# Generalized Scoring Rules and the Frequency of Coalitional Manipulability

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## ABSTRACT

We introduce a class of voting rules called *generalized scoring rules*. Under such a rule, each vote generates a vector of $k$ scores, and the outcome of the voting rule is based only on the sum of these vectors—more specifically, only on the order (in terms of score) of the sum's components. This class is extremely general: we do not know of any commonly studied rule that is not a generalized scoring rule.

We then study the coalitional manipulation problem for generalized scoring rules. We prove that under certain natural assumptions, if the number of manipulators is $O(n^p)$ (for any $p < \frac{1}{2}$), then the probability that a random profile is manipulable is $O(n^{p-\frac{1}{2}})$, where $n$ is the number of voters. We also prove that under another set of natural assumptions, if the number of manipulators is $\Omega(n^p)$ (for any $p > \frac{1}{2}$) and $o(n)$, then the probability that a random profile is manipulable (to any possible winner under the voting rule) is $1 - O(e^{-\Omega(n^{2p-1})})$. We also show that common voting rules satisfy these conditions (for the uniform distribution). These results generalize earlier results by Procaccia and Rosenschein as well as even earlier results on the probability of an election being tied.

## Categories and Subject Descriptors

J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*; I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems

## General Terms

Algorithms, Economics, Theory

## Keywords

Computational social choice, frequency of manipulability, voter/manipulator power, probability of tied elections



## 1. INTRODUCTION

In mechanism design, often, various assumptions about the space of possible outcomes and the agents' preferences are made. For example, it is often assumed that the agents can make payments, and that their utilities are *quasilinear* (that is, the contribution of payment to utility is linear and independent of the outcome chosen). However, these assumptions are not always reasonable. In contrast, in a general *social choice* or *voting* setting, every agent (or *voter*) can rank the outcomes (or *alternatives*) in any possible way. A mechanism (or *voting rule*) takes every agent's reported ranking of the alternatives as input, and produces one of the alternatives as output.

Unfortunately, considering such an unrestricted setting comes at a price. It turns out that any reasonable voting rule is vulnerable to *manipulation*, that is, a voter can sometimes make herself better off by declaring her preferences insincerely. A rule that is not vulnerable to manipulation is called *strategy-proof*. The Gibbard-Satterthwaite theorem [13, 18] states that when there are three or more alternatives, there is no strategy-proof voting rule that satisfies non-imposition (for every alternative, there exist votes that would make that alternative win) and non-dictatorship (the rule does not simply always choose the most-preferred alternative of a single fixed voter). This is in sharp contrast to settings with quasilinear preferences, where, for example, VCG mechanisms [20, 6, 14] are strategy-proof.

Although a manipulation is guaranteed to exist (for reasonable rules), in order for the manipulating agent to use it, she must also be able to find it. Recent research has studied whether finding a manipulation can be made computationally hard, thereby erecting a computational barrier against manipulation. A number of results have been obtained that show that finding a successful manipulation is NP-hard [3, 2, 7, 11, 9, 15]. Some of these results consider manipulation by an individual voter, whereas others consider the more general case of manipulation by a coalition of voters.

However, all of these hardness results are *worst-case* results. That is, they suggest that any algorithm will require superpolynomial time to solve *some* instances. However, this does not mean that there is no efficient algorithm that can find a manipulation for *most* instances. Several recent results seem to suggest that indeed, in various senses, hard instances of the manipulation problem are the exception rather than the rule [17, 8, 16, 21].

The results in this paper add to the body of work that suggests that the manipulation problem is usually easy to solve. For a very large class of voting rules, we show that in most cases, as the number of voters gets large, either the probability that the manipulators can change the outcome is very small, or the probability that they can (easily) make any alternative win is very large. Which of these two cases holds depends on the relative size of the coalition of manipulators, and there is a small boundary between these two cases for which we have no result, when the size of the manipulating coalition is on the order of $\sqrt{n}$, where $n$ is the total number of voters. Hence, for almost all cases, a simple inspection of relative size of the manipulating coalition suffices to decide the manipulation problem (and finding the actual manipulation is not hard).

More specifically, given the nonmanipulators' votes, there is some set of alternatives that can still win. That is, an alternative $c$ is a *possible winner* with respect to a given set of (nonmanipulators') votes and some set of manipulators if there exist votes for the manipulators that make $c$ win. In this paper, we consider a setting in which the nonmanipulators' votes are drawn at random, and we are interested in how large the set of possible winners is. What is the probability that the manipulators cannot change the outcome (there is only one possible winner)? What is the probability that the manipulators can make any alternative win (all alternatives are possible winners)? For a very general class of voting rules, we will show conditions under which the former probability is high, and conditions under which the latter probability is high. Under the latter set of conditions, we also show *how* the manipulators can make any alternative win (with a high probability).

These results are very similar to the results by Proacaccia and Rosenschein [16], but our results are significantly more general. Specifically, Proacaccia and Rosenschein only show their result for positional scoring rules (which we will define shortly). They also mention without proof that they can extend the results to the Copeland and maximin rules, and they conjecture that the results can be extended to other rules as well. Our results serve to prove this informal conjecture: we introduce a new class of voting rules called *generalized scoring rules*, and we prove the results for this class of rules. This class is extremely general: we are not aware of any commonly studied voting rule that cannot be expressed as a generalized scoring rule.

While we feel that our main contribution is to introduce the class of generalized scoring rules and prove the results for these rules, we also characterize the probability of manipulability more precisely rather than saying it converges to 1 or 0.[1] This characterization constitutes a general version of various results on the probability that an election ends up (roughly) in a tie, that is, a single voter can change the winner; this probability is also called the *voting power* [4, 12]. Knowing this probability is also interesting from the perspective of a voter who is determining her incentive to vote. Again, all of the existing results consider only the much smaller class of positional scoring rules. Specifically, Baharad and Neeman [1] showed that under some local correlation conditions, when the number of manipulators is no more than a constant, the probability that manipulation can be done is $O(\frac{1}{\sqrt{n}})$, where $n$ is the number of voters, under any positional scoring rule. Slinko [19] showed that under a particular condition on the probability distribution, under any faithful positional scoring rule (that is, all the scores in the scoring vector are different) the ratio of the number of manipulable profiles to the number of all profiles is $O(\frac{k}{\sqrt{n}})$, where $k$ is the number of manipulators.

The rest of this paper is laid out as follows. After covering some preliminaries in Section 2, in Section 3, we introduce *generalized scoring rules*, in which every vote generates a vector of $k$ scores, and the outcome of the voting rule is based only on the sum of these vectors—more specifically, only on the order (in terms of score) of the sum's components. This class is extremely general: we do not know of any commonly studied rule that is not a generalized scoring rule. In the subsequent sections, we study the coalitional manipulation problem for generalized scoring rules. In Section 4 we prove that under certain natural assumptions, if the number of manipulators is $O(n^p)$ (for any $p < \frac{1}{2}$), then the probability that a random profile is manipulable is $O(n^{p-\frac{1}{2}})$, where $n$ is the number of voters. In Section 5, we prove that, under another set of natural assumptions, if the number of manipulators is $\Omega(n^p)$ (for any $\frac{1}{2} < p < 1$) and $o(n)$, then the probability that a random profile is manipulable (to any possible winning alternative under the rule) is $1 - O(e^{-\Omega(n^{2p-1})})$. Finally, in Section 6, we show how these results apply to any positional scoring rule, Copeland, STV, maximin, and ranked pairs, under the uniform distribution over votes.

[1]For the result where the manipulators can probably make any alternative win, Procaccia and Rosenschein do give an expression for this probability in their proof (for positional scoring rules).

## 2. PRELIMINARIES

Let $\mathcal{C} = \{c_1, \ldots, c_m\}$ be the set of *alternatives* (or *candidates*). A linear order on $\mathcal{C}$ is a transitive, antisymmetric, and total relation on $\mathcal{C}$. The set of all linear orders on $\mathcal{C}$ is denoted by $L(\mathcal{C})$. An $n$-voter profile $P$ on $\mathcal{C}$ consists of $n$ linear orders on $\mathcal{C}$. That is, $P = (V_1, \ldots, V_n)$, where for every $i \leq n$, $V_i \in L(\mathcal{C})$. The set of all profiles on $\mathcal{C}$ is denoted by $P(\mathcal{C})$. In the remainder of the paper, $m$ denotes the number of alternatives and $n$ denotes the number of voters. A *voting rule* $r$ is a function from the set of all profiles on $\mathcal{C}$ to $\mathcal{C}$, that is, $r : P(\mathcal{C}) \to \mathcal{C}$. The following are some common voting rules.

1. *(Positional) scoring rules*: Given a *scoring vector* $\vec{v} = (v(1), \ldots, v(m))$, for any vote $V \in L(\mathcal{C})$ and any $c \in \mathcal{C}$, let $s(V, c) = v(j)$, where $j$ is the rank of $c$ in $V$. For any profile $P = (V_1, \ldots, V_n)$, let $s(P, c) = \sum_{i=1}^{n} s(V_i, c)$. The rule will select $c \in \mathcal{C}$ so that $s(P, c)$ is maximized. Two examples of scoring rules are *Borda*, for which the scoring vector is $(m-1, m-2, \ldots, 0)$, and *plurality*, for which the scoring vector is $(1, 0, \ldots, 0)$.

2. *Copeland*: For any two alternatives $c_i$ and $c_j$, we can simulate a *pairwise election* between them, by seeing how many votes prefer $c_i$ to $c_j$, and how many prefer $c_j$ to $c_i$. Then, an alternative receives one point for each win in a pairwise election. Typically, an alternative also receives half a point for each pairwise tie. The winner is the alternative who has the highest score.

3. *STV*: The election has $|\mathcal{C}|$ rounds. In each round, the alternative that gets the minimal plurality score drops out, and is removed from all of the votes (so that votes for this alternative transfer to another alternative in the next round). The last-remaining alternative is the winner.

4. *Maximin*: Let $N(c_i, c_j)$ denote the number of votes that rank $c_i$ ahead of $c_j$. The winner is the alternative $c$ that maximizes $min\{N(c, c') : c' \in \mathcal{C}, c' \neq c\}$.

5. *Ranked pairs*: This rule first creates an entire ranking of all the alternatives. $N(c_i, c_j)$ is defined as for the maximin rule. In each step, we will consider a pair of alternatives $c_i, c_j$ that we have not previously considered; specifically, we choose the remaining pair with the highest $N(c_i, c_j)$. We then fix the order $c_i > c_j$, unless this contradicts previous orders that we fixed (that is, it violates transitivity). We continue until we have considered all pairs of alternatives (hence we have a full ranking). The alternative at the top of the ranking wins.

In this paper, a *manipulation instance* is defined as follows.

**Definition 1** *A* manipulation instance *is a tuple $(r, P^{NM}, |M|)$, consisting of a voting rule $r$, a profile of nonmanipulators $P^{NM}$, and a number of manipulators $|M|$. A* weighted manipulation instance *is a tuple $(r, P^{NM}, W_{NM}, W_M)$, where $W_{NM}$ and $W_M$ are the weights of the nonmanipulators and the manipluators, respectively.*

## 3. GENERALIZED SCORING RULES

In this section, we define an extremely general class of voting rules that we call *generalized scoring rules*. This is the class for which we will prove our results. We do not know of any example of a commonly studied rule that is not a generalized scoring rule. A generalized scoring rule associates a vector of $k$ real numbers with every vote, for some $k$ that depends on (but is not necessarily equal to) $m$. The decision that the rule makes is based only on the sum of these vectors. Even more specifically, the decision is based only on comparisons among the components in this sum. That is, if we know, for every $i, j \in \{1, \ldots, k\}$, whether the $i$th component in the sum is larger than the $j$th component, the $j$th is larger than the $i$th, or they are the same, then we know enough to determine the winner. Sometimes, the components can be partitioned so that the decision only depends on comparisons within elements of the partition, which will be helpful.

### 3.1 Unweighted generalized scoring rules

Let $k \in \mathbb{N}$, and let $\mathscr{K} = \{K_1, \ldots, K_q\}$ be a partition of $K = \{1, \ldots, k\}$. That is, for any $i \leq q$, $K_i \subseteq K$, $K = \cup_{l=1}^{q} K_l$, and for any $i, j \leq q, i \neq j$, $K_i \cap K_j = \emptyset$. We say that two vectors of length $k$ are equivalent with respect to a partition if, within each element of the partition, they agree on which components are larger.

**Definition 2** *Let $\mathscr{K}$ be a partition of $K$. For any $a, b \in \mathbb{R}^k$, we say that $a$ and $b$ are* equivalent with respect to $\mathscr{K}$, *denoted by $a \sim_{\mathscr{K}} b$, if for any $l \leq q$, any $i, j \in K_l$, $a_i \geq a_j \Leftrightarrow b_i \geq b_j$ (where $a_i$ denotes the $i$th component of the vector $a$,* etc.*).*

For two partitions $\mathscr{K} = \{K_1, \ldots, K_q\}$ and $\mathscr{K}' = \{K'_1, \ldots, K'_p\}$, $\mathscr{K}'$ is a *refinement* of $\mathscr{K}$ if for any $l \leq q$, any $l' \leq p$, $K'_{l'} \cap K_l$ is either $K'_{l'}$ or $\emptyset$. That is, $\mathscr{K}'$ is obtained from $\mathscr{K}$ by partitioning the sets in $\mathscr{K}$. In this case, we say that $\mathscr{K}$ is *coarser* than $\mathscr{K}'$, and $\mathscr{K}'$ is *finer* than $\mathscr{K}$.

**Proposition 1** *For any partitions $\mathscr{K}$, $\mathscr{K}'$ such that $\mathscr{K}'$ is a refinement of $\mathscr{K}$, and any $a, b \in \mathbb{R}^k$, if $a \sim_{\mathscr{K}} b$, then $a \sim_{\mathscr{K}'} b$.*

We note that $\{K\}$ (the partition that only contains $K$ itself) is the coarsest partition.

**Definition 3** *Let $\mathscr{K}$ be a partition of $K$. A function $g : \mathbb{R}^k \to \mathcal{C}$ is* compatible with $\mathscr{K}$ *if for any $a, b \in \mathbb{R}^k$, $a \sim_{\mathscr{K}} b \Rightarrow g(a) = g(b)$.*

That is, for any mapping $g$ that is compatible with $\mathscr{K}$, $g(a)$ is determined (only) by comparisons within each $K_l$, $l \leq q$. Namely, we do not need to compare components across different elements of the partition.

Now we are ready to define generalized scoring rules.

**Definition 4** *Let $k \in \mathbb{N}$, $f : L(\mathcal{C}) \to \mathbb{R}^k$ and $g : \mathbb{R}^k \to \mathcal{C}$, where $g$ is compatible with a partition $\mathscr{K}$ of $K$. $f$ and $g$ determine the* (unweighted) generalized scoring rule $GS(f, g)$ *as follows. For any profile of votes $V_1, \ldots, V_n \in L(\mathcal{C})$, $GS(f, g)(V_1, \ldots, V_n) = g(\sum_{i=1}^{n} f(V_i))$. We say that $GS(f, g)$ is* of order $k$, *and* compatible with $\mathscr{K}$.

The weighted version of generalized scoring rules is defined in Appendix 2. Below, unless otherwise specified, generalized scoring rules refer to unweighted generalized scoring rules. From Proposition 1 we know that for any partitions $\mathscr{K}$, $\mathscr{K}'$ such that $\mathscr{K}'$ is a refinement of $\mathscr{K}$, $GS(f, g)$ is compatible with $\mathscr{K}'$, then $GS(f, g)$ is also compatible with $\mathscr{K}$. Given a profile $P$ of votes, we use $f(P)$ as shorthand for $\sum_{V \in P} f(V)$. We will call $f(P)$ the *total generalized score vector*. By definition, any unweighted generalized scoring rule satisfies *anonymity* (that is, every voter is treated equally) and *homogeneity* (that is, if we add any number of copies of the profile to the profile, the winner does not change). Any generalized scoring rule is compatible with the partition $\{K\}$. Nevertheless, being compatible with $\{K\}$ is not vacuous: if we modified the definition so that $g$ is not required to be compatible with any partition, then any anonymous voting rule would belong to the resulting class of rules. If a generalized scoring rule is compatible with a partition, this effectively means that, within each element of the partition, the scores are of the same "type," so that we can compare them.

We now illustrate how general the class of generalized scoring rules is by showing how some standard rules belong to the class. Many other rules can also be shown to belong to the class.

**Proposition 2** *All positional scoring rules, Copeland, STV, maximin, and ranked pairs are generalized scoring rules.*

**Proof of Proposition 2:** We explicitly give $k, f, g, \mathscr{K}$ for each of these rules. In the remainder of the proof, the number of alternatives is fixed to be $m$. Let $V \in L(\mathcal{C})$ be a vote, and let $P$ be a profile of votes. Because it is ambiguous how ties should be broken for the rules in the proposition, we will also not specify how ties are broken when we describe these rules as generalized scoring rules.

• **Positional scoring rules:** Suppose the scoring vector for the rule is $\vec{v} = (v(1), \ldots, v(m))$. The total generalized score vector will simply consist of the total scores of the individual alternatives. Let
– $k_{\vec{v}} = m$.
– $f_{\vec{v}}(V) = (s(V, c_1), \ldots, s(V, c_m))$.
– $g_{\vec{v}}(f_{\vec{v}}(P)) = \arg\max_i (f_{\vec{v}}(P))_i$.
– $\mathscr{K}_{\vec{v}} = \{K\}$.

• **Copeland:** For Copeland, the total generalized score vector will consist of the scores in the pairwise elections. Let
– $k_{Copeland} = m(m-1)$; the components are indexed by pairs $(i, j)$ such that $i, j \leq m, i \neq j$.
– $(f_{Copeland}(V))_{(i,j)} = \begin{cases} 1 & \text{if } c_i \succ_V c_j \\ 0 & \text{otherwise} \end{cases}$
– $g_{Copeland}$ selects the winner based on $f_{Copeland}(P)$ as follows. For each pair $i \neq j$, if $(f_{Copeland}(P))_{(i,j)} > (f_{Copeland}(P))_{(j,i)}$, then add 1 point to $i$'s Copeland score; if $(f_{Copeland}(P))_{(j,i)} > (f_{Copeland}(P))_{(i,j)}$, then add 1 point to $j$'s Copeland score; if tied, then add 0.5 to both $i$'s and $j$'s Copeland scores. The winner is the alternative that gets the highest Copeland score.
– $q_{Copeland} = \frac{m(m-1)}{2}$ (we recall that $q$ is the number of elements in the partition). The elements of the partition are indexed by $(i, j)$, $i < j$. For any $l = (i, j)$, $i < j$, let $K_l = \{(i, j), (j, i)\}$. Let $\mathscr{K}_{Copeland} = \{K_l : l = (i, j), i < j\}$.

• **STV:** For STV, we will use a total generalized score vector with many components. For every proper subset $S$ of alternatives, for every alternative $c$ outside of $S$, there is a component in the vector that contains the number of times that $c$ is ranked first if all of the alternatives in $S$ are removed. Let
– $k_{STV} = \sum_{i=0}^{m-1} \binom{m}{i}(m-i)$; the components are indexed by $(S, j)$, where $S$ is a proper subset of $\mathcal{C}$ and $j \leq m, c_j \notin S$.
– $(f_{STV}(V))_{(S,j)} = 1$, if after removing $S$ from $V$, $c_j$ is at the top; otherwise, let $(f_{STV}(V))_{(S,j)} = 0$.
– $g_{STV}$ selects the winner based on $f_{STV}(P)$ as follows. In the first round, find $j_1 = \arg\min_j ((f_{STV}(P))_{(\emptyset,j)})$. Let $S_1 = \{c_{j_1}\}$. Then, for any $2 \leq i \leq m-1$, define $S_i$ recursively as follows: $S_i = S_{i-1} \cup \{j_i\}$, where $j_i = \arg\min_j (f_{STV}(P)_{(S_{i-1},j)})$; finally, the winner is the unique alternative in $\mathcal{C} - S_{m-1}$.
– $q_{STV} = 2^m - 1$. The elements of the partition are indexed by the $S \subset \mathcal{C}$. For any $S \subset \mathcal{C}$, let $K_S = \{(S, j) : c_j \notin S\}$. Let $\mathscr{K}_{STV} = \{K_S : S \subset \mathcal{C}\}$.

• **Maximin:** For maximin, we use the same total generalized score vector as for Copeland, that is, the vector of all scores in pairwise elections. Let

– $k_{maximin} = m(m-1)$; the components are indexed by pairs $(i, j)$ such that $i, j \leq m, i \neq j$.

– $(f_{maximin}(V))_{(i,j)} = \begin{cases} 1 & \text{if } c_i \succ_V c_j \\ 0 & \text{otherwise} \end{cases}$

– $g_{maximin}(f_{maximin}(P))$ is the $c_i$ such that for any $i' \leq m$, $i' \neq i$, there exists $j' < m, j' \neq i'$ such that for any $j \leq m, j \neq i$, we have $f_{maximin}(P)_{(i,j)} > (f_{maximin}(P))_{(i',j')}$.

– $\mathscr{K}_{maximin} = \{K\}$.

• **Ranked pairs:** We use the same total generalized score vector as for Copeland and maximin, that is, the vector of all scores in pairwise elections. Let

– $k_{rp} = m(m-1)$; the components are indexed by pairs $(i, j)$ such that $i, j \leq m, i \neq j$.

– $(f_{rp}(V))_{(i,j)} = \begin{cases} 1 & \text{if } c_i \succ_V c_j \\ 0 & \text{otherwise} \end{cases}$

– $g_{rp}$ selects the winner based on $f_{rp}(P)$ as follows. In each step, we consider a pair of alternatives $c_i, c_j$ that we have not previously considered; specifically, we choose the remaining pair with the highest $(f_{rp}(P))_{(i,j)}$. We then fix the order $c_i > c_j$, unless this contradicts previous orders that we fixed (that is, it violates transitivity). We continue until we have considered all pairs of alternatives. The alternative at the top of the ranking wins.

– $\mathscr{K}_{rp} = \{K\}$. □

We showed that STV, also known as instant run-off voting, is a generalized scoring rule. In Appendix 1, we generalize this and show that *any* multiround run-off process where in each round, alternatives are eliminated according to a generalized scoring rule (to be precise, a correspondence) must itself be a generalized scoring rule. (For STV, a version of plurality that just eliminates one alternative is used in every round.)

We stress that the class of generalized scoring rules is not equal to the class of anonymous voting rules. To see this, we recall that any generalized scoring rule satisfies homogeneity. The next example shows an anonymous voting rule that does not satisfy homogeneity.

**Example 1** *Let $r$ be the voting rule that selects an alternative $c$ if the number of times that $c$ is ranked at the top is higher than that of any other alternative by at least 2; if no such alternative exits, then the first (default) alternative $c_1$ is selected.*

*$r$ is anonymous. We note that $r(c_2 \succ c_1) = c_1$ and $r(2(c_2 \succ c_1)) = c_2$. Hence, $r$ does not satisfy homogeneity.*

# 4. CONDITIONS UNDER WHICH COALITIONAL MANIPULABILITY IS RARE

Let $\pi$ be a probability distribution over $L(\mathcal{C})$ that is positive everywhere. Let $\phi_{\pi,n}$ be the distribution over profiles of $n$ voters in which each vote is drawn i.i.d. according to $\pi$. Given a manipulation instance $(r, P^{NM}, |M|)$, if there is only one possible winner, then we say that this manipulation instance is *closed*; otherwise we say this manipulation instance is *open* [16].

**Definition 5** *A manipulation instance $(r, P^{NM}, |M|)$ is* closed *if for any profiles $P_1^M, P_2^M$ for the manipulators, $r(P^{NM} \cup P_1^M) = r(P^{NM} \cup P_2^M)$. An instance is* open *if it is not closed.*

Procaccia and Rosenschein [16] have shown that if
**1.** the rule is a positional scoring rule,
**2.** the number of manipulators $|M|$ is $o(\sqrt{n})$,
**3.** the votes are drawn independently, and
**4.** there exists $d > 0$ such that for each vote's distribution, the variance of the difference in scores for any pair of alternatives is at least $d$,

then when $n \to \infty$, the probability that a weighted manipulation instance is open is 0. In this section, we generalize this result to generalized scoring rules; in addition, we characterize the rate of convergence to 0. (However, unlike Procaccia and Rosenschein, we do assume that votes are drawn i.i.d.; this is needed to obtain the convergence rate. Hence, strictly speaking, our result is not a generalization of their result. We can also obtain a strict generalization of Procaccia and Rosenschein's results to generalized scoring rules, but without proving a convergence rate; we will not do so in this paper.)

Specifically, in this section, we study the probability that a manipulation instance is open when there are $O(n^p)$ ($0 \leq p < \frac{1}{2}$) manipulators, and the nonmanipulator votes are drawn i.i.d. Here, $n$ is the total number of voters, $|NM| + |M|$ (nonmanipulators and manipulators). We will prove that for any generalized scoring rule, this probability is $O(\frac{1}{\sqrt{n}})$. Let $T(r, m, n, \pi, |M|)$ denote this probability. That is,

$$T(r, m, n, \pi, |M|) = Pr_{P^{NM} \sim \phi_{\pi,|NM|}}\{(r, P^{NM}, |M|) \text{ is open}\}$$

**Lemma 1** *Let $N \in \mathbb{N}$. Let $Y_1, \ldots, Y_N$ be i.i.d. random variables with $E(Y_1) < \infty$, $E((Y_1 - E(Y_1))^2) > 0$, and $E(|Y_1 - E(Y_1)|^3) < \infty$. Let $Y = \sum_{\zeta=1}^N Y_\zeta$. For any constant $0 \leq p < \frac{1}{2}$ that does not depend on $N$, and any function $f(N)$ that is $\Omega(1)$, we have that $Pr(|Y| \leq f(N))$ is $O(\frac{f(N)}{\sqrt{N}})$.*

**Proof of Lemma 1:** Let $\Phi(x)$ be the cumulative distribution function of the standard normal distribution $N(0, 1)$. Let $\sigma^2 = E((Y_1 - E(Y_1))^2)$, $\rho = E(|Y_1 - E(Y_1)|^3)$. Then we have:

$$\begin{aligned}
&Pr(|Y| < f(N)) \\
=&Pr(-\frac{E(Y_1)N}{\sigma\sqrt{N}} - \frac{f(N)}{\sigma\sqrt{N}} < \frac{Y - E(Y_1)N}{\sigma\sqrt{N}} < -\frac{E(Y_1)N}{\sigma\sqrt{N}} + \frac{f(N)}{\sigma\sqrt{N}})
\end{aligned}$$

Then by the Berry-Esséen theorem [10],

$$\begin{aligned}
&Pr(|Y| < f(N)) \\
<&\Phi(-\frac{E(Y_1)N}{\sigma\sqrt{N}} + \frac{f(N)}{\sigma\sqrt{N}}) - \Phi(-\frac{E(Y_1)N}{\sigma\sqrt{N}} - \frac{f(N)}{\sigma\sqrt{N}}) + \frac{C\rho}{\sigma^3\sqrt{N}} \\
=&\int_{-\frac{E(Y_1)N}{\sigma\sqrt{N}} - \frac{f(N)}{\sigma\sqrt{N}}}^{-\frac{E(Y_1)N}{\sigma\sqrt{N}} + \frac{f(N)}{\sigma\sqrt{N}}} N(0,1)(x)dx + \frac{C\rho}{\sigma^3\sqrt{N}} \\
<&\frac{2f(N)}{\sigma\sqrt{N}} \times \frac{1}{\sqrt{2\pi}} + \frac{C\rho}{\sigma^3\sqrt{N}}
\end{aligned}$$

which is $O(\frac{f(N)}{\sqrt{N}})$, because $C$ is a constant that does not depend on $N$ and $f(N) = \Omega(1)$. □

**Theorem 1** *Let $r = GS(f, g)$ be a generalized scoring rule of order $k$. For any $m \in \mathbb{N}$, any constant $0 \leq p < \frac{1}{2}$, and any constant $h$ (where both $m$ and $h$ do not depend on $n$), there exists a constant $t_{m,p,h} > 0$ (that does not depend on $n$) such that if $|M| \leq hn^p$, then*

$$T(r, m, n, \pi, |M|) \leq t_{m,p,h} n^{p-\frac{1}{2}}$$

**Proof of Theorem 1:** We recall that each vote is drawn i.i.d. according to the probabilistic distribution $\pi$. For any pair $i_1, i_2 \leq k$, $i_1 \neq j_2$, and any $t > 0$, let

$$R(i_1, i_2, t, \pi, |NM|) = Pr\{|(f(P^{NM}))_{i_1} - (f(P^{NM}))_{i_2}| \leq t\}$$

We recall that $(f(P^{NM}))_i$ is the $i$th component of $f(P^{NM})$. In other words, $R(i_1, i_2, t, \pi, |NM|)$ is the probability of profiles of nonmanipulators' votes $P^{NM}$ such that the difference between the $i_1$th component and the $i_2$th component of $f(P^{NM})$ is no more than $t$, when each vote is drawn i.i.d. according to $\pi$. Let $Y_1^{i_1,i_2}, \ldots, Y_{|NM|}^{i_1,i_2}$ be $|NM|$ i.i.d. random variables, where the distribution for each $Y_\zeta^{i_1,i_2}$ is the same as the distribution for $(f(V))_{i_1} - (f(V))_{i_2}$, where $V$ is drawn according to $\pi$. That is, for any $V \in L(\mathcal{C})$, with probability $\pi(V)$, $Y_1^{i_1,i_2}$ takes value $(f(V))_{i_1} - (f(V))_{i_2}$. Let $Y^{i_1,i_2} = \sum_{\zeta=1}^{|NM|} Y_\zeta$.

Let $v_{max} = \max_{i \le k, V \in L(\mathcal{C})} (f(V))_i$. That is, $v_{max}$ is the maximum component of all score vectors corresponding to a single vote. We note that $v_{max}$ is a constant that does not depend on $n$. We also note that since $|M|$ is $O(n^p)$ and $p < \frac{1}{2}$, it must be that $|NM|$ is $\Omega(n)$, so that $n$ is $O(|NM|)$, $v_{max}hn^p$ is $O(|NM|^p)$. Therefore, by Lemma 1 (in which we let $N = |NM|$), we know that $Pr(|Y^{i_1,i_2}| \le v_{max}hn^p)$ is $O(\frac{v_{max}hn^p}{\sqrt{|NM|}}) = O(|NM|^{p-\frac{1}{2}})$, so it is $O(n^{p-\frac{1}{2}})$. Hence, there exists a constant $t_{i_1,i_2}$ such that

$$Pr(|Y^{i_1,i_2}| \le v_{max}hn^p) < t_{i_1,i_2}n^{p-\frac{1}{2}}$$

We let $t_{max} = \max_{i,j \le k, i \ne j} t_{i,j}$. If a manipulation instance is open, then there exists a profile $P^M$ for the manipulators such that $GS(f,g)(P^M \cup P^{NM}) \ne GS(f,g)(P^{NM})$, which means that $f(P^M \cup P^{NM}) \not\sim f(P^{NM})$. In this case there must exist $i, j$, $i \ne j$, such that $|(f(P^{NM}))_i - (f(P^{NM}))_j| \le v_{max}|M| \le v_{max}hn^p$. Therefore, $T(GS(f,g), m, n, \pi, |M|) \le \sum_{1 \le i < j \le m} R(i, j, v_{max}hn^p, \pi, |NM|)$.

We note that $R(i, j, v_{max}hn^p, \pi, |NM|) = Pr(|Y^{i,j}| \le v_{max}hn^p)$. Therefore, we have

$$T(GS(f,g), m, n, \pi, |M|) \le \sum_{i \ne j} R(i, j, v_{max}hn^p, \pi, |NM|)$$

$$\le \sum_{i \ne j} t_{i,j}n^{p-\frac{1}{2}} \le \frac{k(k-1)}{2} t_{max}n^{p-\frac{1}{2}}$$

Let $t_{m,p,h} = \frac{k(k-1)}{2} t_{max}$. We know that $t_{m,p,h}$ is a constant that does not depend on $n$.

(End of the proof of Theorem 1.) $\square$

From Proposition 2 and Theorem 1, we obtain the following corollary.

**Corollary 1** *Let $r$ be any positional scoring rule, Copeland, STV, maximin, or ranked pairs. For any $m \in \mathbb{N}$, any constant $0 \le p < \frac{1}{2}$, and any constant $h$ (where $m$, $p$, and $h$ do not depend on $n$), there exists a constant $t_{m,p,h} > 0$ (that does not depend on $n$) such that if $|M| \le hn^p$, then*

$$T(r, m, n, \pi, |M|) \le t_{m,p,h}n^{p-\frac{1}{2}}$$

A profile is said to be *tied* if a single additional voter can change the outcome. By letting $p = 0$ and $h = 1$ in Theorem 1, we have that for any generalized scoring rule and any fixed $m$, the number of tied profiles is $O(\frac{1}{\sqrt{n}})$.

We note that Theorem 1 does not apply to *all* anonymous voting rules. For example, let us consider the voting rule $r$ that selects the first candidate, $c_1$, if the number of times it is ranked at the top is even; otherwise, the rule selects the second candidate, $c_2$. For this rule, even when there is only one manipulator, any profile $P^{NM}$ for the non-manipulators is open, because the manipulator can always determine whether the number of times that $c_1$ is ranked at the top in the complete profile (that is, the profile that includes both the non-manipulators and the manipulator) is odd or even, by casting a vote that either ranks $c_1$ at the top or not.

# 5. CONDITIONS UNDER WHICH COALITIONS OF MANIPULATORS ARE ALL-POWERFUL

Let us consider a positional scoring rule and a distribution over nonmanipulator votes. Furthermore, let us consider each alternative's expected score; let $C_{max}$ be the set of alternatives with the highest expected score. Procaccia and Rosenschein [16] have shown that if

1. the number of manipulators is in both $\omega(\sqrt{n})$ and $o(n)$, and
2. votes are drawn i.i.d.,

then, the probability that the manipulators can make any alternative in $C_{max}$ win converges to 1 as $n \to \infty$. Hence, assuming $|C_{max}| > 1$, the probability that the instance is open converges to 1 (however, if $|C_{max}| = 1$, it converges to 0).

In this section, we prove a similar result for generalized scoring rules; in addition, we characterize the rate of convergence to 0. (In fact, in this case, Procaccia and Rosenschein also characterize this rate—for positional scoring rules.)

Specifically, in this section, we study the case where the number of manipulators is $\Omega(n^p)$ ($\frac{1}{2} < p < 1$) and $o(n)$, the votes are drawn i.i.d. according to $\pi$, and a generalized scoring rule is used. We provide a sufficient condition under which the manipulators can make any alternative in a particular set of alternatives win with probability $1 - O(e^{-\Omega(n^{2p-1})})$. (We need the $o(n)$ assumption for a technical reason, as do Procaccia and Rosenschein.)

**Definition 6** *$\pi$ is* compatible with *$\mathscr{K}$ w.r.t. $f$, if, for $V \sim \pi$, for any $l \le q$, any $i, j \in K_l$ ($i \ne j$), $E((f(V))_i) = E((f(V))_j)$.*

That is, $\pi$ is compatible with $\mathscr{K}$ w.r.t. $f$ if within each element of the partition $\mathscr{K}$, the expectation of the components of $f(V)$ are the same (where $V$ is drawn according to $\pi$).

Given $GS(f,g)$, it will be useful to have a profile $P$ such that for some partition $\mathscr{K}$ that $GS(f,g)$ is compatible with, the components of $f(P)$ within each $K_l$ ($l \le q$) are all different. The next definition makes this precise.

**Definition 7** *For any $GS(f,g)$ compatible with $\mathscr{K}$, a profile $P$ is said to be* distinctive *w.r.t. $GS(f,g)$ and $\mathscr{K}$ if for any $l \le q$, any $i, j \in K_l$, $i \ne j$, $(f(P))_i \ne (f(P))_j$.*

The next definition concerns the set of alternatives that can be made to win using a distinctive profile.

**Definition 8** *For any $GS(f,g)$ compatible with $\mathscr{K}$, let $W_{\mathscr{K}}(f,g)$ be a subset of the alternatives defined as follows.*

$W_{\mathscr{K}}(f,g) = \{GS(f,g)(P) : P$ *is distinctive w.r.t $GS(f,g)$ and $\mathscr{K}$*$\}$

For any profile $P^M$ of manipulators and any alternative $c$, we define $T(m, n, \pi, c, P^M) = Pr(GS(f,g)(P^M \cup P^{NM}) = c)$. That is, given a profile of votes $P^M$ of the manipulators, $T(m, n, \pi, c, P^M)$ is the probability that the winner of the profile $P^M \cup P^{NM}$ is $c$, when the number of alternatives is $m$, the number of voters is $n$, and the nonmanipulators' votes $P^{NM}$ are drawn i.i.d. according to $\pi$. Now we are ready to present the theorem.

**Theorem 2** *Let $GS(f,g)$ be a generalized scoring rule that is compatible with $\mathscr{K}$. Let $\pi_{\mathscr{K}}$ be a distribution over $L(\mathcal{C})$ such that $\pi_{\mathscr{K}}$ is compatible with $\mathscr{K}$ w.r.t. $f$. For any $m > 0$, there exist constants $t_m > 0$ and $u_m > 0$ (neither of which depend on $n$) such that for any constant $h > 0$ (that does not depend on $n$) and any alternative $c \in W_{\mathscr{K}}(f,g)$, if the number of manipulators is at least $hn^p$ ($\frac{1}{2} < p < 1$) (as well as $o(n)$), then there exists a coalitional* *manipulation $P^M$ such that*

$$T(m,n,\pi_{\mathscr{K}},c,P^M) > 1 - t_m e^{-u_m n^{2p-1}}$$

Theorem 2 states that when the number of alternatives is held fixed, if the number of manipulators is large ($\Omega(n^p)$ for $p > \frac{1}{2}$, as well as $o(n)$) then for any alternative $c \in W_{\mathscr{K}}(f,g)$, there exists a manipulation $P^M$ such that when the nonmanipulators' votes are drawn i.i.d. according to $\pi_{\mathscr{K}}$, then $c$ is the winner with a probability of $1 - O(e^{-\Omega(n^{2p-1})})$.

**Proof of Theorem 2:** Let $|M| \geq hn^p$. If $W_{\mathscr{K}}(f,g) = \emptyset$, then Theorem 2 vacuously holds. So we assume that $W_{\mathscr{K}}(f,g) \neq \emptyset$. For each $c \in W_{\mathscr{K}}(f,g)$, we associate $c$ with a distinctive profile (w.r.t. $f$ and $\mathcal{X}$), denoted by $P_c^*$, such that $c = GS(f,g)(P_c^*)$. We recall that $P_c^*$ is distinctive if and only if for any $l \leq q$, $i,j \in K_l$, $i \neq j$, $(f(P_c^*))_i \neq (f(P_c^*))_j$. Let

$$d_{min} = \min_{l \leq q, i,j \in K_l, i \neq j, c \in W_{\mathscr{K}}(f,g)} (|(f(P_c^*))_i - (f(P_c^*))_j|).$$

That is, $d_{min}$ is the minimal difference between any two components within the same element of $\mathscr{K}$ of $f(P_c^*)$, taken over all $c \in W_{\mathscr{K}}(f,g)$. Since $|W_{\mathscr{K}}(f,g)| < m$ (which does not depend on $n$), and $P_c^*$ is distinctive, we know that $d_{min} > 0$ and does not depend on $n$. Let $p_{max} = \max_{c \in C} |P_c^*|$. That is, for all $c \in W_{\mathscr{K}}(f,g)$, the number of votes in $P_c^*$ is no more than $p_{max}$. We note that $p_{max}$ does not depend on $n$.

For any $c \in C$, define a profile of the manipulator votes $P_c^M$ as follows. $P_c^M$ consists of two parts:

1. $\lfloor \frac{|M|}{|P_c^*|} \rfloor P_c^*$, and
2. an arbitrary profile for the remaining $|M| - \lfloor \frac{|M|}{|P_c^*|} \rfloor |P_c^*|$ votes.

That is, $P_c^M$ consists mostly of $\lfloor \frac{|M|}{|P_c^*|} \rfloor$ copies of $P_c^*$; the remaining votes (at most $|P_c^*|$) are chosen arbitrarily. We note that $|P_c^*|$ is a constant that does not depend on $n$, so that the second part becomes negligible when $n \to \infty$.

The next claim provides a lower bound on the difference between any two components of $f(P_c^M)$.

**Claim 1** *There exists a constant $d_c$ that does not depend on $n$ such that the minimum difference between components of $f(P_c^M)$ is at least $d_c n^p$, when $n \to \infty$.*

**Proof of Claim 1:** Since the minimal difference between any two components of $P_c^*$ is at least $d_{min}$, the minimal difference between any two components of $f(P_c^M)$ is at least $\lfloor \frac{|M|}{|P_c^*|} \rfloor d_{min}$. We note that the number of arbitrarily assigned votes in $P_c^M$ is no more than $|P_c^*|$, and the difference between any two components in a vote is no more than $v_{max}$. Therefore the minimal difference between any two components of $f(P^M)$ is at least

$$\lfloor \frac{|M|}{|P_c^*|} \rfloor d_{min} - v_{max}|P_c^*| \geq (\frac{|M|}{p_{max}} - 1) d_{min} - v_{max} p_{max},$$

which is $\Omega(n^p)$ because $p_{max}$, $d_{min}$, and $v_{max}$ are constants that do not depend on $n$, and $|M|$ is $\Omega(n^p)$. Therefore, there exists a $d_c$ that does not depend on $n$ such that the minimal difference between any two components of $f(P_c^M)$ is at least $d_c n^p$, when $n \to \infty$.

(End of the proof of Claim 1.) $\square$

The next lemma is known as *Chernoff's inequality* [5].

**Lemma 2 (Chernoff's inequality)** *Let $N \in \mathbb{N}$. Let $Y_1, \ldots, Y_N$ be $N$ i.i.d. random variables with variance $\sigma^2$. Let $Y = \sum_{\zeta=1}^N Y_\zeta$. For any $0 \leq k \leq 2\sqrt{N}\sigma$, $Pr(|Y - E(Y)| \geq k\sqrt{N}\sigma) \leq 2e^{-k^2/4}$.*

For any profile $P^{NM}$ for the nonmanipulators, any $i_1, i_2 \leq k$, $i_1 \neq i_2$, let $D(P^{NM}, i_1, i_2) = |(f(P^{NM}))_{i_1} - (f(P^{NM}))_{i_2}|$.

The next claim states that if each vote of $P^{NM}$ is drawn i.i.d. according to $\pi_{\mathscr{K}}$, then for any different $i_1, i_2$ within the same element $K_l$ of the partition $\mathscr{K}$, the probability that the difference between the $i_1$th and the $i_2$th component of $f(P^{NM})$ is larger than $d_c n^p$ is $O(e^{-\Omega(n^{2p-1})})$.

**Claim 2** *For any $l \leq q$ and any $i_1, i_2 \in K_l$ ($i_1 \neq i_2$), there exists a constant $d_{c,i_1,i_2} > 0$ that does not depend on $n$ such that*

$$Pr(D(P^{NM}, i_1, i_2) > d_c n^p) \leq 2e^{-d_{c,i_1,i_2} n^{2p-1}}$$

**Proof of Claim 2:** Let $Y_1^{i_1,i_2}, \ldots, Y_{|NM|}^{i_1,i_2}$ be $|NM|$ i.i.d. random variables such that the distribution for each $Y_\zeta^{i_1,i_2}$ is the same as the distribution for $(f(V))_{i_1} - (f(V))_{i_2}$, where $V$ is drawn according to $\pi$. That is, for any $V \in L(\mathcal{C})$, with probability $\pi(V)$, $Y_1^{i_1,i_2}$ takes value $(f(V))_{i_1} - (f(V))_{i_2}$. Let $Y^{i_1,i_2} = \sum_{\zeta=1}^{|NM|} Y_\zeta^{i_1,i_2}$. Then, $Pr(D(P^{NM}, i_1, i_2) > d_c n^p) = Pr(Y^{i_1,i_2} > d_c n^p)$.

Since $\pi_{\mathscr{K}}$ is compatible with $\mathscr{K}$, for any $l \leq q$, $i_1, i_2 \in K_l$, we know that $E((f(V))_{i_1}) = E((f(V))_{i_2})$, where $V$ is drawn according to $\pi$. Therefore, $E(Y_1^{i_1,i_2}) = 0$. Let $\sigma_{i_1,i_2}^2$ be the variance of $Y_1^{i_1,i_2}$. We note that $\sigma_{i_1,i_2}$ does not depend on $n$. If $\sigma_{i_1,i_2}^2 = 0$, then for any $V \in L(\mathcal{C})$, $(f(V))_{i_1} = (f(V))_{i_2}$ (because for any $V \in L(\mathcal{C})$, $\pi_{\mathscr{K}}(V) > 0$), which means that $W_{\mathscr{K}}(f,g) = \emptyset$. This contradicts the assumption that $W_{\mathscr{K}}(f,g) \neq \emptyset$. Hence $\sigma_{i_1,i_2}^2 > 0$. Since $|M| = o(n)$, $|NM| = \Omega(n)$, and for sufficiently large $n$ we have $\frac{d_c n^p}{\sigma_{i_1,i_2}\sqrt{|NM|}} \leq 2\sigma_{i_1,i_2}\sqrt{|NM|}$. Therefore, we can use Lemma 2 (in which we let $N = |NM|$) to bound $Pr(D(P^{NM}, i_1, i_2) > d_c n^p)$ above as follows.

$$\begin{aligned}
&Pr(D(P^{NM}, i_1, i_2) > d_c n^p) \\
=&Pr(|Y^{i_1,i_2}| > d_c n^p) \\
=&Pr(|Y^{i_1,i_2}| > \frac{d_c n^p}{\sigma_{i_1,i_2}\sqrt{|NM|}} \times \sigma_{i_1,i_2}\sqrt{|NM|}) \\
\leq&2e^{-(\frac{d_c n^p}{\sigma_{i_1,i_2}\sqrt{|NM|}})^2/4} && \text{Lemma 2} \\
\leq&2e^{-\frac{d_c^2}{4\sigma_{i_1,i_2}^2} n^{2p-1}} && |NM| \leq n
\end{aligned}$$

We note that $\frac{d_c^2}{4\sigma_{i_1,i_2}^2}$ is a constant that does not depend on $n$. Therefore, there exists $u_{c,i_1,i_2} > 0$ such that $Pr(D(P^{NM}, i_1, i_2) > d_c n^p) \leq 2e^{-u_{c,i_1,i_2} n^{2p-1}}$.

(End of the proof of Claim 2.) $\square$

Let $u_c = \min_{l \leq q, i,j \in K_l, i \neq j} u_{c,i,j}$. Then $u_c > 0$ and is a constant (that does not depend on $n$). We note that for any $P^{NM}$, if $f(P^{NM} \cup P_c^M) \not\sim_{\mathscr{K}} f(P_c^M)$, then there exists $l \leq q$, $i,j \in K_l$, $i \neq j$, such that $|(f(P^{NM}))_i - (f(P^{NM}))_j| > |(f(P_c^M))_i - (f(P_c^M))_j| > d_c n^p$. Therefore, we can bound the probability of $f(P^{NM} \cup P_c^M) \sim_{\mathscr{K}} f(P_c^M)$ below as follows.

$$\begin{aligned}
&Pr(f(P^{NM} \cup P_c^M) \sim_{\mathscr{K}} f(P_c^M)) \\
=&1 - Pr(f(P^{NM} \cup P_c^M) \not\sim_{\mathscr{K}} f(P_c^M))
\end{aligned}$$

$$\geq 1 - Pr((\exists l \leq q)(\exists i,j \in K_l) D(P^{NM}, i, j) > d_c n^p)$$
$$\geq 1 - \sum_{l \leq q} \sum_{i,j \in K_l, i \neq j} Pr(D(P^{NM}, i, j) > d_c n^p)$$
$$\geq 1 - \sum_{l \leq q} \sum_{i,j \in K_l, i \neq j} 2e^{-u_{c,i,j} n^{2p-1}}$$
$$\geq 1 - \sum_{l \leq q} \sum_{i,j \in K_l, i \neq j} 2e^{-u_c n^{2p-1}} \geq 1 - \frac{m(m-1)}{2} \times 2e^{-u_c n^{2p-1}}$$

When $n$ is sufficiently large, $f(P_c^M) \sim_{\mathscr{K}} f(P_c^*)$. Therefore, we know that there exists a constant $t_c > 0$ (that does not depend on $n$) such that $Pr(f(P^{NM} \cup P_c^M) \sim_{\mathscr{K}} f(P_c^*)) \geq 1 - t_c e^{-u_c n^{2p-1}}$. Hence

$$T(m, n, \pi_{\mathscr{K}}, c, P^M)$$
$$\geq Pr(f(P^{NM} \cup P_c^M) \sim_{\mathscr{K}} f(P_c^*)) \geq 1 - t_c e^{-u_c n^{2p-1}}$$

(End of the proof of Theorem 2.) □

# 6. ALL-POWERFUL MANIPULATORS IN COMMON RULES

We already showed how Theorem 1, which states a condition under which manipulability is rare, can be applied to common voting rules in Corollary 1. We have not yet done so for Theorem 2; we will do so in this section. Specifically, we prove that if the number of alternatives is fixed, then for any positional scoring rule, Copeland, STV, ranked pairs, and maximin, if the number of manipulators is $\Omega(n^p)$ ($p > \frac{1}{2}$) and $o(n)$, and the nonmanipulators' votes are drawn i.i.d. according to the uniform distribution, then for any alternative $c$, there exists a coalitional manipulation that will make $c$ win with a probability of $1 - O(e^{-\Omega(n^{2p-1})})$.

The next theorem provides a necessary and sufficient condition for $W_{\mathscr{K}}(f,g)$ to be nonempty.

**Theorem 3** *Let $G(f,g)$ be compatible with $\mathscr{K}$. $W_{\mathscr{K}}(f,g) \neq \emptyset$ if and only if for any $l \leq q$, any $i,j \in K_l$, $i \neq j$, there exists a vote $V \in L(\mathcal{C})$ such that $(f(V))_i \neq (f(V))_j$.*

**Proof of Theorem 3:** First we prove the "if" part. Suppose that for any $l \leq q$, any $i,j \in K_l$, $i \neq j$, there exists a vote $V \in L(\mathcal{C})$ such that $(f(V))_i \neq (f(V))_j$. For any $l \leq q$, let $h_{l,max} = \max_{i,j \in K_l, V \in L(\mathcal{C})} \{|(f(V))_i - (f(V))_j|\}$, $h_{l,min} = \min_{i,j \in K_l, V \in L(\mathcal{C})} \{|(f(V))_i - (f(V))_j| : |(f(V))_i - (f(V))_j| > 0\}$. That is, $h_{l,max}$ is the maximum difference between any two components within $K_l$, for any $f(V)$; $h_{l,min}$ is the minimum *positive* difference between any two components within $K_l$, for any $f(V)$. Then, for any $l \leq q$, $h_{l,max} \geq h_{l,min} > 0$. Let $h$ be a natural number such that for any $l \leq q$, $h > \frac{h_{l,max}}{h_{l,min}} + 1$. Suppose $L(\mathcal{X}) = \{L_1, \ldots, L_{m!}\}$. Then, let $P = \sum_{s=1}^{m!} h^{m!-s} L_s$. We now show that $P$ is distinctive w.r.t. $GS(f,g)$ and $\mathscr{K}$.

For any $l \leq q$, any $i,j \in K_l$, let $t$ be the minimum natural number such that $(f(L_t))_i \neq (f(L_t))_j$. W.l.o.g. let $(f(L_t))_i > (f(L_t))_j$. Then

$$(f(P))_i - (f(P))_j = \sum_{s=1}^{m!} h^{m!-s}((f(L_s))_i - (f(L_s))_j)$$
$$= h^{m!-t}((f(L_t))_i - (f(L_t))_j) + \sum_{s=t+1}^{m!} h^{-s}((f(L_s))_i - (f(L_s))_j)$$
$$\geq h^{m!-t} h_{l,min} - \sum_{s=t+1}^{m!} h^{m!-s} h_{l,max}$$
$$= h^{m!-t}\left(h_{l,min} - \frac{1}{h} \frac{1 - \frac{1}{h^{m!-t}}}{1 - \frac{1}{h}} h_{l,max}\right)$$
$$> h^{m!-t}\left(h_{l,min} - \frac{1}{h-1} h_{l,max}\right) > 0$$

The last inequality holds because $h > \frac{h_{l,max}}{h_{l,min}} + 1$. Therefore, we know that for any $l \leq q$, any $i,j \in K_l$, $i \neq j$, $(f(P))_i \neq (f(P))_j$. Hence, $P$ is distinctive w.r.t. $GS(f,g)$ and $\mathscr{K}$, completing the proof of the "if" part.

Now we prove the "only if part. Suppose there exist $l \leq q$, $i,j \in K_l$ such that for any $V \in L(\mathcal{C})$, $(f(V))_i = (f(V))_j$. Then, for any profile $P$, $(f(P))_i = (f(P))_j$, which means that $P$ is not distinctive w.r.t. $GS(f,g)$ and $\mathscr{K}$. Therefore $W_{\mathscr{K}}(f,g) = \emptyset$, completing the proof of the "only if" part.

(End of the proof of Theorem 3.) □

Now we show how the conditions in Theorem 2 are satisfied for any positional scoring rule, STV, Copeland, maximin, and ranked pairs, when the nonmanipulator votes are drawn from the uniform distribution.

**Proposition 3** *Let $\pi_u$ be the uniform distribution. For any rule $r$ that is a positional scoring rule, Copeland, STV, maximin, or ranked pairs, let $k_r$, $GS(f_r, g_r)$ and $\mathscr{K}_r$ be defined as in Proposition 2. Then, $\pi_u$ is compatible with $\mathscr{K}_r$, and for any $l \leq q_r$ and any $i,j \leq K_l$ ($i \neq j$), there exists a vote $V \in L(\mathcal{C})$ such that $(f_r(V))_i \neq (f_r(V))_j$.*

**Proof of Proposition 3:** By simple calculation we have that when $r$ is a

**positional scoring rule** with scoring vector $\vec{v}$: for any $i \leq m$, $E_{V \sim \pi_u}((f_{\vec{v}}(V))_i) = \frac{\sum_{j=1}^m v(j)}{m}$.

**Copeland, maximin, or ranked pairs**: for any $i \leq m$, $j \leq m$, $i \neq j$, $E_{V \sim \pi_u}((f_r(V))_{(i,j)}) = \frac{1}{2}$.

**STV**: for any $(S,j)$ such that $S \subset \mathcal{C}$, $|S| = i$, $c_j \notin S$, $E_{V \sim \pi_u}((f_{STV}(V))_{(S,j)}) = \frac{1}{m-i}$.

Now we show, for any two given components (that lie within the same element of the partition), the vote that makes these two components different. When $r$ is a

**positional scoring rule** with scoring vector $\vec{v}$: for any $i,j \leq m$, $i \neq j$, let $V$ be the vote that ranks $c_i$ at the top and $c_j$ at the bottom; then, $(f_{\vec{v}}(V))_i = v(1) \neq v(m) = (f_{\vec{v}}(V))_j$.

**Copeland, maximin, or ranked pairs**: for any $i_1, i_2 \leq m$, $j_1, j_2 \leq m$, $i_1 \neq j_1$, $i_2 \neq j_2$, and $(i_1, j_1) \neq (i_2, j_2)$, let $V$ be any vote in which $c_{i_1} \succ_V c_{j_1}$ and $c_{j_2} \succ_V c_{i_2}$. Because $(i_1, j_1) \neq (i_2, j_2)$, such a $V$ exists. Then,

$$(f_r(V))_{(i_1,j_1)} = 1 \neq 0 = (f_r(V))_{(i_2,j_2)}$$

**STV**: for any $S \subset \mathcal{C}$, $j_1 \neq j_2$ such that $c_{j_1} \notin S$, $c_{j_2} \notin S$, let $V$ be the vote in which $c_{j_1}$ is at the top. Then $(f_{STV}(V))_{(S,j_1)} = 1 \neq 0 = (f_{STV}(V))_{(S,j_2)}$.

(End of the proof of Proposition 3.) □

By combining Proposition 3 and Theorem 3, we know that for any of the rules in Proposition 3, there exists a distinctive profile; hence, $W_{\mathscr{K}_r}(f,g)$ is nonempty (some alternative will win under the distinctive profile, without any tie). Also, all of these rules are neutral (they treat every alternative in the same way) when restricted to profiles that do not cause a tie, so if $W_{\mathscr{K}_r}(f,g)$ is nonempty, it must be that $W_{\mathscr{K}_r}(f,g) = \mathcal{C}$.

**Corollary 2** *Let $\pi_u$ be the uniform distribution over $L(\mathcal{C})$. For any rule $r$ that is a positional scoring rule, Copeland, STV, maximin, or ranked pairs, if the number of manipulators is $\Omega(n^p)$ ($\frac{1}{2} < p \leq 1$) as well as $o(n)$, then for any $c \in \mathcal{C}$, there exists a coalitional manipulation $P^M$ such that the probability that $r(P^M \cup P^{NM}) = c$ is $1 - O(e^{-\Omega(n^{2p-1})})$.*

# 7. CONCLUSIONS

In this paper, we introduced generalized scoring rules. All of the common voting rules we know are generalized scoring rules. We studied the coalitional manipulation problem under generalized scoring rules, and we proved that when the number of manipulators is small ($O(n^p)$, $p < \frac{1}{2}$), and the votes are drawn i.i.d. from a distribution that is positive everywhere, then the probability of a manipulable instance is $O(n^{p-\frac{1}{2}})$. We also proved that when the number of manipulators is large ($O(n^p)$, $p > \frac{1}{2}$, and $o(n)$), and the votes are drawn i.i.d. from a distribution satisfying some natural assumptions with respect to the rule, then with a probability of $1 - O(e^{-\Omega(n^{2p-1})})$, the manipulators can make any alternative win (assuming that it is possible for the alternative to win under the rule). To show that the assumptions used in the results are natural, we proved that they are satisfied by any positional scoring rule, Copeland, STV, maximin, and ranked pairs under the uniform distribution over votes.

While in this paper, we have focused on the frequency of coalitional manipulability, generalized scoring rules offer a general framework within which to study common voting rules. The idea of generalized scoring can be easily extended to social welfare functions (for example, the Kemeny and Slater social welfare functions). We pointed out that the class of generalized scoring rules is not equal to the class of anonymous voting rules; we believe that we know how to show that it is not equal to the class of anonymous voting rules that also satisfy homogeneity. Finding alternative characterizations of the class of generalized scoring rules is an exciting direction for future research.

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## Appendix 1: Run-off rules that use generalized scoring rules

In a multiround run-off rule (or, more generally, a *correspondence*, which possibly selects more than one winner), there are multiple rounds; in each round, some of the alternatives are removed from all of the votes based on a (sub)correspondence, after which the next round proceeds with the rankings of the remaining alternatives. After the last round, some number of alternatives (typically one) is left; these alternatives are the winners. STV is an example of a multiround run-off rule where in each round the "plurality loser" is eliminated.

We prove that for any voting correspondence with finitely many run-off rounds, if in each step the correspondence used is a generalized scoring correspondence (it is straightforward to generalize the definition of generalized scoring rules to correspondences), then the multiround run-off correspondence is a generalized scoring correspondence. In fact, we only need to show this for a run-off correspondence with two rounds; the result will follow for an arbitrary (fixed) number of rounds by induction.

In the remainder of this appendix, we assume that the correspondence that we use in the second round is neutral, that is, it treats all alternatives equally; it seems unnatural to have a multiround run-off correspondence for which this is not the case.

**Definition 9** *Let $r_1, r_2$ be two voting correspondences (where $r_2$ is neutral), both defined on any set of alternatives. For any set of alternatives $\mathcal{C}$, any profile $P$, the run-off correspondence $R(r_1, r_2)(P)$ is defined as follows.*

*1. Let $C_1 = r_1(P)$. Let $P_1 = P|_{C_1}$. That is, $P_1$ is obtained by removing the alternatives not in $C_1$ from each vote in $P$.*

*2. Let $R(r_1, r_2)(P) = r_2(P_1) = r_2(P|_(r_1(P)))$.*

**Theorem 4** *If $r_1$ and $r_2$ are both generalized scoring correspondences, then $R(r_1, r_2)$ is also a generalized scoring rule correspondence.*

**Proof of Theorem 4:** Let $m$ be the number of alternatives. For $i = 1, 2$, let $r_i = GS(f_i, g_i)$ be of order $k_{i,m}$ and compatible with $\mathscr{K}_{i,m}$. The proof is similar to the one that shows that STV is a generalized scoring rule. Unlike STV, $R(r_1, r_2)$ consists of only two rounds, and in each round, multiple alternatives are eliminated (in contrast, STV eliminates one alternative in each round). Let

- $k_{R(r_1,r_2)} = k_{1,m} + \sum_{i=1}^{m} \binom{m}{i} k_{2,i}$. The vector consists of two parts: the first part $K_1$ is indexed by $i_1 \leq k_{1,m}$, and the second part $K_2$ is indexed by $S, i_2$ where $S \subseteq \mathcal{C}$, $S \neq \emptyset$, and $i_2 \leq k_{2,|S|}$. (Here, $S$ corresponds to the set of alternatives that survive the first round.)
- For any $i_1 \leq k_{1,m}$, $(f_{R(r_1,r_2)}(V))_{i_1} = (f_1(V))_{i_1}$. That is, for any vote $V$, the first part of the vector $f_{R(r_1,r_2)}(V)$ is exactly $f_1(V)$. For any $S \subseteq \mathcal{C}$, $S \neq \emptyset$, let $V|_S$ denote the *restriction* of $V$ to $S$, that is, $V|_S$ is obtained from $V$ by removing all alternatives not in $S$. Let $(f_{R(r_1,r_2)}(V))_{(S,i_2)} = (f_2(V|_S))_{i_2}$.
- $g_{R(r_1,r_2)}$ selects the winner based on $f_{R(r_1,r_2)}(P)$ as follows. In the first round, let $S_1 = g_1((f_{R(r_1,r_2)}(P))_{K_1})$, where $(f_{R(r_1,r_2)}(P))_{K_1}$ is the score vector that consists of all components in $K_1$ of $f_{R(r_1,r_2)}(P)$. Then, let

$$g_{R(r_1,r_2)}(f_{R(r_1,r_2)}(P)) = g_2((f_{R(r_1,r_2)}(P))_{(S_1,K_2)})$$

where $(f_{R(r_1,r_2)}(P))_{(S_1,K_2)}$ is the score vector that consists of all components $(S_1, i_2)$ ($i_2 \leq k_{2,|S_1|}$) of $f_{R(r_1,r_2)}(P)$.
- $q_{R(r_1,r_2)} = 2^m$. For any $S \subseteq \mathcal{C}$, $S \neq \emptyset$, let $K_S = \{(S, i_2) : i_2 \leq k_{2,|S|}\}$. Let $\mathscr{K}_{R(r_1,r_2)} = \{K_1\} \cup \{K_S : S \subseteq \mathcal{C}, S \neq \emptyset\}$. $\square$

The purpose of our next result is to show that if the conditions in Theorem 2 hold for the individual correspondences used in the run-off (for the uniform distribution), then these conditions are also satisfied for the run-off rule as a whole.

**Theorem 5** *Let $\pi_{u,m}$ be the uniform distribution over the set of linear orders of $m$ alternatives. For $i = 1, 2$, let $r_i = GS(f_i, g_i)$ be of order $k_{i,m}$ and compatible with $\mathscr{K}_{i,m}$. Then, there exists a partition $\mathscr{K}'_{R(r_1,r_2)}$ that $R(r_1, r_2)$ is compatible with, such that*

*1. If for all $m$, $i = 1, 2$, $\pi_{u,m}$ is compatible with $\mathscr{K}_{i,m}$, then for all $m$, $\pi_{u,m}$ is compatible with $\mathscr{K}'_{R(r_1,r_2)}$.*

*2. If for $i = 1, 2$, $W_{\mathscr{K}_i}(f_i, g_i) \neq \emptyset$, then*

$$W_{\mathscr{K}'_{R(r_1,r_2)}}(f_{R(r_1,r_2)}, g_{R(r_1,r_2)}) \neq \emptyset$$

**Proof of Theorem 5:** All of the notation is defined in the same way as in the proof of Theorem 4. First we show how to construct $\mathscr{K}'_{R(r_1,r_2)}$. For any $S \subseteq \mathcal{C}$, $S \neq \emptyset$, there exists a way to partition $K_{2,|S|}$—the set of components in the score vector under $r_2$ when there are $|S|$ alternatives—into $\mathscr{K}_{2,|S|}$ such that $\pi_{u,|S|}$ is compatible with $\mathscr{K}_{2,|S|}$ w.r.t. $f_2$. That is, $\mathscr{K}_{2,|S|}$ is the partition that $r_2$ is compatible with when the set of alternatives is $S$. Let $\mathscr{K}'_{R(r_1,r_2)}$ be a refinement of $\mathscr{K}_{R(r_1,r_2)}$ such that $K_1$ is refined according to $\mathscr{K}_{1,m}$, that is, the partition that $r_1$ is compatible with, and for any $S \subseteq \mathcal{C}$, $S \neq \emptyset$, $K_S$ is refined according to $\mathscr{K}_{2,|S|}$.

Because $\pi_{u,m}$ is compatible with $\mathscr{K}_1$, for any $i_1, i'_1 \in K_1$, we know that $E_V((f_{R(r_1,r_2)}(V))_{i_1}) = E_V((f_{R(r_1,r_2)}(V))_{i'_1})$. For any $S \subseteq \mathcal{C}$, $S \neq \emptyset$, any $(S, i_2), (S, i'_2)$ in the same element of $\mathscr{K}_{2,S}$ (here, $\mathscr{K}_{2,S}$ is the partition of the components of the run-off score vector that $r_2$ uses when only the alternatives $S$ remain; this partition has the same structure as $\mathscr{K}_{2,|S|}$), we have the following chain of equalities:

$$E_{V\sim\pi_{u,m}}((f_{R(r_1,r_2)}(V))_{(S,i_2)}) = E_{V_{|S|}\sim\pi_{u,|S|}}((f_2(V_{|S|}))_{i_2})$$
$$= E_{V_{|S|}\sim\pi_{u,|S|}}((f_2(V_{|S|}))_{i'_2}) = E_{V\sim\pi_{u,m}}((f_{R(r_1,r_2)}(V))_{(S,i'_2)})$$

Therefore, we know that $\pi_{u,m}$ is compatible with $\mathscr{K}'_{R(r_1,r_2)}$ w.r.t. $R(r_1, r_2)$, proving the first part of the theorem.

If for $i = 1, 2$, $W_{\mathscr{K}_i}(f_i, g_i) \neq \emptyset$, then by Theorem 3, for any $i_1, i'_1$ in the same element of the partition of $K_1$, there exists a vote $V \in L(\mathcal{C})$ such that $(f_1(V))_{i_1} \neq (f_1(V))_{i'_1}$, which means that $(f_{R(r_1,r_2)}(V))_{i_1} \neq (f_{R(r_1,r_2)}(V))_{i_2}$. Again by Theorem 3, for any $S \subseteq \mathcal{C}$, $S \neq \emptyset$, any $i_2, i'_2$ in the same element of the partition of $K_{2,|S|}$, there exists $V_S \in L(S)$ such that $(f_2(V_S))_{i_2} \neq (f_2(V_S))_{i'_2}$. Let $V \in L(\mathcal{C})$ be any vote such that $V|_S = V_S$. Then, we have that $(f_{R(r_1,r_2)}(V))_{(S,i_2)} \neq (f_{R(r_1,r_2)}(V))_{(S,i'_2)}$. By Theorem 3 we know that $W_{\mathscr{K}'_{R(r_1,r_2)}}(f_{R(r_1,r_2)}, g_{R(r_1,r_2)}) \neq \emptyset$. This proves the second part of the theorem.

(End of the proof of Theorem 5.) $\square$

This implies that, if we take some of the correspondences for which we have shown in Proposition 3 that the conditions of Theorem 2 hold, and construct a run-off correspondence for them, then the resulting run-off correspondence also satisfies the conditions of Theorem 2. (In Proposition 3, we referred to voting rules, but in fact all of these become correspondences if we do not break ties. So, an example would be to run plurality first, then eliminate from the votes all the alternatives except those that are tied for the win, then run ranked pairs on the remaining alternatives, and so on.)

**Corollary 3** *For any multiround run-off voting correspondence with finitely many run-off rounds, if in each round the voting correspondence used is one of those in Proposition 3, then Theorem 2 applies to the multiround run-off voting correspondence.*

## Appendix 2: Weighted generalized scoring rules

*Weighted generalized scoring rules* are a slight generalization of unweighted generalized scoring rules. Let $n \in \mathbb{N}$ be the number of voters, and let $w : \{1, \ldots, n\} \to \mathbb{R}^+$ be a function assigning weights to the voters. We define a weighted generalized scoring rule by $GS(w, f, g)(V_1, \ldots, V_n) = g(\sum_{i=1}^{n} w(i) f(V_i))$. When all the weights are equal (that is, the rule is unweighted), generalized scoring rules are anonymous. However, the converse is not true, that is, a rule can still be anonymous even if the weights are not equal. This is illustrated in the following example.

**Example 2** *Let $m = 3$, $n = 3$, $f(c_1 \succ c_2 \succ c_3) = f(c_1 \succ c_3 \succ c_2) = (1, 0, 0)$, $f(c_2 \succ c_1 \succ c_3) = f(c_2 \succ c_3 \succ c_1) = (0, 3, 0)$, $f(c_3 \succ c_1 \succ c_2) = f(c_3 \succ c_2 \succ c_1) = (0, 0, 9)$. For any profile $P$, $g(f(P))$ is the $c_i$ such that $i$ is the maximum component of $f(P)$. (Effectively, the rule is a version of plurality that is biased towards $c_3$ and biased against $c_1$.) It is easy to check that for any profile consisting of three votes, the maximum component of $f(P)$ is higher than any other component by at least 1.*

*Let $w = (1.1, 1, 1)$, so that the weight of the first voter is slightly higher than that of the other two voters. The additional $0.1$ weight of the first voter will only affect the difference between any two components by at most $9 \times 0.1 < 1$. Therefore, for any $P = (V_1, V_2, V_3)$, $\sum_{i=1}^{3} w(i) f(V_i) \sim \sum_{i=1}^{3} f(V_i)$, so that $GS(w, f, g) = GS(f, g)$. Hence $GS(w, f, g)$ is anonymous.*

We can also extend the definition of weighted generalized scoring rules so that voters are allowed to divide their votes into fractions, that is, submit *fractional votes*. In such a setting, each voter submits a convex combination of linear orders (that is, an element of $Conv(L(\mathcal{C}))$. Such a vote $V_i$ is given as $V_i = \sum_{V \in L(C)} t_i^V V$ where $t_i^V \geq 0$ and $\sum_{V \in S} t_i^V = 1$. Let $P = (V_1, \dots, V_n)$ be a profile of fractional votes. We define

$$f_w(P) = \sum_{i=1}^{n} w(i) \sum_{V \in L(\mathcal{C})} t_i^V f(V)$$

We let $GS(w, f, g)(P) = g(f_w(P))$. Now, in contrast to the previous result, we show that if voters are allowed to submit fractional votes, then (under an assumption) a weighted generalized scoring rule is anonymous if and only if all the weights are the same.

**Theorem 6** *Suppose there exist two profiles $P_1, P_2$ such that:*
*1. $GS(w, f, g)(P_1) \neq GS(w, f, g)(P_2)$, and*
*2. the components of $f_w(P_1)$ are all different, and so are the components of $f_w(P_2)$,*
*then $GS(w, f, g)$ is anonymous if and only if for any $i, j \leq n$, $w(i) = w(j)$.*

**Proof of Theorem 6:** The "if" part is obvious. We now prove the "only if" part. Suppose there exist $i_1, i_2 \leq n$ such that $w(i_1) > w(i_2)$. Let $f(L(\mathcal{C})) = \{f(V) : V \in L(\mathcal{C})\}$, that is, $f(L(\mathcal{C}))$ is the set of all (generalized) score vectors. Let $w_s = \sum_{i=1}^{n} w(i)$, that is, $w_s$ is the sum of all weights. The next claim states that the set of sums of weighted score vectors that can be obtained by a fractional profile is exactly the convex hull of $f(L(\mathcal{C}))$ multiplied by $w_s$.

**Claim 3** $f_w(Conv(L(\mathcal{C}))^n) = w_s Conv(f(L(\mathcal{C})))$. *Here, $w_s Conv(f(L(\mathcal{C})))$ is obtained by multiplying each vector in $Conv(f(L(\mathcal{C})))$ by a factor of $w_s$.*

**Proof of Claim 3:** First we prove that $f_w(Conv(L(\mathcal{C}))^n) \subseteq w_s Conv(f(L(\mathcal{C})))$. For any $P \in Conv(L(\mathcal{C}))^n$, let $P = (V_1, \dots, V_n)$ and $V_i = \sum_{V \in L(\mathcal{C})} t_i^V V$ such that $t_i^V \geq 0$ and $\sum_{V \in L(\mathcal{C})} t_i^V = 1$. Then $f_w(P)$ can be written as follows.

$$f_w(P) = \sum_{i=1}^{n} \sum_{V \in L(\mathcal{C})} w(i) t_i^V f(V)$$
$$= \sum_{V \in L(\mathcal{C})} \sum_{i=1}^{n} w(i) t_i^V f(V) = w_s \sum_{V \in L(\mathcal{C})} \frac{\sum_{i=1}^{n} w(i) t_i^V}{w_s} f(V)$$

For any $V \in L(\mathcal{C})$, let $t^V = \frac{\sum_{i=1}^{n} w(i) t_i^V}{w_s}$. Then:

$$\sum_{V \in L(\mathcal{C})} t^V = \sum_{V \in L(\mathcal{C})} \frac{\sum_{i=1}^{n} w(i) t_i^V}{w_s} = \frac{1}{w_s} \sum_{i=1}^{n} \sum_{V \in L(\mathcal{C})} w(i) t_i^V$$
$$= \frac{1}{w_s} \sum_{i=1}^{n} w(i) \sum_{V \in L(\mathcal{C})} t_i^V = \frac{1}{w_s} \sum_{i=1}^{n} w(i) = 1$$

Therefore, $f_w(P) = w_s(\sum_{V \in L(\mathcal{C})} t^V f(V)) \in w_s Conv(f(L(\mathcal{C})))$, which means that $f_w(Conv(L(\mathcal{C}))^n) \subseteq w_s Conv(f(L(\mathcal{C})))$.

Next, we prove that $w_s Conv(f(L(\mathcal{C}))) \subseteq f_w(Conv(L(\mathcal{C}))^n)$. For any element $\vec{p}$ in $w_s Conv(f(L(\mathcal{C})))$, suppose $\vec{p} = w_s \sum_{V \in L(\mathcal{C})} t^V f(V)$, where $t^V \geq 0$ and $\sum_{V \in L(\mathcal{C})} t^V = 1$. Then, let $V_1 = \ldots = V_n = \sum_{V \in L(\mathcal{C})} t^V V$.

$$f_w(V_1, \dots, V_n) = \sum_{i=1}^{n} w(i) f(V_1, \dots, V_n)$$
$$= w_s \sum_{V \in L(\mathcal{C})} t^V f(V) = w_s \vec{p}$$

Therefore, we know that $\vec{p} \in f_w(Conv(L(\mathcal{C}))^n)$, which means that $w_s Conv(f(L(\mathcal{C}))) \subseteq f_w(Conv(L(\mathcal{C}))^n)$. Hence $w_s Conv(f(L(\mathcal{C}))) = f_w(Conv(L(\mathcal{C}))^n)$.

(End of the proof of Claim 3.) □

Suppose there exist $P_1, P_2$ such that $GS(w, f, g)(P_1) \neq GS(w, f, g)(P_2)$, and all the components are different within $f_w(P_1)$, as well as within $f_w(P_2)$, respectively. Let $\vec{v}_P = f_w(P_1) - f_w(P_2)$. Let $l(\lambda) = f_w(P_2) + \lambda \vec{v}_P$, $0 \leq \lambda \leq 1$. From Claim 3 we know that $f_w(Conv(L(\mathcal{C}))^n)$ is convex. Therefore, for any $0 \leq \lambda \leq 1$, we have $l(\lambda) \in f_w(Conv(L(\mathcal{C}))^n) = w_s Conv(f(L(\mathcal{C})))$, which means that there exists a fractional vote $V_\lambda$ such that $w_s f(V_\lambda) = l(\lambda)$. For any total preorder $O$ (that is, a binary relation that is complete, reflexive, and transitive) over the components, define $S(O)$ to be a subspace of $\mathbb{R}^k$ such that the order of the components of any element in $S(O)$ is $O$. That is, $S(O) = \{\vec{v} \in \mathbb{R}^k : (\forall i, j)((\vec{v})_i \geq (\vec{v})_j \Leftrightarrow i \succeq_O j)\}$. Then it is easy to check that for any $O$, $S(O)$ is convex, and for any $\vec{v}_1, \vec{v}_2 \in S(O)$, $\vec{v}_1 \sim \vec{v}_2$, which means that $g(\vec{v}_1) = g(\vec{v}_2)$. Let $g(S(O)) = g(\vec{v}_1)$. For any $O$, let $I(O)$ be the intersection of $S(O)$ and the line $l(\lambda)$. Since both of them are convex, $I(O)$ is convex. Therefore, either $I(O)$ is an interval (that is, $I(O)$ is described by two ends $0 \leq \lambda_1 < \lambda_2 \leq 1$, such that for all $\lambda_1 < \lambda < \lambda_2$, $l(\lambda) \in I(O)$, and for any $\lambda < \lambda_1$ or $\lambda_2 < \lambda$, $l(\lambda) \notin I(O)$) or a single point. We note that for any $O_1 \neq O_2$, $S(O_1) \cap S(O_2) = \emptyset$, hence $I(O_1) \cap I(O_2) = \emptyset$.

For any $O$ that is a linear order, $I(O)$ is an interval in $l(\lambda)$ (that is, $I(O)$ has more than one point), because for any $\vec{v} \in S(O)$, a small perturbation would not change the order. Let $O_{P_1}$, $O_{P_2}$ be the linear orders that $f_w(P_1)$ and $f_w(P_2)$ are compatible with, respectively. Since $g(O_{P_1}) \neq g(O_{P_2})$ and there are only finitely many total preorders over $K$, there must exist two adjacent intervals $I(O_1)$ and $I(O_2)$, such that $g(O_1) \neq g(O_2)$. More precisely, there exist $0 < \lambda' < 1$ and $d > 0$ such that for all $0 < \epsilon < d$, $l(\lambda' - \epsilon) \in I(O_1)$ and $l(\lambda' + \epsilon) \in I(O_2)$. Let $0 < d^* < d$. Then, let $V^+$ be the fractional vote such that $w_s f(V^+) = l(\lambda' + d^*)$ (the existence of such a $V^+$ is guaranteed by Claim 3); let $V^-$ be the fractional vote such that $w_s f(V^-) = l(\lambda' - d^*)$; let $V^0$ be the fractional vote such that $w_s f(V^0) = l(\lambda')$. Then, let $P_1$ be the profile where $V_{i_1} = V^-$, $V_{i_2} = V^+$ (we recall that $w(i_1) > w(i_2)$), and for any $1 \leq s \leq n$, $s \neq i_1, s \neq i_2$, let $V_s = V^0$. Then we have

$$f_w(V_1, \dots, V_n)$$
$$= (w_s - w(i_1) - w(i_2)) f(V^0) + w(i_1) f(V^-) + w(i_2) f(V^+)$$
$$= w_s f(V^0) + w(i_1)(f(V^-) - f(V^0)) + w(i_2)(f(V^+) - f(V^0))$$
$$= l(\lambda) - \frac{w(i_1) d^*}{w_s} \vec{v}_P + \frac{w(i_2) d^*}{w_s} \vec{v}_P = l(\lambda - \frac{w(i_1) - w(i_2)}{w_s} d^*)$$

Since $0 < \frac{w(i_1) - w(i_2)}{w_s} d^* \leq d^* < d$, it must be that $f_w(P_1) \in I(O_1)$, which means that $GS(f, g)(P_1) = g(O_1)$. Let $P_2$ be the profile where $V_{i_1} = V^+$, $V_{i_2} = V^-$, and for any $1 \leq s \leq n$, $s \neq i_1, s \neq i_2$, let $V_s = V^0$. Similarly, we have that $f_w(P_2) \in I(O_2)$ and $GS(f, g)(P_2) = g(O_2)$. Hence, $GS(f, g)(P_1) \neq GS(f, g)(P_2)$. However, $P_2$ is obtained by exchanging the votes of voter $i_1$ and $i_2$. Therefore, $GS(f, g)$ does not satisfy anonymity.

(End of the proof of Theorem 6.) □