cannot be uniquely determined with a rigid body model. One way to resolve this indeterminacy is to incorporate a model of contact compliance [11]. However, we warn the reader that this “remedy” suffers from its own set of problems. For example, the original differential-algebraic system becomes a system of *stiff* differential equations whose solution then depends on the contact stiffnesses, which, in turn, depend on the global geometries of the parts in contact.

As alluded to above, the differential algebraic system arising from the rigid body assumption cannot be formulated at times when the impending contact state is not known *a priori*. In simulation, the usual approach to predicting the ensuing contact state is to assume it will be the same as the current one. Then after solving the corresponding differential-algebraic system, the solution is checked against a contact model. If the normal force at a contact has become negative, then the simulation is backed up to the time when the component became zero and the system equations are reformulated under the assumption that the corresponding contact has separated. Analogous logic is applied to determine all other possible contact state transitions.

The above approach clearly relies on force

history information, which of course, is often absent for the first time step in a simulation. It also depends implicitly upon the assumption that the contact forces are continuous functions of time. This is violated whenever a collision occurs, since the contacts experience impulsive forces at those times. Clearly one cannot base his assumption of the contact state just after a collision on the contact forces immediately before it. In this situation, one would have to resort to other means to “guess” the impending contact state. For example, one could test all possible new contact states and choose one which is consistent. In problems with small numbers of contacts, it is practical to enumerate all possible contact states until a consistent solution is found. However, since the number of possible contact states grows exponentially with the number of contacts, this approach is limited by computing speed and algorithm efficiency.

The difficulties described above suggest that it is important to understand the multi-rigid-body contact problem formulated *without* assuming prior knowledge of the ensuing contact states. This philosophy leads to the appearance of kinematic inequality constraints in the system model, and then naturally to a complementarity problem combining the Newton-Euler equations, the kinematic constraints, and a contact friction model [6, 9, 10] in the unknown accelerations and contact forces. In this paper, we extend previous results for systems having isolated point contacts to those having one or more contacts distributed over locally planar, finite areas [5]. The limit surface formalism developed by Goyal is used to model the load-motion behavior at the contacts [3].

The contribution of this paper is a set of new existence and uniqueness results, that provide strict theoretical guidelines for the use of our model in the analysis of multi-rigid-body dynamic systems.

2 The Model

The derivation of our mathematical model describing the motion of a system rigid bodies with locally planar, finite areas of contact is analogous to the model with isolated points contacts developed previously[10]. Therefore, in this paper, we will only detail the aspects of the friction limit surface model.

The mathematical model describing the dynamic, three-dimensional, multi-rigid-body contact problem with point and distributed contacts consists of four sets of equations and inequalities. In order to describe the model, let n_c denote the number of contacts at the time instant (*i.e.*, the current time) for which the model is formulated. By definition, the normal components of the relative linear velocity at these contacts, $v_{in}, i = 1, \dots, n_c$, are all zero; the two orthogonal components of the relative linear velocity in the contact tangent plane are denoted by (v_{it}, v_{io}) . The relative angular velocity component in the direction of the i^{th} contact normal is v_{ir} . We assume that a distributed contact forms as a result of compressive normal loading between two contacting bodies; we are not concerned with the geometry of the distributed contacts, their relative angular velocities and accelerations, nor the components of the contact moments in the tangent planes of the contacts. The modeling of these details will be necessary in some situations, but we do not consider them here.

Let us classify a contact i as *rolling* if $v_{it} = v_{io} = v_{ir} = 0$ and *non-rolling* otherwise. Let \mathcal{R} and \mathcal{N} denote, respectively, the set of rolling and non-rolling contacts; these two index sets partition $\{1, \dots, n_c\}$. Let \mathcal{M} and \mathcal{J} denote, respectively, the system inertia and constraint Jacobian matrices; we note that the former is a symmetric positive definite matrix whereas the latter is defined by the contact geometry. Let

$$\mathbf{A} \equiv \mathcal{J}^T \mathcal{M} \mathcal{J}.$$

Letting n_d and n_p be the numbers of distributed and point contacts, respectively, one can show

that \mathbf{A} is a $(3n_c + n_d) \times (3n_c + n_d)$ symmetric positive semidefinite matrix; its null space coincides with the null space of \mathcal{J} . In particular, \mathbf{A} is positive definite if and only if \mathcal{J} has linearly independent columns. The unknown vectors of relative accelerations and contact forces and moments are denoted by (a_n, a_t, a_o, a_r) and (c_n, c_t, c_o, c_r) respectively. Here each vector subscripted by n, t , or o is of order n_c and refers to the normal or tangential components of the relative linear accelerations or forces at the contacts. The vectors subscripted by r are of order n_d and represent the relative angular accelerations or transmitted moments about normals of the distributed contacts.

The four sets of defining equations and inequalities of the basic model are:

- (i) the combined kinematic/Newton-Euler equations of motion,

$$\begin{bmatrix} a_n \\ a_t \\ a_o \\ a_r \end{bmatrix} = \mathbf{A} \begin{bmatrix} c_n \\ c_t \\ c_o \\ c_r \end{bmatrix} + \begin{bmatrix} b_n \\ b_t \\ b_o \\ b_r \end{bmatrix},$$

where (b_n, b_t, b_o, b_r) is a constant vector that contains the known external forces applied to the system and velocity product forces;

- (ii) the nontensile restrictions on the contact forces, unilateral kinematic constraints, and the complementarity conditions on the normal contact forces and accelerations,

$$(a_n, c_n) \geq 0, \quad (a_n)^T c_n = 0;$$

- (iii) the Coulomb friction limit surface condition suggested by Howe and Cutkosky [5],

$$\frac{c_{it}^2}{e_{it}^2} + \frac{c_{io}^2}{e_{io}^2} + \frac{c_{ir}^2}{e_{ir}^2} \leq \mu_i^2 c_{in}^2, \quad i = 1, \dots, n_c, \quad (1)$$

where e_{it}, e_{io} , and e_{ir} are given positive constants and μ_i is the coefficient of friction (assumed positive); and

(iv) the maximum work inequality: for each $i \in \mathcal{N}$,

$$(c_{it}, c_{io}, c_{ir}) \in \operatorname{argmax} \left\{ \begin{array}{l} -(v_{it}c'_{it} + v_{io}c'_{io} + v_{ir}c'_{ir}) : \\ \left(\frac{c'_{it}}{e_{it}}\right)^2 + \left(\frac{c'_{io}}{e_{io}}\right)^2 + \left(\frac{c'_{ir}}{e_{ir}}\right)^2 \leq \mu_i^2 c_{in}^2 \end{array} \right\},$$

and for each $i \in \mathcal{R}$,

$$(c_{it}, c_{io}, c_{ir}) \in \operatorname{argmax} \left\{ \begin{array}{l} -(a_{it}c'_{it} + a_{io}c'_{io} + a_{ir}c'_{ir}) : \\ \left(\frac{c'_{it}}{e_{it}}\right)^2 + \left(\frac{c'_{io}}{e_{io}}\right)^2 + \left(\frac{c'_{ir}}{e_{ir}}\right)^2 \leq \mu_i^2 c_{in}^2 \end{array} \right\},$$

where $\operatorname{argmax} \{f(x) : x \in X\}$ denotes the set of optimal solutions of the maximization problem:

$$\begin{array}{l} \text{maximize } f(x) \\ \text{subject to } x \in X. \end{array}$$

Equivalently, the maximum work inequality can be formulated as the following equations:

$$\left. \begin{array}{l} e_{it}^2 \mu_i c_{jn} \sigma_{it} + \sqrt{e_{it}^2 \sigma_{it}^2 + e_{io}^2 \sigma_{io}^2 + e_{ir}^2 \sigma_{ir}^2} c_{it} = 0 \\ e_{io}^2 \mu_i c_{in} \sigma_{io} + \sqrt{e_{it}^2 \sigma_{it}^2 + e_{io}^2 \sigma_{io}^2 + e_{ir}^2 \sigma_{ir}^2} c_{io} = 0 \\ e_{ir}^2 \mu_i c_{in} \sigma_{ir} + \sqrt{e_{it}^2 \sigma_{it}^2 + e_{io}^2 \sigma_{io}^2 + e_{ir}^2 \sigma_{ir}^2} c_{ir} = 0 \end{array} \right\} \\ \forall i = 1, \dots, n_c;$$

where

$$(\sigma_{it}, \sigma_{io}, \sigma_{ir}) = \begin{cases} (v_{it}, v_{io}, v_{ir}) & \text{if } i \in \mathcal{N}, \\ (a_{it}, a_{io}, a_{ir}) & \text{if } i \in \mathcal{R}. \end{cases}$$

In order to be able to handle other kinds of Coulomb friction laws, we introduce a generalized model in which we replace the quadratic friction cone defined by (1) by an abstract closed convex cone and modify the maximum work inequality accordingly. Specifically, for each $i = 1, \dots, n_c$, let $\mathcal{F}_i : R_+ \rightarrow R^3$ be a set-valued map with the property that for each scalar $\tau \geq 0$, the image $\mathcal{F}_i(\tau)$ is a closed convex cone in the 3-dimensional Euclidean space R^3 and that $\mathcal{F}_i(0) = \{0\}$. The latter property of \mathcal{F}_i stipulates that at each contact, if the normal

force is zero, then so is the friction force and the transmitted moment.

Consider the following generalized friction conditions:

(iii)' for each $i = 1, \dots, n_c$, $(c_{it}, c_{io}, c_{ir}) \in \mathcal{F}_i(\mu_i c_{in})$;

(iv)' the maximum work inequality: for each $i \in \mathcal{N}$,

$$(c_{it}, c_{io}, c_{ir}) \in \operatorname{argmax} \{ -(v_{it}c'_{it} + v_{io}c'_{io} + v_{ir}c'_{ir}) : (c'_{it}, c'_{io}, c'_{ir}) \in \mathcal{F}_i(\mu_i c_{in}) \},$$

and for each $i \in \mathcal{R}$,

$$(c_{it}, c_{io}, c_{ir}) \in \operatorname{argmax} \{ -(a_{it}c'_{it} + a_{io}c'_{io} + a_{ir}c'_{ir}) : (c'_{it}, c'_{io}, c'_{ir}) \in \mathcal{F}_i(\mu_i c_{in}) \}.$$

The generalized dynamic multi-rigid-body problem with concurrent distributed frictional contacts is to find contact forces $(c_{in}, c_{it}, c_{io}, c_{ir})$ and accelerations $(a_{in}, a_{it}, a_{io}, a_{ir})$ satisfying conditions (i), (ii), (iii)', and (iv)'.

Examples of $\mathcal{F}_i(\tau)$ include (a) the quadratic cone (1):

$$\mathcal{F}_i(\tau) \equiv \left\{ (c_{it}, c_{io}, c_{ir}) \in R^3 : \frac{c_{it}^2}{e_{it}^2} + \frac{c_{io}^2}{e_{io}^2} + \frac{c_{ir}^2}{e_{ir}^2} \leq \tau^2 \right\},$$

where e_{it}, e_{io} , and e_{ir} are some given positive scalars; (b) approximations of such a cone by a convex polyhedron:

$$\mathcal{F}_i(\tau) \equiv \left\{ (c_{it}, c_{io}, c_{ir}) \in R^3 : \alpha_{ij}c_{it} + \beta_{ij}c_{io} + \gamma_{ij}c_{ir} \leq \tau, j = 1, \dots, m_i \right\},$$

where α_{ij}, β_{ij} and γ_{ij} are some given scalars and m_i is a positive integer; and (c) mixtures of elliptic and polyhedral friction constraints: *e.g.*,

$$\mathcal{F}_i(\tau) \equiv \left\{ (c_{it}, c_{io}, c_{ir}) \in R^3 : \frac{c_{it}^2}{e_{it}^2} + \frac{c_{io}^2}{e_{io}^2} \leq \tau^2, |c_{ir}| \leq \tau \right\}.$$

For planar problems, we can let

$$\mathcal{F}_i(\tau) \equiv \{(c_{it}, 0, 0) \in \mathbb{R}^3 : |c_{it}| \leq \tau\}.$$

Examples (a) and (c) pertain to axi-symmetric friction laws; whereas (b) do not necessary correspond to such laws. Other axi-asymmetric friction laws can also be modeled by using the friction map \mathcal{F}_i .

3 Existence and Uniqueness of Solutions

Employing a unified approach, we provide sufficient conditions for the existence and uniqueness of solutions to the basic model presented in the last section. Similar results can be established for variations of this model, such as those based on the abstract friction maps \mathcal{F}_i . Due to space limitation, we will focus our discussion on the basic model under the Coulomb friction limit surfaces.

Let \mathcal{F} consist of all force tuples (c_n, c_t, c_o, c_r) such that $c_n \geq 0$,

$$\left. \begin{aligned} e_{it}^2 \mu_i c_{in} v_{it} + \sqrt{e_{it}^2 v_{it}^2 + e_{io}^2 v_{io}^2 + e_{ir}^2 v_{ir}^2} c_{it} &= 0 \\ e_{io}^2 \mu_i c_{in} v_{io} + \sqrt{e_{it}^2 v_{it}^2 + e_{io}^2 v_{io}^2 + e_{ir}^2 v_{ir}^2} c_{io} &= 0 \\ e_{ir}^2 \mu_i c_{in} v_{ir} + \sqrt{e_{it}^2 v_{it}^2 + e_{io}^2 v_{io}^2 + e_{ir}^2 v_{ir}^2} c_{ir} &= 0 \end{aligned} \right\} \quad \forall i \in \mathcal{N},$$

and

$$\frac{c_{it}^2}{e_{it}^2} + \frac{c_{io}^2}{e_{io}^2} + \frac{c_{ir}^2}{e_{ir}^2} \leq \mu_i^2 c_{in}^2, \quad \forall i \in \mathcal{R}.$$

Let

$$\mathcal{F}_{\mathcal{J}} \equiv \mathcal{F} \cap \text{null space of } \mathcal{J}.$$

The main result of this paper is summarized in the following theorem.

Theorem 1 *Let $\mathbf{A} \equiv \mathcal{J}^T \mathcal{M} \mathcal{J}$ with \mathcal{M} being symmetric positive definite.*

(A) *If \mathbf{A} is positive definite, then there exists a scalar $\bar{\mu} > 0$ such that whenever $\mu_i \in [0, \bar{\mu}]$ for all $i \in \mathcal{N}$, there exist*

$$(a_n, a_t, a_o, a_r) \quad \text{and} \quad (c_n, c_t, c_o, c_r)$$

solving the rigid-body contact model defined by conditions (i)–(iv). If in addition $\mu_i \in [0, \bar{\mu}]$ for all $i \in \mathcal{R}$, then the solution is unique.

(B) *If $\mathcal{N} = \emptyset$, and*

$$\begin{bmatrix} b_n \\ b_t \\ b_o \\ b_r \end{bmatrix}^T \begin{bmatrix} c_n \\ c_t \\ c_o \\ c_r \end{bmatrix} \geq 0 \quad \text{for all} \quad \begin{bmatrix} c_n \\ c_t \\ c_o \\ c_r \end{bmatrix} \in \mathcal{F}_{\mathcal{J}},$$

then for any positive $\{\mu_i : i = 1, \dots, n_c\}$, the first conclusion of (A) holds.

The conditions in the two statements (A) and (B) are different. The conditions in (A) require the entire matrix \mathbf{A} be positive definite and the friction coefficients at the non-rolling contacts be small; in this case if the friction coefficients at the rolling contacts are also sufficiently small, then the solution must be unique. Part (B) pertains to the all-rolling case; in this case, there is no condition imposed on the friction coefficients; also \mathbf{A} is not required to be positive definite. The proofs of Theorem 1 is a simple extension of those in the papers [8, 10]; background results needed in the proofs are in [2, 1].

4 Conclusion

We have formulated the dynamic equations of a general, spatial, multi-rigid-body system with multiple distributed contacts as a complementarity problem. The condition guaranteeing existence of solution constrains both the maximum coefficient of friction at the non-rolling contacts and the linear independence of the

kinematic constraints associated with the contact geometry. If the coefficients of friction at the rolling contacts are also small, then the solution is unique. The second condition guaranteeing existence pertains to problems in which all contacts are initially rolling (without twisting). We emphasize that this condition does *not* restrict the coefficients of friction.

The ultimate goal of this work is to develop efficient algorithms for formulating and solving the complementarity multi-rigid-body contact problem. Algorithms exist for solving for general complementarity problems that include our formulation as a special case, but we anticipate that these algorithms will not be able to compete with one specialized to our problem. In animation applications, a specialized algorithm should be able to perform even better, if we exploit solution coherence over time. However, we must also keep in mind that the model developed here does not always have a unique solution and approaches must be developed to find all solutions, particularly when force history information is does not exist (in the first time step of integration) or is not useful (the first time step after an impulsive force occurs).

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