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A Quantitative Test for Form Closure Grasps[1]

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Abstract - *Grasp and manipulation planning of slippery objects often relies on the "form closure" grasp, which can be maintained regardless of the external force applied to the object. Despite its importance, a quantitative test for form closure valid for any number of contact points is not available. The primary contribution of this paper is the introduction of such a test formulated as a linear program, of which the optimal objective value provides a measure of how far a grasp is from losing form closure. While the test is formulated for frictionless grasps, we discuss how it can be modified to identify grasps with "frictional form closure."*

I. Introduction

In grasp and manipulation planning, the two most important classes of grasps are known as "form closure" and "force closure" grasps. These terms are borrowed from the field of machine design in which they have been in use since 1875, when Reuleaux [11] studied the mechanics of some "early machines." One machine was the water wheel, whose axel was usually laid in a groove of semi-circular cross-section. Proper operation required the gravitational force of the wheel to maintain or "close" the contact between the groove and the axel. Thus the terminology "force closure" came to describe contacts whose maintenance depended on an externally applied force. If instead, the contact was maintained by virtue of the geometry of the contacting elements, (as would be the case of an axel in a cylindrical hole), then the term "form closure" was adopted. This terminology is still in use today in the mechanisms research community (see [5]) and was first introduced into the robotics research community by Salisbury [12]. Since then, motivated by the mathematical interpretation of vector "closure," [3] some authors (notably, Nguyen [9], Mishra [8], and Sastry [6]) have chosen to use "force closure" to mean what Reuleaux and Salisbury meant by form closure. In this paper, we follow the president set by Reuleaux and Salisbury by adopting the following definitions.

Definition: Form Closure: A fixed set of contacts on a rigid body is said to exhibit *form closure* if the body's equilibrium is maintained despite the application of any possible externally applied wrench (force and moment). Equivalently, the contacts prevent all motions of the body, including infinitesimal motions.

Definition: Force Closure: A fixed set of contacts on a rigid body is said to exhibit *force closure* if the maintenance of the body's equilibrium requires the application an externally applied wrench. Equivalently, the contacts do not prevent all motions of the body.

While the open literature abounds with papers on grasping and grasp planning (see [10] for a good bibliography of grasping literature published before 1988), an efficient quantitative test for form closure valid for any number of contact points is not available. Reuleaux [11] studied the form closure problem for rigid lamina restricted to move in a plane. He showed that at least four higher-pair (point) contacts were required to prevent all motion of a lamina. He also provided a graphical technique to test a set of four contacts for form closure. These ideas were used by Nguyen to develop algorithms to synthesize form closure grasps of given rigid lamina and were extended for use with three-dimensional objects [9]. The conditions for form closure of an arbitrary three-dimensional rigid body were first given by Somoff, in 1900 [13], who established that a minimum of seven point contacts was necessary. Much later, Lakshminarayana [5] described an approach to synthesizing form closure grasps of three-dimensional frictionless objects and gave an insightful physical interpretation of the associated equations.

Mishra *et al.* [8] were the first to place an upper bound on the number of contact points needed for a form closure grasp of a frictionless object. They showed that if the object was "nonexceptional" (*i.e.*, the object's surface was not one of revolution), then twelve contact points were sufficient to balance all possible external wrenches. This bound however, seemed loose to Markenscoff *et al.* who succeeded in "closing the gap." [7] They proved the stronger result that any nonexceptional frictionless object can be grasped in form closure with

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only seven contact points by considering infinitesimal perturbations of the contact points away from the maximal inscribed sphere. They stated that their proof could be used as the basis for algorithms for synthesizing form closure grasps which would imply that form closure tests could also be developed, but no algorithms were presented. For the purpose of grasp synthesis, Nguyen [9] and Mishra [8] developed grasp tests that indicated only the existence or nonexistence of form closure. However, the binary nature of the tests motivated Kirkpatrick [4] to formulate a quantitative test for "positive grips" with form closure based on Steinitz's Theorems. Unfortunately, these results are restricted to frictionless grasps of polyhedra with at least 12 contacts occurring only at "nonsingular" points on the object's surface, where singular points are those for which the surface normal is ill-defined. These restrictions are seen as significant drawbacks, since in dexterous manipulation it is common (and occasionally desirable) for fewer contacts to occur and for some of them to be on vertices of the object.

A. Contributions

The primary contribution of this paper is the formulation of a quantitative test for detecting form closure in frictionless grasps. This test takes the form of a linear program that produces a crude measure, qualitatively similar to Kirkpatrick's, of how "far" a grasp is from losing form closure. In contrast to Kirkpatrick's test, our test is valid for any number of frictionless contacts as long as their locations and normal directions are known. The problem of contacts occurring at nondifferentiable surface points is not a consideration here, because a unique, computable normal is available in all but the ephemeral and pathological cases of a convex vertex in contact with either a convex edge (spatial case only) or another convex vertex. Since the test relies on geometric information, it is valid for frictional grasps, but it does not quantify friction's stabilizing effects. However, we discuss how to modify the test to explicitly include friction effects, so that grasps which do not have form closure due to their geometry can be tested for "frictional form closure." [1]

The secondary contribution of this paper is the development of a binary test for the identification of frictionless grasps belonging to the subclass of force closure grasps called *strong force closure* grasps. This subclass is identical to grasps of "partial restraint" considered by Lakshminarayana [5] and deserves special recognition because maintaining strong force closure grasps during dexterous manipulation requires compliant control of the fingers, which is not the case for other force closure grasps. Given Lakshminarayana's terminology, "partial restraint," one might also refer to this subclass of grasps as *partial form closure* grasps. However, we prefer the name *strong force closure* to emphasize that the grasps have force closure.

B. Paper Layout

In the remaining sections, three linear programs are introduced which can be used to quantify and detect grasp qualities. The test developed in Section 2 is able to detect and quantify form closure in both planar and spatial frictionless grasps. This test then serves as the basis for the frictional form closure and frictionless strong force closure tests introduced in Sections 2.1 and 3 respectively. The introduction of each test is followed by an illustrative example problem.

II. IDENTIFICATION OF FORM CLOSURE

According to the definition of Reuleaux and Salisbury, given in Section 1, a grasp has form closure if and only if object equilibrium is possible regardless of the external wrench. If we assume a point contact model with Coulomb friction, then for a grasp with n_c contacts, the equilibrium equations and Coulomb friction constraints may be written as follows (see [14]).

$$\mathbf{W}\mathbf{c} \geq -\mathbf{g}_{ext} \qquad \text{for all } \mathbf{g}_{ext} \in E^6 \tag{1}$$

$$\mathbf{c}^T \mathbf{D}\, \mathbf{c} \geq 0 \tag{2}$$

$$\mathbf{c}_n \geq \mathbf{0} \tag{3}$$

where E^6 represents the 6-dimensional Euclidean space (6 is the number of degrees of freedom of the uncontacted object), $\mathbf{g}_{ext}$ is the external wrench applied to the object, $\mathbf{W}$ is the $6 \times 3n_c$ wrench matrix formed by the horizontal concatenation of the individual contact wrench matrices, $\mathbf{W}_i$,

$$\mathbf{W}_i = \begin{bmatrix} \hat{\mathbf{n}}_i & \hat{\mathbf{t}}_i & \hat{\mathbf{o}}_i \\ \mathbf{r}_i \times \hat{\mathbf{n}}_i & \mathbf{r}_i \times \hat{\mathbf{t}}_i & \mathbf{r}_i \times \hat{\mathbf{o}}_i \end{bmatrix}, \tag{4}$$

$\mathbf{r}_i$ is the position of the i^{th} contact point, $\hat{\mathbf{n}}_i$ is the contact's unit normal directed inward with respect to the object, $\hat{\mathbf{t}}_i$ and $\hat{\mathbf{o}}_i$ are orthogonal unit vectors defining the contact tangent plane, $\mathbf{D}$ is the $3n_c \times 3n_c$ diagonal Coulomb friction matrix formed by the block diagonal concatenation of the individual Coulomb friction matrices, $\mathbf{D}_i$ (see equation (5)), $\mathbf{c}$ is the vector of wrench intensities of length $3n_c$ formed by the vertical concatenation of the individual wrench intensity vectors, $\mathbf{c}_i$,

$$\mathbf{D}_i = \begin{bmatrix} \mu_i^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{c}_i = \begin{bmatrix} c_{in} \\ c_{it} \\ c_{io} \end{bmatrix}, \tag{5}$$

and $\mathbf{c}_n$ is the normal wrench intensity vector of length n_c formed by the vertical concatenation of the individual normal wrench intensities, c_{in}.

Partitioning equilibrium equation (1) to expose the tangential and orthogonal wrench intensity vectors, $\mathbf{c}_t$ and $\mathbf{c}_o$, which describe the friction wrenches yields

$$\begin{bmatrix} \mathbf{W}_n & \mathbf{W}_t & \mathbf{W}_o \end{bmatrix} \begin{bmatrix} \mathbf{c}_n \\ \mathbf{c}_t \\ \mathbf{c}_o \end{bmatrix} = -\mathbf{g}_{ext} . \tag{6}$$

where $\mathbf{c}_t$ and $\mathbf{c}_o$ are formed by vertically concatenating the elements c_{it} and c_{io}, respectively, and the normal, tangential, and orthogonal wrench matrices, $\mathbf{W}_n$, $\mathbf{W}_t$, and $\mathbf{W}_o$ are formed in correspondence with the definitions of $\mathbf{c}_n$, $\mathbf{c}_t$, and $\mathbf{c}_o$.

The form closure conditions when friction is absent can be obtained from relationships (1), (2), and (3) by setting Coulomb friction matrix, $\mathbf{D}$, and the tangential and orthogonal wrench intensity vectors, $\mathbf{c}_t$ and $\mathbf{c}_o$, to zero, yielding

$$\mathbf{W}_n \mathbf{c}_n = -\mathbf{g}_{ext} ; \qquad \text{for all } \mathbf{g}_{ext} \in E^6 \tag{7}$$

$$\mathbf{c}_n \geq \mathbf{0} . \tag{3}$$

From Somoff's work, [13] two necessary conditions for form closure are that the normal wrench matrix, $\mathbf{W}_n$, be full rank and have more columns than rows. Therefore, $\mathbf{W}_n$ has a nontrivial null space which allows us to rewrite the form closure requirements in terms of the row and null space components of the normal wrench intensity vector as follows

$$\mathbf{W}_n \mathbf{c}_{n,row} = -\mathbf{g}_{ext} ; \qquad \text{for all } \mathbf{g}_{ext} \in E^6 \tag{8}$$

$$\mathbf{W}_n \mathbf{c}_{n,null} = \mathbf{0} \tag{9}$$

$$\mathbf{c}_{n,row} + \mathbf{c}_{n,null} \geq \mathbf{0} , \tag{10}$$

where $\mathbf{c}_{n,null}$ and $\mathbf{c}_{n,row}$ are the null and row space components $\mathbf{c}_n$, respectively. With relationships (8-10) in mind, Salisbury showed that a sufficient condition for form closure is the existence of a vector, $\mathbf{c}_{n,null}$, with all positive elements, which is equivalent to Mishra's result requiring that the origin of the wrench space lie strictly within the convex hull defined by the columns $\mathbf{W}_n$ [8]. Next we use the facts that $\mathbf{W}_n$ and its pseudoinverse provide one-to-one and onto mappings between the spaces of $\mathbf{g}_{ext}$ and $\mathbf{c}_{n,row}$, and that $\mathbf{g}_{ext}$ is arbitrary. As such, any or all elements of $\mathbf{c}_{n,row}$ can be made negative by the proper choice of $\mathbf{g}_{ext}$. This places the onus of $\mathbf{c}_n$'s nonnegativity squarely on $\mathbf{c}_{n,null}$. Therefore, form closure requires that equation (9) admit at least one strictly positive solution, *i.e.*, the following relationships must be feasible

$$\mathbf{W}_n \mathbf{c}_{n,null} = \mathbf{0} \tag{9}$$

$$\mathbf{c}_{n,null} > \mathbf{0} . \tag{11}$$

If no such solution exists, then one can easily find an external wrench, $\mathbf{g}_{ext}$, that cannot be balanced. If a strictly positive $\mathbf{c}_{n,null}$ does exist, then it may be arbitrarily scaled to make $\mathbf{c}_n$ nonnegative for any finite choice of $\mathbf{g}_{ext}$. In fact, in this case, all wrench intensities may be increased without bound, which in turn, implies that the joint torques may be increased without bound, too. This observation turns out to be quite useful in trajectory planning for dexterous manipulation, as it implies that we can squeeze as hard as we like without disturbing the form closure character of the grasp; a fact which considerably reduces the accuracy required of the force controller. However, it is important to note that Cutkosky has shown that compliant effects can cause grasp instability as the joint torques increase [2].

The form closure measure we propose, is the scalar value, d^*, of the minimum element of $\mathbf{c}^*_{n,null}$, where $\mathbf{c}^*_{n,null}$ is the null space vector with largest minimum element. If d^* is strictly positive, then the grasp has form closure, otherwise the grasp has force closure. This measure is the optimal objective value of the following linear program

$$\underset{\mathbf{c}_{n,null}}{\text{Maximize}} \qquad d \tag{12}$$

$$\text{Subject to:} \qquad \mathbf{W}_n \mathbf{c}_{n,null} = \mathbf{0} \tag{9}$$

$$\mathbf{c}_{n,null} - \mathbf{d} \geq \mathbf{0} \tag{13}$$

$$d \geq 0 \tag{14}$$

$$\mathbf{A}\mathbf{c}_{n,null} \geq \mathbf{h} \tag{15}$$

where d is a slack variable and $\mathbf{d}$ is a vector with all elements equal to d. Inequality (15) may be any set of constraints that is feasible for $\mathbf{c}_{n,null} = \mathbf{0}$ and that prevents the linear program from becoming unbounded. Note that if inequality (15) approximates the unit ball, then our measure is quite similar to the grasp "efficiency" given by Kirkpatrick [4]. However, Kirkpatrick's measure is valid only for grasps with 12 or more contact points (6 or more in the planar case), whereas our measure is valid for grasps with any number of contact points.

A. Example 1

Consider a rectangle subjected to a planar grasp with four frictionless contact points as shown in Figure 1. It can be shown analytically, that this grasp has form closure if the intersection of the third and fourth contact normals lie inside the rectangle, *i.e.* α lies in the interval $(1.052, \frac{\pi}{2})$. If $\alpha = \frac{\pi}{2}$, then the object may translate vert-

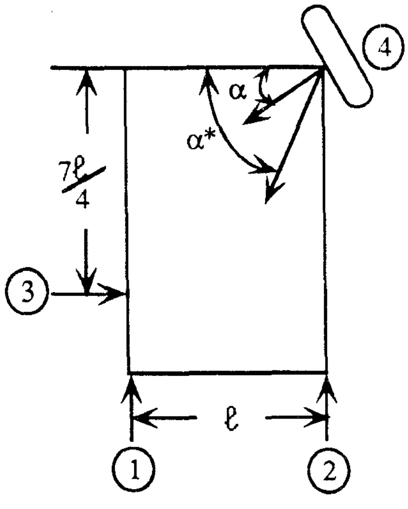

Figure 1: Rectangle with Four Frictionless Contact Points.

ically. For this Example, the linear program defined by statements (9) and (12-15) was used to quantify form closure for various values of α. **A** and **h** were chosen so that inequality (15) would represent a cube with edges of length 2, centered on the origin of the space defined by $\mathbf{c}_{n,null}$

$$\mathbf{A} = \begin{bmatrix} -\mathbf{I} \\ \mathbf{I} \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} -\mathbf{1} \\ -\mathbf{1} \end{bmatrix},$$

where **I** is the 4×4 identity matrix and **1** is the four-vector with all elements equal to 1. The normal wrench matrix was formed using the coordinate directions shown and summing the moments about the upper right-hand corner of the rectangle yielding

$$\mathbf{W}_n = \begin{bmatrix} 0 & 0 & 1 & -cos(\alpha) \\ 1 & 1 & 0 & -sin(\alpha) \\ -l & 0 & \frac{7l}{4} & 0 \end{bmatrix}.$$

Table 1, below, summarizes the results. Note that the grasp "furthest" from losing form closure is the one for which $\alpha^* \approx 1.2$ Radians.

α	d^*	closure type
1.04	0.0	force
1.06	0.020	**form**
1.1	0.097	**form**
1.2	**0.300**	**form**
1.3	0.267	**form**
1.4	0.170	**form**
1.5	0.071	**form**
1.56	0.010	**form**
1.5707	0.0	force
1.58	0.0	force

Table 1: Quantification of Frictionless Form Closure.

Note that the computation time required to compute each d^* is small since its value requires the solution of a linear program with $nullity(\mathbf{W}) + 1$ (in this example, 2) variables and $n_{dof} + 1 + nullity(\mathbf{W}) + n_{bound}$ (in this example, 15) constraints where n_{dof} is the number of degrees of freedom of the uncontacted object, $nullity(\mathbf{W})$ is the nullity of the normal wrench matrix, and n_{bound} is the number of bounding planes used to approximate the unit ball. However, this computation time cannot be compared to Kirkpatrick's since there are too few contacts for his method to apply.

B. Frictional Form Closure

Frictionless form closure hinges upon the ability of the elements of the normal wrench intensity vector, $\mathbf{c}_n$, to increase indefinitely. If under the frictionless assumption, a grasp does not have form closure, then one may test the grasp for "frictional form closure" [1] through the following procedure. First, using equations (1-3), determine which set of elements of $\mathbf{c}_n$ can be increased without bound[2]. Second, include the corresponding columns of $\mathbf{W}_t$ and $\mathbf{W}_o$ and elements of $\mathbf{c}_t$ and $\mathbf{c}_o$ in the wrench matrix and wrench intensity vector in the form closure test given by equations (9) and (12-15) (see Example 2 below for clarification). However, because the individual tangential and orthogonal wrench intensities, c_{it} and c_{io}, may be positive or negative, inequality (13) should not be modified. Thus our test for frictional

[2] Determining the set or sets of element of $\mathbf{c}_n$ could be accomplished by approximation the friction cone constraints (2) by systems of linear inequalities and solving one or more linear programs designed to drive elements of $\mathbf{c}_{n,null}$ toward infinity. However, the computational complexity of such a test could make it impractical.

form closure takes the following form

$$\underset{\mathbf{c}_{n,null}}{\text{Maximize}} \quad d \tag{12}$$

$$\text{Subject to:} \quad \mathbf{W}\mathbf{c}_{null} = \mathbf{0} \tag{18}$$

$$\mathbf{c}_{n,null} - \mathbf{d} \geq \mathbf{0} \tag{13}$$

$$d \geq 0 \tag{14}$$

$$\mathbf{A}\mathbf{c}_{null} \geq \mathbf{h} \tag{19}$$

where $\mathbf{W}$ and $\mathbf{c}_{null}$ are $\mathbf{W}_n$ and $\mathbf{c}_{n,null}$ augmented by adding the columns of $\mathbf{W}_t$ and $\mathbf{W}_o$ and the elements of $\mathbf{c}_t$ and $\mathbf{c}_o$ corresponding to the elements of $\mathbf{c}_n$ which can be increased indefinitely. This test procedure amounts to determining which friction wrenches can act as rigid structural restraints when squeezing sufficiently tightly.

C. Example 2

Consider the three-point grasp of the triangle shown in Figure 2 below.

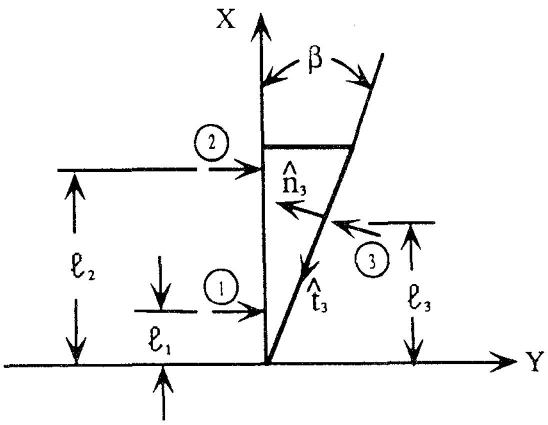

Figure 2: Triangle with Frictional and Frictionless Contacts.

If all contacts are assumed to be frictionless, then the grasp does not have form closure. However if we assume that contacts 1 and 2 (on the vertical edge) are frictionless and contact 3 has coefficient of friction $\mu > 0$, then it can be shown geometrically [9] that as long as $\tan^{-1}\mu > \beta$ and $l_1 > l_3 > l_2$, then all external wrenches, $\mathbf{g}_{ext}$, may be balanced by squeezing tightly enough. This implies that the tangential wrench and intensity corresponding to the third contact should be included in the frictional form closure test defined by statements (12-14), (18), and (19).

The augmented wrench matrix and wrench intensity vector are given as follows

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & -\cos(\beta) & -\sin(\beta) \\ 0 & 0 & \sin(\beta) & -\cos(\beta) \\ -l_1 & -l_2 & \dfrac{l_3}{\cos(\beta)} & 0 \end{bmatrix} \tag{20}$$

$$\mathbf{c}_{n,null} = \begin{bmatrix} c_{1n,null} \\ c_{2n,null} \\ c_{3n,null} \end{bmatrix} \qquad \mathbf{c}_{null} = \begin{bmatrix} \mathbf{c}_{n,null} \\ c_{3t,null} \end{bmatrix}. \tag{21}$$

For this Example, we applied the frictional form closure test for different positions of the third contact, l_3, with all other parameters fixed: $l_1 = 0$, $l_2 = 1$, $\mu = 0.3$, and $\beta = 15°$. Table 2 below summarizes the results. $\mathbf{A}$ and $\mathbf{h}$ had the same values as in Example 1.

l_3	d^*	closure type	$c^*_{1n,null}$	$c^*_{2n,null}$
-0.001	0.0	force		
0.001	0.001	**frictional form**	0.001	1.0
0.1	0.104	**frictional form**	0.104	0.932
0.3	0.311	**frictional form**	0.311	0.725
0.5	**0.518**	**frictional form**	**0.518**	**0.518**
0.7	**0.311**	**frictional form**	0.725	0.311
0.9	0.104	**frictional form**	0.932	0.104
0.999	0.001	**frictional form**	1.0	0.001
1.001	0.0	force		

Table 2: Quantification of Frictional Form Closure.

When the grasp has frictional form closure, the null space components of $\mathbf{c}_n$ are all positive. This indicates that the object can be squeezed as tightly as one likes. In the case of the force closure grasps, one or more elements of $\mathbf{c}_{n,null}$ are negative, so if force closure is possible at all, squeezing is limited. Note that the elements of $\mathbf{c}_{n,null}$ are not shown for the force closure grasps, because the subroutine used to solve the frictional form closure test only returned the flag "infeasible" and not a null space basis vector. In addition, $c^*_{3n,null}$ and $c^*_{4n,null}$ don't change appreciably with l_3 and so are not listed here.

III. Identification of Strong Force Closure

All elements of the normal wrench intensity vector of a form closure grasp can be increased indefinitely without disturbing the equilibrium of the grasp. For most force closure grasps, all wrench intensities have finite bounds. Equivalently, manipulation maintaining form closure requires compliant finger motion, whereas maintaining force closure usually does not. In this Section, we define the subclassification of force closure grasps which we call *strong force closure* grasps. This subclassification is equivalent to Lakshminarayana's grasps of "partial restraint." Grasps in this subclass deserve recognition, because their maintenance during manipulation requires compliant motion control, as would be the case for form closure grasps, but they can become unstable since they are, in fact, force closure grasps. As in the previous Section, we concentrate on the frictionless case and discuss the inclusion of friction effects later.

Definition: A frictionless grasp is said to have *strong force closure* if it does not have form closure and a subset of the elements of the normal wrench intensity vector can be increased without bound.

Since a strong force closure grasp does not have form closure, no strictly positive solution of equation (9) may exist (*i.e.*, relationships (9) and (11) are infeasible). However, given our definition, at least one nonnegative solution must exist (*i.e.*, relationship (11) is relaxed by allowing equality with zero). If a nonnegative solution does not exist, then no element of the normal wrench intensity vector can be increased indefinitely.

Theorem: A frictionless grasp has strong force closure if and only if it does not have form closure and there exists a nontrivial vector, $\mathbf{c}_{n,null}$, in the null space of $\mathbf{W}_n$ with all nonnegative elements, such that if the i^{th} element of $\mathbf{c}_{n,row}$ is negative, then the i^{th} element of $\mathbf{c}_{n,null}$ is positive.

Proof: The quasi-static assumption implies that the grasp under consideration is in equilibrium and therefore satisfies the relationships (7) and (3). Solving equation (7) and substituting into inequality (3) yields

$$\mathbf{c}_n = \mathbf{c}_{n,row} + \mathbf{c}_{n,null} \geq \mathbf{0}, \tag{10}$$

where $\mathbf{c}_{n,row} = -\mathbf{W}_n{}^{+}\mathbf{g}_{ext}$, $\mathbf{c}_{n,null} = (\mathbf{W}_n{}^{+}\mathbf{W}_n - \mathbf{I})\hat{\mathbf{k}}_c\alpha$, α is a positive scalar, and $\hat{\mathbf{k}}_c$ is an arbitrary unit vector of compatible length.

If $\mathbf{c}_{n,null} > \mathbf{0}$ exists, then by definition, the grasp has form closure: not force closure. Next, note that $\mathbf{c}_{n,null}$ with all nonpositive elements, (*i.e.*, $\mathbf{c}_{n,null} \leq \mathbf{0}$), prevents the unbounded increase of $\mathbf{c}_{n,null}$. Therefore, for a grasp to have strong force closure, it is necessary that a nontrivial $\mathbf{c}_{n,null}$ exists such that $\mathbf{c}_{n,null} \geq \mathbf{0}$. Finally, let $c_{in,row}$ be the i^{th} element of $\mathbf{c}_{n,row}$. It is clear from equation (10) that if $c_{in,row}$ is negative, then $c_{in,null}$ must be positive. *q.e.d.*

In light of the above proof, a test for frictionless strong force closure, must allow some of the elements of $\mathbf{c}_{n,null}$ to remain zero while encouraging others to be positive. This can be accomplished with the following linear program

$$\underset{\mathbf{c}_{n,null}}{\text{Maximize}} \quad y = \mathbf{1}^T\mathbf{c}_{n,null} \tag{22}$$

$$\text{Subject to:} \quad \mathbf{W}_n\mathbf{c}_{n,null} = \mathbf{0} \tag{9}$$

$$\mathbf{W}_n\mathbf{c}_{n,row} = -\mathbf{g}_{ext} \tag{23}$$

$$\mathbf{c}_{n,row} + \mathbf{c}_{n,null} \geq \mathbf{0} \tag{10}$$

$$\mathbf{c}_{n,null} \geq \mathbf{0} \tag{24}$$

$$\mathbf{A}\mathbf{c}_{n,null} \geq \mathbf{h}\,. \tag{15}$$

where y^* is the optimal value.

During manipulation under compliant control the matrix $\mathbf{W}_n$, which has dimension $(6 \times n_c)$, is typically of full rank and usually has 7 or more columns. If the nullity of $\mathbf{W}_n$ is one, then the solution of the above linear program indicates precisely which elements of $\mathbf{c}_n$ may be increased and which may not: if $c^*_{in,null}$ is zero, then $c_{i,n}$ may not be increased; all others may (see Example 3 below). When the nullity is greater than one, then the solution returned is just one of many possible and will not necessarily reflect which intensities may be increased. Brute force circumvention of this problem could be achieved by solving the linear program one time for each contact with the objective being to maximize the corresponding component of $\mathbf{c}_{n,null}$, rather than the sum of the elements.

As illustrated in Example 3 below, our strong force closure test is binary in nature. This comes from the fact that the null space components of some elements of $\mathbf{c}_{n,null}$ are zero-valued. However, a quantitative result similar to that produced by the form closure test could be formulated in two stages. First apply the binary test to identify the nonzero components of $\mathbf{c}_{n,null}$ and then apply the form closure test with the slack variable, d, added only to those components.

A. Example 3

Consider a rectangle subjected to a planar grasp with four frictionless contact points as shown in Figure 3. It can be shown that this grasp has strong force closure if $-1.00 \leq a \leq 1.00$. It does not have form closure, since the object may translate vertically. Using the coordinate directions shown and summing the moments about the object's center of mass yields the following normal wrench matrix

$$\mathbf{W}_n = \begin{bmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & a \end{bmatrix}. \tag{25}$$

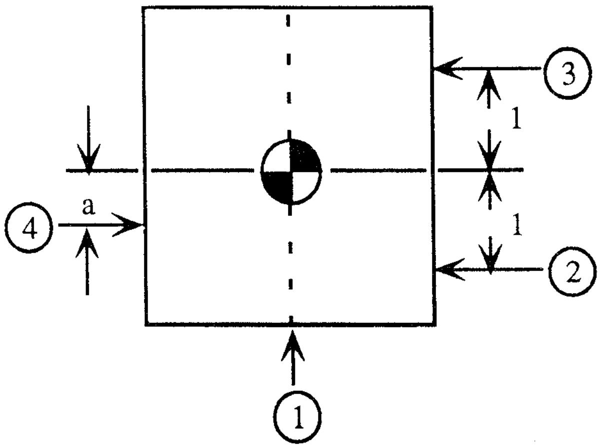

Figure 3: Square in Frictionless Strong Force Closure Grasp.

Table 3 below illustrates how the components of $\mathbf{c}_{n,null}$ and $\mathbf{c}_{n,row}$ vary with a. **A** and **h** had the same values as in Example 1.

a	y^*	closure type	$c^*_{n,null,2}$	$c^*_{n,null,3}$
-1.001	0.0	force		
-1.000	2.0	**strong force**	0.0	1.0
-0.5	2.0	**strong force**	0.25	0.75
-0.0	**2.0**	**strong force**	**0.5**	**0.5**
0.4	2.0	**strong force**	0.7	0.3
0.7	2.0	**strong force**	0.85	0.15
1.00	2.0	**strong force**	1.0	0.0
1.001	0.0	force		

Table 3: Detecting Frictionless Strong Force Closure.

The values of $c^*_{n,null,1}$ and $c^*_{n,null,4}$ are not shown since $c^*_{n,null,1} = 0$ and $c^*_{n,null,4} = 1$ for all values of a. This implies that $c_{n,1}$ cannot be altered by squeezing and so should not be included in optimizing the grasp (with respect to a). Letting d^* be defined as the maximum, minimum element of $c^*_{n,null,2}$, $c^*_{n,null,3}$, and $c^*_{n,null,4}$ over all values of a indicates that centering contact 4 between contacts 2 and 3 is "optimal."

B. Frictional Strong Force Closure

To include the effects of friction in the strong force closure test, one could proceed as for the form closure test by identifying the frictional contacts whose normal wrench intensity components can be increased without bound. However, this test would only be useful for grasps which do not have strong force closure without considering friction and do not have form closure when including friction, but **do** have strong force closure when friction is considered. Although it has not been proven, we suspect that any grasp which has frictional strong force closure has frictional form closure. Thus, if our suspicions are true, then a test for frictional strong force closure would be unnecessary.

IV. Conclusion

A desire to solve dexterous manipulation planning problems has highlighted the need for computational procedures for identifying and quantifying form closure grasps. Toward this end, we have developed a linear program whose optimal objective value provides a measure of how far a grasp is from losing form closure. For situations in which it would be desirable to maintain form closure at all times (*e.g.*, in micro-gravity environments), our test would be particularly useful for both planning and monitoring grasp "health." To our knowledge, our form closure test is the first quantitative test valid for any number of contact points.

We have also derived a second test, binary in nature, which one can use to identify frictionless grasps of "partial restraint," which we call *strong force closure* grasps. The distinction between form closure and strong force closure is important, because while form closure grasps can always be maintained (given sufficient hand strength), strong force closure grasps cannot. Thus during manipulation planning, if a grasp has form closure, one need not be terribly concerned with the external wrench applied to the object, whereas otherwise, knowledge of the external wrench is crucial. It is also important to recognize strong force closure grasps, because unlike force closure grasps, their maintenance during dexterous manipulation requires some form of complaint control.

It is possible to apply the work presented above to optimal grasping in a rather straight forward way, exhaustively evaluating a subset of all possible grasps. Such an approach would be exceedingly computationally intensive and would therefore be of little use in real-time grasping and manipulation tasks. One fruitful avenue of research would invlolve the development of grasp systhesis algorithms which use the quantitative results given above, but do not require evaluation of huge numbers of potential grasps. This should be possible by applying geometric forms of the necessary and sufficient conditions for form closure [9] to large regions (*i.e.*, planar faces) of the grasped object.

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