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On minimal models of the Region Connection Calculus

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Abstract. Region Connection Calculus (RCC) is one primary formalism of qualitative spatial reasoning. Standard RCC models are continuous ones where each region is infinitely divisible. This contrasts sharply with the predominant use of finite, discrete models in applications. In a recent paper, Li et al. (2004) initiate a study of countable models that can be constructed step by step from finite models. Of course, some basic problems are left unsolved, for example, how many non-isomorphic countable RCC models are there? This paper investigates these problems and obtains the following results: (i) the *exotic* RCC model described by Gotts (1996) is isomorphic to the minimal model given by Li and Ying (2004); (ii) there are continuum many non-isomorphic minimal RCC models, where a model is minimal if it can be isomorphically embedded in each RCC model.

Keywords: Qualitative Spatial Reasoning; Region Connection Calculus; minimal RCC models.

1. Introduction

Region Connection Calculus (RCC) is one primary formalism of qualitative spatial reasoning. As a first order theory, RCC is based on one primitive contact relation that satisfies several axioms. Models of RCC have been studied by several authors, see e.g. [3, 9, 1, 2, 4, 5, 6].

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Given a topological space X, write $\mathsf{RC}(X)$ for the regular closed Boolean algebra of X. For two regions a, b, that is, two nonempty sets in $\mathsf{RC}(X)$, a is said to be in *contact* with b, written $a\mathbf{C}_X c$, if $a \cap b \neq \varnothing$. If X happens to be connected and regular, then Gotts [3] shows that $\mathsf{RC}(X)$ is indeed an RCC model. Later, Li and Ying [4] also show that $\mathsf{RC}(X)$ is an RCC model if and only if X is connected and the open set lattice of X is inexhaustible in the sense of Stell [9]. Recently, Düntsch and Winter [2] show that, under the mild condition that X is semi-regular, $\mathsf{RC}(X)$ is an RCC model if and only if X is weakly regular connected.[1] They further show that any RCC model can be isomorphically embedded in a connected weakly regular space.

Standard models of RCC are continuous, that is, regions where are infinitely divisible. This contrasts sharply with finite, discrete models predominantly used in application. In a recent paper, Li et al. [6] initiate a study of countable models that can be constructed step by step from finite models. Of course, some basic problems are left unsolved, for example, how many non-isomorphic countable RCC models are there? This paper investigates these problems and obtains the following results: (i) the exotic RCC model described by Gotts [3] is isomorphic to the minimal model given by Li and Ying [5]; (ii) there are continuum many non-isomorphic minimal RCC models, where a model is minimal if it can be isomorphically embedded in each RCC model.

The structure of this paper is as follows. Section 2 recalls the concept of RCC models and introduces the minimal RCC model $\mathfrak{B}_\omega$ given in Li and Ying [5]. Then in Section 3 we show that Gotts' model is isomorphic to Li and Ying's minimal model. Section 4 introduces a sequence of sub-RCC models of $\mathfrak{B}_\omega$. The key notion of k-chain is also defined here. We then show these sub-RCC models are non-isomorphic. Based on this result, we in Section 5 construct for each binary ω-string a sub-RCC model of $\mathfrak{B}_\omega$, and show that they are non-isomorphic. Conclusions are given in Section 6.

2. A minimal RCC model

2.1. Models of the Region Connection Calculus

The RCC theory was initially described in [7, 8], we here adopt an equivalent formulation given by Stell [9].

Definition 2.1. ([9])
An RCC model is a Boolean algebra A containing more than two elements, together with a binary contact relation $\mathbf{C}$ on $A - \{\bot\}$ that satisfies the following conditions:

A1. $\mathbf{C}$ is reflexive and symmetric;

A2. $(\forall x \in A - \{\bot, \top\})\mathbf{C}(x, -x)$;

A3. $(\forall xyz \in A - \{\bot\})\mathbf{C}(x, y \vee z) \leftrightarrow \mathbf{C}(x, y)$ or $\mathbf{C}(x, z)$;

A4. $(\forall x \in A - \{\bot, \top\})(\exists z \in A - \{\bot, \top\})\neg\mathbf{C}(x, z)$.

[1] A topological space is called semi-regular if X has a basis of regular open sets; and a semi-regular space is called weakly regular if for each nonempty open set u there is another nonempty open set v such that the closure of v is contained in u [2]. This last condition is precisely the so called "inexhaustibility" of the open lattice of X given in [4].

where $\bot$ and $\top$ are, respectively, the bottom and the top element of A, $-x$ is the complement of x in A, $x \vee z$ is the least upper bound (lub) of x and z in A.

Given an RCC model $\langle A, \mathbf{C} \rangle$, a *region* in A is an element other than $\bot$. For two regions $a, b \in A$, we write $\mathbf{DC}(a, b)$ if they are not in contact; write $\mathbf{EC}(a, b)$ if a, b are in *external contact*, i.e. $\mathbf{C}(a, b)$ but $a \wedge b = \bot$. We call a a *part* of b, write $\mathbf{P}(a, b)$, if $a \leq b$, and call a a *proper part* of b, write $\mathbf{PP}(a, b)$, if $a < b$. For $a < b$, we say a is a *tangential proper part* (tpp for short) of b, write $\mathbf{TPP}(a, b)$, if there is a third region c such that $\mathbf{EC}(c, a)$ and $\mathbf{EC}(c, b)$; otherwise, we say a is a *non-tangential proper part* (ntpp for short) of b, write $\mathbf{NTPP}(a, b)$ (or $a \ll b$ for short). It is easy to show that for any $b \neq \top$, $a \ll b$ if and only if $\mathbf{DC}(a, -b)$.

Lemma 2.1. For regions a, b, c in an RCC model $\langle A, \mathbf{C} \rangle$, if $a \ll b$, then $\mathbf{EC}(c, b)$ only if $\mathbf{EC}(c, b - a)$.

Proof:
Suppose $\mathbf{EC}(c, b)$ and $\mathbf{DC}(c, b - a)$. Then by A3. of Definition 2.1, we have $\mathbf{EC}(c, a)$ since $b = a \vee (b - a)$. This contradicts $a \ll b$. □

In what follows we call $b - a$ a boundary of b if $a \ll b$. In other words, for two regions a, b, we call a a *boundary* of b if $b - a \ll b$. Notice that by Lemma 2.1 we know a boundary of b is in contact with all regions that are in external contact with b. This justifies the word *boundary*.

Another important concept is connectedness of a region.

Definition 2.2. A region a in an RCC model $\langle A, \mathbf{C} \rangle$ is *connected* if

$$\text{For any two regions } b, c \text{ in } A, \text{ if } b \vee c = a \text{ then } \mathbf{C}(b, c) \tag{1}$$

By the definition of RCC models, we know particularly $\top$, the universe, is connected.

Lemma 2.2. Given an RCC model $\langle A, \mathbf{C} \rangle$, suppose $b_1, \cdots, b_m$ are m regions in A with $\mathbf{DC}(b_i, b_j)$ for $1 \leq i \neq j \leq m$. For a connected region $a \in A$, if a is covered by $b_1, \cdots, b_m$, i.e. $a \leq \bigvee_{l=1}^{m} b_l$, then a is a part of some b_l.

Proof:
If not so, then the set $\{a \wedge b_i : a \wedge b_i \neq \bot\}$ contains at least two regions. Any two such regions are clearly not in contact, this contradicts the assumption that a is connected. □

Given an RCC model $\langle A, \mathbf{C} \rangle$, suppose B is a subalgebra of A containing more than two elements. We now consider the sub-structure on B. Using $\mathbf{C}_B$, the restriction of $\mathbf{C}$ on B, and $\leq$, we can define all aforementioned RCC relations on B. Interestingly, these relations on B coincide with the corresponding relation defined in A. In other words, for any two regions $b_1, b_2 \in B$, $\mathbf{R}(b_1, b_2)$ holds in B if and only if $\mathbf{R}(b_1, b_2)$ holds in A, where $\mathbf{R} \in \{\mathbf{C}, \mathbf{EC}, \mathbf{DC}, \mathbf{TPP}, \mathbf{NTPP}, \mathbf{PP}\}$ [6, Proposition 3.1].[2]

For a sub-model B of A, we say a region $b \in B$ is *connected* in B if for any two regions $b_1, b_2 \in B$, $b_1 \vee b_2 = b$ only if $\mathbf{C}(b_1, b_2)$. For $b \in B$, it is clear that b is connected in B only if it is connected in A. The converse, however, is not always true.

[2]This property, however, cannot apply to all binary relations definable by **C** [6].

2.2. Standard RCC models

Given a topological space X, denote by $\mathsf{RC}(X)$ the complete Boolean algebra of regular closed subsets of X. Two regions, that is, two nonempty regular closed sets, a and b in $\mathsf{RC}(X)$, are in *contact* if $a \cap b \neq \varnothing$. If this is the case, we write $\mathbf{C}_X(a, b)$. It is well-known that, if X is a connected regular topological space, then $\mathsf{RC}(X)$ is an RCC model [3]. We call $\mathsf{RC}(X)$, together with the contact relation $\mathbf{C}_X$, the *standard* RCC model over X. Note that each standard RCC model is a complete (uncountable) Boolean algebra. This suggests that regions in these models can not be constructed from basic ones in finite steps. In [5], Li and Ying provide a general approach for constructing countable (thus *non-standard*) RCC models step by step from sequences of finite models. We next recall the minimal RCC model described there.

2.3. A representation of the atomless countable Boolean algebra

Let $\{0, 1\}^*$ be the set of finite binary strings over $\{0, 1\}$. As usual, we write ϵ for the empty string. For a string $s \in \{0, 1\}^*$, we denote by $|s|$ its length. Now for each string $s \in \{0, 1\}^*$, we associate a left-closed-and-right-open sub-interval of $[0, 1)$ as follows: Take

$x_\epsilon = [0, 1)$; $x_0 = [0, 1/2)$, $x_1 = [1/2, 1)$;

$x_{00} = [0, 1/4)$, $x_{01} = [1/4, 1/2)$, $x_{10} = [1/2, 3/4)$, $x_{11} = [3/4, 1)$; and so on.

In general, suppose x_s has been defined for a string $s \in \{0, 1\}^*$, we define x_{s0} to be the first half left-closed-and-right-open sub-interval of x_s, and x_{s1} the second half. Write $X_\omega = \{x_s : s \in \{0, 1\}^*\}$. Then X_ω, with the ordering of set inclusion, can be visualized as an infinite complete binary tree (See Figure 1).

Write $\mathfrak{B}_\omega$ for the subalgebra of the powerset algebra $2^{[0,1)}$ generated by X_ω. Clearly $\mathfrak{B}_\omega$ is countable.

Lemma 2.3. For each region $a \in \mathfrak{B}_\omega$, there exists some $n \geq 1$ such that either $x_s \subseteq a$ or $a \cap x_s = \varnothing$ for any string s with $|s| = n$.

Proof:
Follows directly from the definition of $\mathfrak{B}_\omega$. □

Definition 2.3. ([5])
Define $\mathbf{C}_\omega$ to be the smallest binary reflexive and symmetric relation on $\mathfrak{B}_\omega$ that satisfies the following conditions:

- $\mathbf{C}_\omega 1$ For two different binary strings s, t with same length, $\mathbf{C}_\omega(x_s, x_t)$ if and only if there exists a binary string s_1 such that $\{s, t\} = \{s_1 01^n,\ s_1 11^n\}$ for some $n \geq 0$, where 1^n denotes the (sub-)string $\underbrace{1 \cdots 1}_{n}$.
- $\mathbf{C}_\omega 2$ For any two nonempty $a, b \in \mathfrak{B}_\omega$, $\mathbf{C}_\omega(a, b)$ if and only if $a \cap b \neq \varnothing$ or there exist binary strings s, t such that $x_s \subseteq a$, $x_t \subseteq b$ and $\mathbf{C}_\omega(x_s, x_t)$.

Then $\langle \mathfrak{B}_\omega, \mathbf{C}_\omega \rangle$ is a countable RCC model [5]. In what follows we simply write $\mathfrak{B}_\omega$ for this model. It was shown in [5] that $\mathfrak{B}_\omega$ is a minimal model, that is, it can be isomorphically embedded in any RCC model.

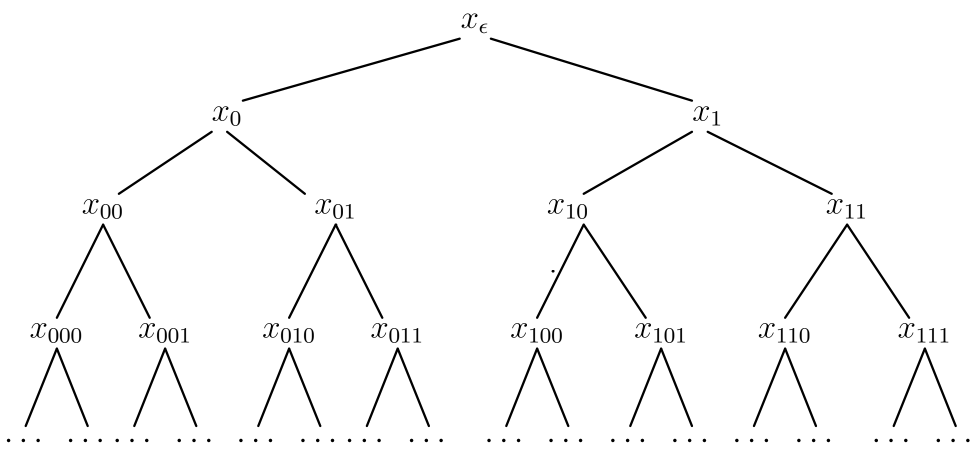


Figure 1. The infinite complete binary tree

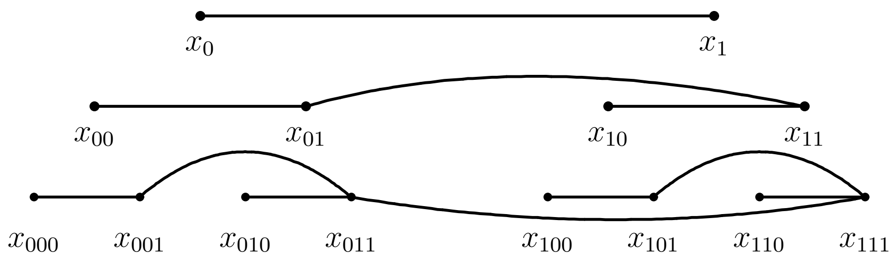


Figure 2. Construction of the contact relation $\mathbf{C}_\omega$

Proposition 2.1. For any nonempty binary string s and any region $a \in \mathfrak{B}_\omega$, $a\mathbf{EC}x_{s0}$ if and only if $x_{s0} \cap a = \varnothing$ and $x_{s1^l} \subseteq a$ for some $l \geq 1$.

As a consequence, any two regions that are in external contact with the same x_{s0} intersect.

Proposition 2.2. ([6])
For any binary string s, x_s is connected in $\mathfrak{B}_\omega$.

Proposition 2.3. Given a nonempty binary string s and a region $x \subseteq x_s$ in $\mathfrak{B}_\omega$, x is a boundary of x_s, i.e. $x_s - x \ll x_s$, iff $x_{s1^l} \subseteq x$ for some $l \geq 1$; or, equivalently, x is an ntpp of x_s iff $x_{s1^l} \not\subseteq x$ for any $l \geq 1$.

Proof:
Notice that $x_s\mathbf{EC}y$ only if $x_{s1^l}\mathbf{EC}y$ for any $y \in \mathfrak{B}_\omega$ and any $l \geq 1$. This follows from the definition of boundary. □

Proposition 2.4. Given a nonempty binary string s and n regions $y_1, \cdots, y_n$ in $\mathfrak{B}_\omega$, suppose $\{y_1, \cdots, y_n\}$ forms a partition of x_s. Then there is exactly one i such that y_i is a boundary of x_s.

Proof:
Follows from Proposition 2.3. □

Proposition 2.5. Suppose x is a connected region in $\mathfrak{B}_\omega$, and s is a binary string such that $x_{s0} \subset x$. Then $x - x_{s0}$ is also connected in $\mathfrak{B}_\omega$.

Proof:
Because of the connectedness of x, we know $\mathbf{C}(x_{s0}, x - x_{s0})$. Now suppose $x - x_{s0}$ is disconnected. Then we can find $a_1, a_2 \in \mathfrak{B}_\omega$ such that $x - x_{s0} = a_1 \cup a_2$ but $\mathbf{DC}(a_1, a_2)$. Because x is connected, we have $\mathbf{EC}(x_{s0} \cup a_1, a_2)$ and $\mathbf{EC}(x_{s0} \cup a_2, a_1)$. Thus by Definition 2.1 A3, x_{s0} is in contact with both a_1 and a_2. This is impossible, since, by Proposition 2.1, both a_1 and a_2 would contain a boundary of x_{s1}. □

3. Gotts' model

The first countable (hence *non-standard*) RCC model was constructed by Gotts in [3]. We in this section show this model turns out to be isomorphic to the minimal model $\mathfrak{B}_\omega$.

Let $U = [0,1]^2$ be the unit Euclidean square. Set $\mathsf{RC}(U)$ to be the standard RCC model on U. Recall regions in this model are just nonempty regular closed subsets of U, and two regions $a, b \in \mathsf{RC}(U)$ are *in contact* if $a \cap b \neq \varnothing$. We next describe how to construct Gotts' model step by step.

Figure 3 (left) illustrates the result of the first few stages in building this model. The first stage in construction is to draw a unit square U. The second is to draw a sub-square of U at the center with length $\frac{1}{3}$. In the third stage five more sub-squares of length $\frac{1}{3^2}$ are added; the fourth adds 25 more, smaller still. Continue this process and write G for the set of all squares produced after some finite number of stages. The Gotts' model, denoted by $\mathfrak{G}$, is therefore the sub-model of $\mathsf{RC}(U)$ generated by these squares in G. That is, a region in $\mathfrak{G}$ will be any square in G, any finite sum of such squares, and the difference between any two such sums of squares. Notice that each nonempty region in $\mathfrak{G}$ contains at least one square in G as a non-tangential proper part. This sub-model clearly is an RCC model.

To show $\mathfrak{B}_\omega$ is isomorphic to $\mathfrak{G}$, we need to establish a Boolean isomorphism $\eta : \mathfrak{B}_\omega \to \mathfrak{G}$. To this end, we now give a correspondence between squares in G and binary strings.

First, write $\gamma(U)$ for all sub-squares in G that are maximally connected. That is, for each sub-square g in $\gamma(U)$, no square other than U contains g. Clearly, $\gamma(U)$ is countable infinite. We write these squares as a sequence

$$g_0, g_1, g_2, g_3, \cdots, g_n, \cdots .$$

Figure 3 (right) gives an illustration of such an ordering.

We associate a binary string for each $i \geq 0$ as $\eta(g_i) = 1^i 0$.

For each square $a \in G$, write $\gamma(a)$ for all sub-squares contained in a that are maximally connected. Clearly there is a natural corresponding between squares in $\gamma(a)$ and squares in $\gamma(U)$. We write squares in $\gamma(a)$ as

$$g_0^a, g_1^a, g_2^a, g_3^a, \cdots, g_n^a, \cdots,$$

where g_i^a is the corresponding square of g_i in $\gamma(a)$. Note also that any $b \in \gamma(a)$ is a non-tangential proper part of a.

Next, we classify squares in G into groups. We denote $G_0 = \gamma(U)$ and $G_1 = \bigcup_{a \in G_0} \gamma(a)$, and, in general, if G_k has been defined for some $k \geq 1$, we define $G_{k+1} = \bigcup_{a \in G_k} \gamma(a)$. Then it is clear $G = \bigcup_{k \in \mathbb{N}} G_k$ and these G_k are pairwise disjoint.

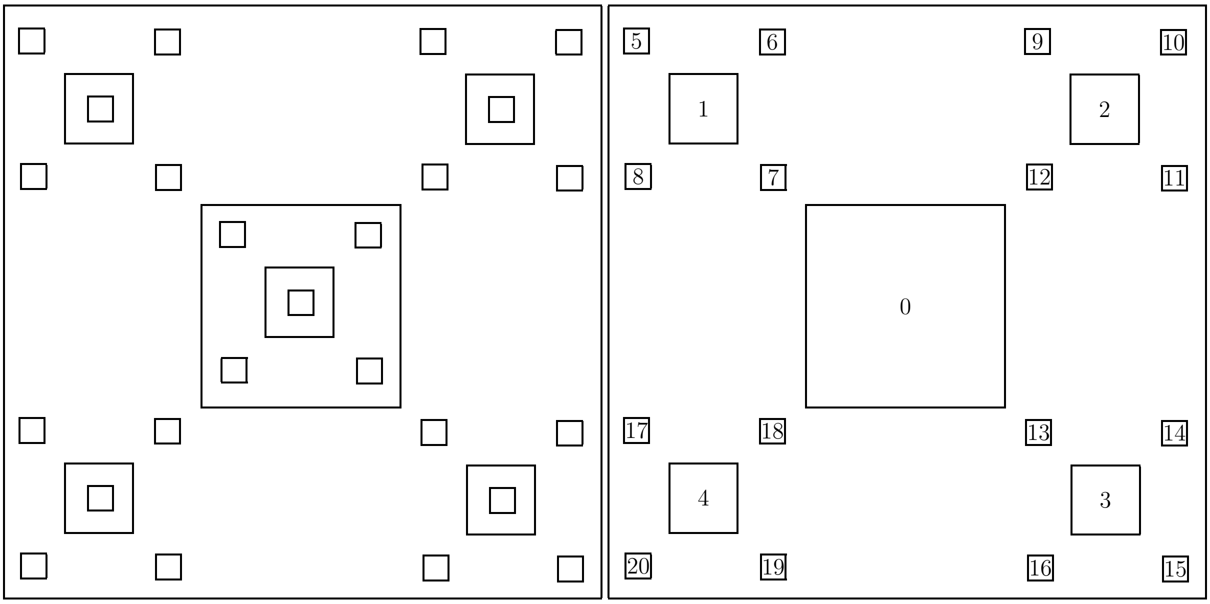


Figure 3. First few steps of Gotts' model (left) and an ordering of maximal squares (right).

For each region a in G, suppose $a \in G_k$ for some $k \geq 0$. Then there exist $a_0 \in G_0 = \gamma(U)$, $a_1 \in G_1, \cdots, a_{k-1} \in G_{k-1}$ such that $a \in \gamma(a_{k-1})$ and $a_i \in \gamma(a_{i-1})$ for any $1 \leq i < k$. Suppose a_0 is the $\lambda(0)$-th square in $\gamma(U)$, a_i is the $\lambda(i)$-th square in $\gamma(a_{i-1})$ for $1 \leq i < k$, and suppose a is the $\lambda(k)$-th square in $\gamma(a_{k-1})$. Then this region a is uniquely determined by the sequence $\lambda(0)\lambda(1)\lambda(2)\cdots\lambda(k)$. Define $\eta(a) = 1^{\lambda(0)}01^{\lambda(1)}01^{\lambda(2)}0\cdots 1^{\lambda(k)}0$. This gives an injection $\eta : G \to \{0,1\}^*$.

On the other hand, given a nonempty binary string s. If s ends with a 0, then s has form

$$1^{\lambda(0)}01^{\lambda(1)}01^{\lambda(2)}0\cdots 1^{\lambda(k)}0; \tag{2}$$

if s ends with a 1 and there is no 0 appearing in s, then $s = 1^{\lambda(0)}1$; if s ends with a 1 and there is some 0 appearing in s, then s has form

$$1^{\lambda(0)}01^{\lambda(1)}01^{\lambda(2)}0\cdots 1^{\lambda(k)}01^{\lambda(k+1)}1 \tag{3}$$

where $\lambda(i) \geq 0$ for all $0 \leq i \leq k+1$.

Now define $f : X_\omega \to \mathfrak{G}$ as follows, where $X_\omega = \{x_s : s \in \{0,1\}^*\}$:

- $f(x_\epsilon) = U$;
- $f(x_s) = a$ if s is a string with form (2) and $\eta(a) = s$;
- $f(x_s) = U - g_0 - g_1 - \cdots - g_{\lambda(0)}$ if $s = 1^{\lambda(0)}1$;
- $f(x_s) = a - g_0^a - g_1^a - \cdots - g_{\lambda(k+1)}^a$ if s is with form (3) and $\eta(a) = s_1$, where $s_1 = 1^{\lambda(0)}01^{\lambda(1)}01^{\lambda(2)}0\cdots 1^{\lambda(k)}0$.

It is straightforward to show that f satisfies the following conditions:

(a) $f(x_s) \subseteq f(x_t)$ iff $x_s \subseteq x_t$, i.e., iff t is an initial segment of s; and

(b) $x_s \mathbf{C}_\omega x_t$ iff $f(x_s) \cap f(x_t) \neq \varnothing$, i.e., iff $f(x_s)\mathbf{C}f(x_t)$.

Notice that $\mathfrak{B}_\omega$ is generated by X_ω, f can be extended to $\mathfrak{B}_\omega$ in a natural way. Write also f for the resulted mapping. Then it's straightforward to show that $f : \mathfrak{B}_\omega \to \mathfrak{G}$ is a Boolean isomorphism and $f(a)\mathbf{C}f(b)$ iff $a\mathbf{C}_\omega b$ for any $a, b \in \mathfrak{B}_\omega$.

4. How many minimal RCC models are there?

A minimal RCC model is a model that can be embedded into any other models. Li and Ying [5] show that the RCC model $\mathfrak{B}_\omega$ (see Definition 2.3) is a minimal model. Naturally, we ask "*Is* $\mathfrak{B}_\omega$ *the unique minimal model?*" and "*How many minimal models are there?*" The rest of this paper is devoted to the solution of this problem.

We first note that each minimal model is countable. Recall also that there is up to isomorphism only one atomless countable Boolean algebra. Now, since there are at most continuum many different contact relations on the atomless countable Boolean algebra, there are at most continuum many different countable RCC models. As a result, there are at most continuum many minimal RCC models.

In the rest of this paper, we show that there are exactly continuum many non-isomorphic minimal RCC models. This is justified by constructing for each binary ω-string \$ a minimal RCC model $\mathfrak{B}_\$$.

To make the construction more comprehensible, we give a sketch of the proof.

4.1. A sketch of the proof

The basic idea is to construct for each binary ω-string \$ over $\{0,1\}$ an RCC model $\mathfrak{B}_\$$, which is a sub-model of the minimal model $\mathfrak{B}_\omega$. This model is clearly minimal. We then need show that any two models $\mathfrak{B}_{\$_1}$ and $\mathfrak{B}_{\$_2}$ are non-isomorphic if $\$_1 \neq \$_2$.

To this end, we introduce the key notion of k-chain ($k \geq 1$). This is an invariant property of RCC models, i.e. for two isomorphic RCC models A and B, A contains a k-chain if and only if B does.

Next, we construct a sequence of sub-RCC models

$$\mathfrak{B}_\omega = A_1, A_2, \cdots, A_k, \cdots \tag{4}$$

of $\mathfrak{B}_\omega$, such that, for two $k, \hat{k} \geq 2$, A_k contains a $\hat{k}$-chain if and only if $k = \hat{k}$.

Given a binary ω-string $\$ = w_1w_2w_3\cdots$, the minimal model $\mathfrak{B}_\$$ can be constructed, roughly speaking, as follows: for each $i \geq 1$, if $w_i = 1$, then replace the local structure of $\mathfrak{B}_\omega$ at $x_{1^{i-1}0}$ with A_i. Then we prove $\mathfrak{B}_\$$ contains a k-chain ($k \geq 2$) if and only if $w_k = 1$. As a result, for two string $\$ = w_1w_2w_3\cdots$ and $\$' = w_1'w_2'w_3'\cdots$, if they differ at the k-th ($k \geq 2$) symbol, i.e. $w_k \neq w_k'$, then $\mathfrak{B}_\$$ and $\mathfrak{B}_{\$'}$ are non-isomorphic since one contains a k-chain but the other doesn't.

In the following sections, we first construct sub-models $\{A_k\}_{k\geq 1}$ of $\mathfrak{B}_\omega$, then we show these models can be differentiated by the notion of k-chain. Using these non-isomorphic models, we construct for each binary ω-string a minimal model.

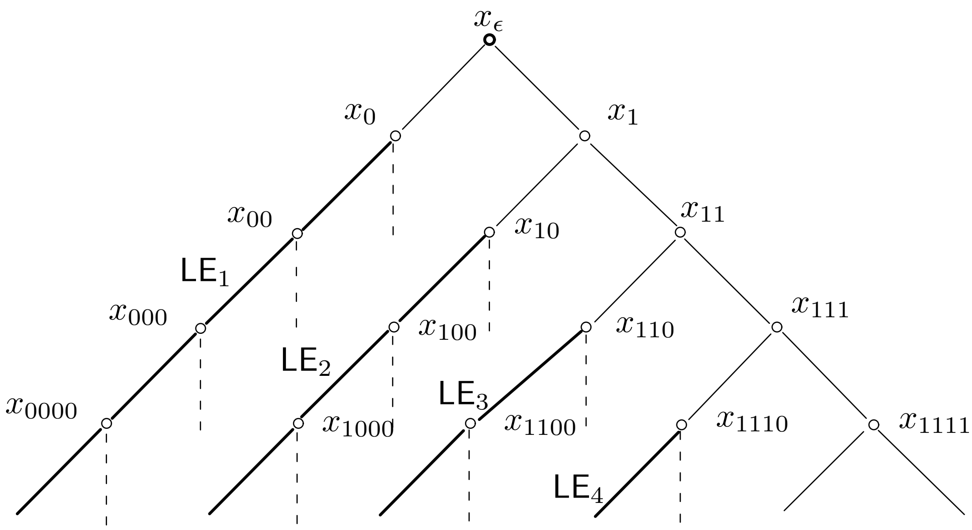


Figure 4. Left edges of the complete binary tree

4.2. A sequence of sub-models of $\mathfrak{B}_\omega$

For each $k \geq 1$, we in this subsection construct a subalgebra A_k of $\mathfrak{B}_\omega$.

We need some additional notations concerning $\mathfrak{B}_\omega$. For each $i \geq 1$, write

$$\mathsf{LE}_i = \{x_{1^{i-1}0^n} : n \geq 1\}.$$

Note in Fig 4, LE_i contains regions in the i-th left edge of the complete binary tree. For example, the second left edge LE_2 contains regions $x_{10}, x_{100}, \ldots$

For each $i \geq 1$ and any region $a \in \mathfrak{B}_\omega$, define $a^{(i)}$ to be the largest region (possibly empty) in LE_i contained in a, i.e.

$$a^{(i)} = \bigcup \{x_s \in \mathsf{LE}_i : x_s \subseteq a\}.$$

Clearly $a^{(i)} \in \mathsf{LE}_i$ if and only if $a^{(i)} \neq \varnothing$. Notice that no region other than the universe x_ϵ can contain some x_{1^i} $(i \geq 0)$ as an ntpp. We call each x_{1^i} a *boundary* of the universe. For any region $a \in \mathfrak{B}_\omega$, write

$$a^{(\infty)} = \bigcup \{x_{1^i} : x_{1^i} \subseteq a, i \geq 0\}.$$

Then $a^{(\infty)} \neq \varnothing$ if and only if a contains a boundary of the universe.

The following two simple properties will be useful in later discussion.

Lemma 4.1. For any region $a \in \mathfrak{B}_\omega$, any $i \geq 1$, $a^{(i)} = \varnothing$ if and only if $(-a)^{(i)} \neq \varnothing$, where $-a$ is the complement of a in $x_\epsilon = [0, 1)$.

Proof:
Notice that by Lemma 2.3, there exists some $n \geq 0$ such that $x_s \subseteq a$ or $a \cap x_s = \varnothing$ for any string s with length $\geq n$. In particular, $x_{1^{i-1}0^n} \subseteq a$ or $x_{1^{i-1}0^n} \cap a = \varnothing$. That is, $x_{1^{i-1}0^n}$ is contained in either a or $-a$. This shows one of $a^{(i)}$ and $-a^{(i)}$ is nonempty. Since $a \cap -a = \varnothing$, only one can be nonempty. $\square$

Lemma 4.2. For two regions $a, b \in \mathfrak{B}_\omega$, any $i \geq 1$, $(a \cup b)^{(i)} \neq \varnothing$ iff either $a^{(i)}$ or $b^{(i)}$ is nonempty, and $(a \cap b)^{(i)} \neq \varnothing$ iff both $a^{(i)}$ and $b^{(i)}$ are nonempty.

Proof:
This follows from the similar argument as given in Lemma 4.1. □

Next, we define a subalgebra of $\mathfrak{B}_\omega$ for each $k \geq 1$.

Definition 4.1. For each $k \geq 1$, define A_k to be a subset of $\mathfrak{B}_\omega$ such that a region $a \in \mathfrak{B}_\omega$ is in A_k if one of the following two conditions is satisfied:

(i) For all $1 \leq j \leq k$, $a^{(j)} \neq \varnothing$; or

(ii) For all $1 \leq j \leq k$, $a^{(j)} = \varnothing$.

By Lemma 4.1 and Lemma 4.2, we know each A_k is a subalgebra of $\mathfrak{B}_\omega$, and A_{k+1} is a subalgebra of A_k.

Proposition 4.1. For each $k \geq 1$, A_k is a subalgebra of $\mathfrak{B}_\omega$. Moreover, we have $A_1 = \mathfrak{B}_\omega$ and A_{k+l} is a subalgebra of A_k for any $k, l \geq 1$.

The key character of A_k is its *uniformity* on left edges $\mathsf{LE}_1, \mathsf{LE}_2, \cdots, \mathsf{LE}_k$. By the definition of A_k, for a region $a \in \mathfrak{B}_\omega$, a is in A_k if and only if $\{a^{(i)} : 1 \leq i \leq k\}$ are collectively empty or nonempty. Take A_2 as example. Then region $a = x_{01} \cup x_{100}$ is not in A_2. This is because $a^{(1)} = \varnothing$, but $a^{(2)} = x_{100}$. But $b = x_{00} \cup x_{10}$ is in A_2, since $b^{(1)} = x_{00}$ and $b^{(2)} = x_{10}$.

What should be stressed is, a connected region in A_{k+1} may be disconnected in A_k, hence disconnected in $\mathfrak{B}_\omega$. Take the above $b = x_{00} \cup x_{10}$ as an example. Since x_{00} is not in contact with x_{10} in $\mathfrak{B}_\omega$, we know b is not a connected region in $\mathfrak{B}_\omega$. But the next proposition shows that b is a connected region in A_2.

Proposition 4.2. For each $k \geq 1$ and any region $a \in A_k$, we have

(1) If $a^{(1)} = a^{(2)} = \cdots = a^{(k)} = \varnothing$, then a is connected in A_k if and only if a is connected in $\mathfrak{B}_\omega$.

(2) If $a = a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(k)}$, then a is connected in A_k.

Proof:
For the first case, if $a = b \cup c$, then we know b, c are also in A_k. This is because $b^{(i)} = c^{(i)} = \varnothing$ for each $1 \leq i \leq k$. Therefore b and c are in contact in $\mathfrak{B}_\omega$ if and only if they are so in A_k. So, by the definition of connectedness, a is connected in $\mathfrak{B}_\omega$ if and only if it is connected in A_k.

For the second case, suppose $b, c \subset a$ are two regions in A_k such that $b \cup c = a$. By Lemma 4.2 and $a^{(1)} \neq \varnothing$, we know either $b^{(1)}$ or $c^{(1)}$ is nonempty.

Suppose both $b^{(1)}$ and $c^{(1)}$ are nonempty. Then by $a^{(1)}$ is a connected region in $\mathfrak{B}_\omega$, we know $b^{(1)}$ and $c^{(1)}$ are in contact in $\mathfrak{B}_\omega$. So are b and c. Moreover, since $b, c \in A_k$, they are also in contact in A_k.

Suppose $b^{(1)}$ is empty. Then by $b, c \in A_k$ all $b^{(i)}$ are empty and all $c^{(i)}$ are nonempty for $1 \leq i \leq k$. Clearly $c^{(i)} = a^{(i)}$ cannot be true for all $1 \leq i \leq k$. This is because $a = a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(k)}$ and $c \neq a$. Without loss of generality, we assume $c^{(1)} \subsetneq a^{(1)}$. Then $c \cap a^{(1)} \neq \varnothing$, since $c^{(1)} \subseteq c$. Note that

$b \cap a^{(1)}$ is also nonempty. This is because otherwise we shall have $c \cap a^{(1)} = a^{(1)}$, hence $c^{(1)} = a^{(1)}$. Now, since $a^{(1)}$ is connected and $a^{(1)} = (b \cap a^{(1)}) \cup (c \cap a^{(1)})$, we know $b \cap a^{(1)}$ is in contact with $c \cap a^{(1)}$. This suggests b and c are in contact in $\mathfrak{B}_\omega$ and, hence, in A_k.

The situation when $c^{(1)}$ is empty is similar. □

Definition 4.2. (Basic regions in A_k)
For a region $a \in A_k$, we call a basic if $a = x_s$ for some binary string $s \neq \epsilon$ or $a = a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(k)}$. For easy of reference, we call regions in A_k like x_s Type-1 regions, and call the others Type-2 regions.

Note first that each Type-1 region x_s is connected in A_k. This is because x_s is connected $\mathfrak{B}_\omega$. By Proposition 4.2, we know Type-2 regions in A_k are also connected.

The following lemma then shows that each region in A_k can be decomposed into a collection of basic regions.

Lemma 4.3. For a basic region $a \in A_k$, a can be decomposed into a collection of disjoint basic regions, where 'disjoint' is in the sense of set theory. In particular, if $a^{(1)} \neq \varnothing$, then a can be represented as the union of one Type-2 region and a collection of Type-1 regions.

Proof:
Set $z_0 = a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(k)}$. Then z_0 is either empty or a Type-2 region. Since $(a - z_0)^{(i)} = \varnothing$ for $1 \leq i \leq k$, $a - z_0$ is also in A_k. Suppose $\{x_{s_1}, x_{s_2}, \cdots, x_{s_n}\}$ is a disjoint decomposition of $a - z_0$ in $\mathfrak{B}_\omega$, where s_i are binary strings. Then each x_{s_i} is in A_k, since it is a sub-region of $a - z_0$. In this way we decompose a into a collection of disjoint basic regions. □

Now we turn to the contact relation on A_k inherited from $\mathfrak{B}_\omega$. It is easy to show that the sub-model $\langle A_k, \mathbf{C}_\omega|_{A_k}\rangle$ is indeed an RCC model.

Theorem 4.1. For $k \geq 1$, $\langle A_k, \mathbf{C}_\omega|_{A_k}\rangle$ is an RCC model.

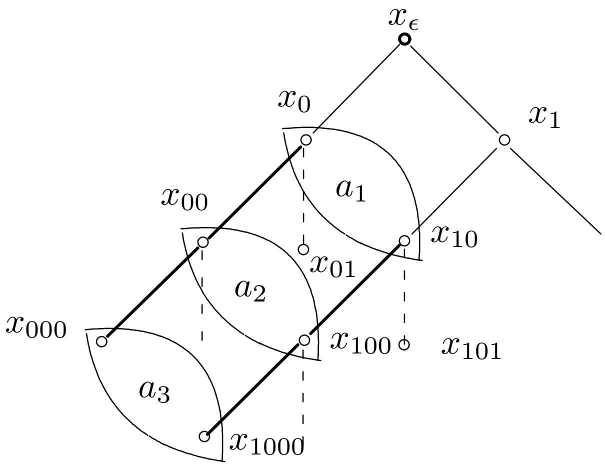
Proof:
We need only to show that, for any region $a \in A_k$, there exists $b \in A_k$ such that $b \ll a$. If all $a^{(i)}$ $(1 \leq i \leq k)$ are empty, let b be any region in $\mathfrak{B}_\omega$ such that $b \ll a$. Notice that $b \in A_k$ since all $b^{(i)}$ are empty. On the other hand, if all $a^{(i)}$ $(1 \leq i \leq k)$ are nonempty, that is $a^{(i)} = x_{s_i}$ for some $x_{s_i} \in \mathsf{LE}_i$. Let $b = \bigcup_{i=0}^{k-1} x_{s_i 0}$. Then $b \ll a$ and $b^{(i)} = x_{s_i 0} \neq \varnothing$ $(1 \leq i \leq k-1)$. Therefore we have some $b \in A_k$ with $b \ll a$ in both cases. □

As a consequence, we obtain a sequence of sub-RCC models of $\mathfrak{B}_\omega$. For convenience we write A_k for $\langle A_k, \mathbf{C}_\omega|_{A_k}\rangle$. It is natural to ask "*Are these models all different?*" In the next subsection, we prove that this is true.

4.3. Ntpp-chain and k-chain

In an RCC model, each region contains a non-tangential proper part. That is, for each region a, we can find another region a' such that $a'\mathbf{NTPP}a$, or $a' \ll a$ for short. This procedure can be continued for any length of steps. The following definition of *ntpp-chain* captures this notion.

Table 1. A segment of 2-chain in A_2



$$\begin{aligned} a_1 &= x_0 \cup x_{10} \\ a_2 &= x_{00} \cup x_{100} \\ a_3 &= x_{000} \cup x_{1000} \\ a_1 - a_2 &= x_{01} \cup x_{101} \end{aligned}$$

- $a_1 \gg a_2 \gg a_3$
- a_1, a_2, a_3 are connected in A_2
- x_{01}, x_{101} are connected in A_2
- $\mathbf{DC}(x_{01}, x_{101})$
- $\mathbf{C}(x_{01}, x_{11})$, $\mathbf{C}(x_{101}, x_{11})$
- $-a_1 = x_{11}$

Definition 4.3. (ntpp-chain)
Suppose A is an RCC model. A sequence of regions $\{a_i : i \geq 1\}$ in A is called an *ntpp-chain* if

$$1 > a_1 \gg a_2 \gg \cdots \gg a_i \gg a_{i+1} \gg \cdots .$$

An ntpp-chain is said to be *connected* if all a_i are connected.

The collection of RCC models $A_1, A_2, \cdots$ constructed in the above subsection can be differentiated by specific connected ntpp-chains.

Fix a $k \geq 2$, for each $i \geq 1$, let

$$a_i = x_{0^i} \cup x_{10^i} \cup \cdots \cup x_{1^{k-1}0^i}.$$

Then $\{a_i\}_{i \geq 1}$ is an ntpp-chain. By Proposition 4.2, we know each a_i is connected in A_k. Therefore $\{a_i\}_{i \geq 1}$ is a connected ntpp-chain in A_k.

We first note that, for any $k' > k$, each a_i is not a region in $A_{k'}$; and for any $k' < k$, each a_i is a disconnected region in $A_{k'}$. The latter statement follows from the observation that any two regions in $\{x_{0^i}, x_{10^i}, \cdots, x_{1^{k-1}0^i}\}$ are not in contact.

Second, we note that for $i < j$, $a_i \setminus a_j = \bigcup_{l=1}^{k} (x_{1^{l-1}0^i} - x_{1^{l-1}0^j})$. If we set $b_{i,j}^l = x_{1^{l-1}0^i} - x_{1^{l-1}0^j}$, then by Proposition 2.5 each $b_{i,j}^l$ is a connected region in A_k. By the definition of the contact relation $\mathbf{C}_\omega$ (Definition 2.3), we know any pair of $b_{i,j}^l$ are not in contact. Moreover, by $\mathbf{C}_\omega(x_{1^{l-1}0^i}, x_{1^{l-1}0^{i-1}1})$ and $x_{1^{l-1}0^j} \ll x_{1^{l-1}0^i}$, we know $\mathbf{C}_\omega(b_{i,j}^l, x_{1^{l-1}0^{i-1}1})$. But since $x_{1^{l-1}0^{i-1}1} \subset -a_i$, the complement of a_i, we know each $b_{i,j}^l$ is in contact with $-a_i$.

In summary, we have a connected ntpp-chain $\{a_i\}_{i \geq 1}$ in A_k such that the difference of any two a_i and a_j ($i < j$) contains precisely k connected components $\{b_{i,j}^l\}_{l=1}^k$ in A_k, where any two components are not in contact. Moreover, each component is in contact with the complement of a_i.

An illustration is given in Table 1 for the case $k = 2$.

We call the above specific ntpp-chain a *k-chain*.

Definition 4.4. (k-chain)
Suppose A is an RCC model. For $k \geq 1$, a connected ntpp-chain $\{a_i\}$ in A is called a k-chain, if for any $i < j$ there exists a partition $D^k_{i,j} = \{b^1_{i,j}, b^2_{i,j}, \cdots, b^k_{i,j}\}$ of $a_i - a_j$ satisfying:

1. each $b^l_{i,j}$ is a connected region in A;
2. $a_i - a_j = \bigcup_{l=1}^k b^l_{i,j}$ and any two $b^l_{i,j}$ are not in contact;
3. each $b^l_{i,j}$ is in contact with $-a_i$.

The above observation then can be summarized as the following theorem.

Theorem 4.2. For $k \geq 1$ and any $i \geq 1$, let $a_i = x_{0^i} \cup x_{10^i} \cup \cdots \cup x_{1^{k-1}0^i}$. Then $\{a_i\}_{i \geq 1}$ is a k-chain in A_k.

It is also clear that each A_k contains a 1-chain. For each $i \geq 1$, set $a_i = x_{1^k 0^i}$. Then it is routine to check that $\{a_i\}_{i \geq 1}$ is a 1-chain in A_k. So in what follows, we only consider k-chains for $k \geq 2$.

In the next subsection, we show the notion of k-chain can be used to differentiate A_k from all the other $A_{\hat{k}}$.

4.4. A_k has a $\hat{k}$-chain only if $k = \hat{k}$

To prove that A_k contains no $\hat{k}$-chain for any $1 < \hat{k} \neq k$, we need several lemmas.

Suppose $\{a_j\}_{j \geq 1}$ is a $\hat{k}$-chain in A_k. Set z_0 to be the union of all $a_1^{(i)}$ ($1 \leq i \leq k$). Then we claim (in Proposition 4.3) that z_0 is nonempty and the chain $\{a_j\}_{j \geq 1}$ will finally contained in z_0.

We achieve this result by two steps. Suppose $\{x_{s_1}, \cdots, x_{s_n}\}$ is a partition of $a_1 - z_0$, where s_i is a binary string. In this way, we decompose a_1 into disjoint basic regions in A_k (see Lemma 4.3). Then Lemma 4.4 asserts that the chain $\{a_j\}_{j \geq 1}$ will finally contained in either z_0 or some x_{s_i}, and Lemma 4.5 suggests that the latter is impossible because $a_j^{(1)} \neq \varnothing$ for any j.

Lemma 4.4. Given a connected ntpp-chain $\{a_j : j \geq 1\}$ in A_k, set $z_0 = \bigcup_{i=1}^k a_1^{(i)}$, and suppose $\{x_{s_1}, \cdots, x_{s_n}\}$ is a partition of $a_1 - z_0$, where s_i is a binary string. For $1 \leq i \leq n$, write $z_i = x_{s_i}$. Then there is a unique $f \in \{0, 1, \cdots, n\}$ such that $\{a_j\}$ is finally contained in z_f, that is, $a_j \subseteq z_f$ for all j large enough.

Proof:
Notice first that z_0, $a_1 - z_0$, $z_i = x_{s_i}$ ($1 \leq i \leq n$) are all regions in A_k.

For $i \geq 1$, we define

$$N_i = \{l : 0 \leq l \leq n, z_l \cap a_i \neq \varnothing\}.$$

Then N_i is nonempty and $N_i \subseteq N_{\hat{i}}$ if $i > \hat{i}$. Because $|N_i| \leq n+1$, there exists $m_0 > 0$ so that $N_m = N_{m_0}$ for any $m > m_0$. Write

$$N_\infty = \{l : 0 \leq l \leq n, (\forall i) z_l \cap a_i \neq \varnothing\}.$$

Then $N_\infty = \bigcap_{i=1}^{\infty} N_i = N_{m_0} \neq \varnothing$. Our aim is to prove $|N_\infty| = 1$. We show this by reduction to absurdity.

Suppose $|N_\infty| > 1$. There exists some $f \in N_\infty$ such that $f > 0$. We assert that $\forall j \geq 1$ and for any $0 < f \in N_\infty$, a_j contains a boundary of z_f.

From the definition of N_{m_0}, $a_j \cap z_f \neq \varnothing$. If a_j contains no boundary of z_f, then by Proposition 2.3 $a_j \cap z_f \ll z_f$, hence $\mathbf{DC}(a_j \cap z_f, -z_f)$. Notice that $a_j - z_f \neq \varnothing$ since $|N_\infty| > 1$. So we have $\mathbf{DC}(a_j \cap z_f, a_j - z_f)$. Since both $a_j \cap z_f$ and $a_j - z_f$ are regions in A_k, this contradicts the connectedness of a_j.

Denote $z_\infty = \bigcup\{z_f : f \in N_\infty, f > 0\}$. Notice that $z_\infty \cup z_0 \subseteq a_1 \neq x_\epsilon$, we have $-z_0 - z_\infty = -(z_0 \cup z_\infty) \neq \varnothing$. For each $i \geq 0$, recall $a_1^{(i)} = \varnothing$ or $a_1^{(i)} = x_{s0}$ for some s. By Proposition 2.5, we know $-z_0 = x_\epsilon - a_1^{(1)} - \cdots - a_1^{(k)}$ is also connected in $\mathfrak{B}_\omega$, hence connected in A_k. By $\varnothing \neq z_\infty \subseteq -z_0$, we have $\mathbf{EC}(z_\infty, -z_0 - z_\infty)$. Now since $z_\infty = \bigcup\{z_f : f \in N_\infty, f > 0\}$, we have $\mathbf{EC}(z_f, -z_0 - z_\infty)$ for some positive $f \in N_\infty$. Recall that each a_j contains a boundary of z_f for any positive $f \in N_\infty$. From $\mathbf{EC}(z_f, -z_0 - z_\infty)$ we know $\mathbf{EC}(a_j \cap z_f, -z_0 - z_\infty)$ holds for any j. This contradicts the assumption that $\{a_j\}$ is a connected ntpp-chain. □

Lemma 4.5. Suppose $\{a_j : j \geq 1\}$ is a $\hat{k}$-chain in A_k, $k \geq 2$. Then $a_j^{(l)} \neq \varnothing$ holds for any $j \geq 1$ and any $1 \leq l \leq k$.

Proof:
Suppose there exist some $j^* \geq 1$ and some $1 \leq l^* \leq k$ such that $a_{j^*}^{(l^*)} = \varnothing$. By the definition of $\hat{k}$-chain, we know for any $j \geq j^*$ and any $1 \leq l \leq k$, $a_j^{(l)} = \varnothing$. Set $b_j = a_{j+j^*-1}$ ($j \geq 1$). Then $\{b_j : j \geq 1\}$ is a $\hat{k}$-chain such that $b_j^{(l)} = \varnothing$ holds for any $j \geq 1$ and any $1 \leq l \leq k$. So we can assume without loss of generality that $a_j^{(l)} = \varnothing$ holds for any $j \geq 1$ and any $1 \leq l \leq k$.

Let $\{x_{s_1}, x_{s_2}, \cdots, x_{s_n}\}$ be a decomposition of a_1. Then $x_{s_i} \notin \mathsf{LE}_l$ for any $1 \leq l \leq k$. Write $z_i = x_{s_i}$. Then $z_i \in A_k$ ($1 \leq i \leq n$). By Lemma 4.4, we know $\{a_j\}$ are eventually contained in, say z_1. In other words, there exists some m such that $a_j \ll z_1$ for any $j \geq m$.

By the definition of $\hat{k}$-chain, we can decompose $a_1 - a_m$ into $\hat{k}$ separate connected components $\{b_{1,m}^i : 1 \leq i \leq \hat{k}\}$ such that $a_1 - a_m = \bigcup\{b_{1,m}^i : 1 \leq i \leq \hat{k}\}$ and $\mathbf{C}(b_{1,m}^i, -a_1)$ ($1 \leq i \leq \hat{k}$).

We first prove that $b_{1,m}^i \cap z_1 \neq \varnothing$ for all i. This is because, if $b_{1,m}^{i^*} \cap z_1 = \varnothing$ for some i^*, then by $a_m \ll z_1$, we know $\mathbf{DC}(b_{1,m}^{i^*}, a_m)$. Since any two $b_{1,m}^i$ are not in contact, we have $\mathbf{DC}(b_{1,m}^{i^*}, a_1 - a_m - b_{1,m}^{i^*})$, hence $\mathbf{DC}(b_{1,m}^{i^*}, a_1 - b_{1,m}^{i^*})$. This contradicts the connectedness of a_1. So for any $1 \leq i \leq \hat{k}$ we have $b_{1,m}^i \cap z_1 \neq \varnothing$.

From $a_m \ll z_1 \subseteq a_1$ and $a_1 - a_m = \bigcup_{i=1}^{\hat{k}} b_{1,m}^i$, we know $\{a_m, b_{1,m}^1 \cap z_1, \cdots, b_{1,m}^{\hat{k}} \cap z_1\}$ is a partition of z_1. Noticing z_1 has form x_{s_1}, by Proposition 2.4 and $a_m \ll z_1$, there exists i^* such that $b_{1,m}^{i^*}$ contains a boundary of z_1, and $b_{1,m}^i \cap z_1 \ll z_1$ for all $i \neq i^*$. Notice that if $b_{1,m}^i - z_1$ is nonempty either, then $\mathbf{EC}(b_{1,m}^i - z_1, b_{1,m}^i \cap z_1)$ and $\mathbf{EC}(b_{1,m}^i - z_1, z_1)$ since $b_{1,m}^i$ is connected. This suggests that $b_{1,m}^i \cap z_1 \ll z_1$ cannot hold. Hence $b_{1,m}^i - z_1 = \varnothing$ for all $i \neq i^*$. Therefore $b_{1,m}^i \subseteq z_1$, hence $b_{1,m}^i \ll z_1$, for all $i \neq i^*$. Note that, since $\hat{k} \geq 2$, there exists $i^\star \neq i^*$. Now because $z_1 \subseteq a_1$, we have $b_{1,m}^{i^\star} \ll a_1$, hence $\mathbf{DC}(b_{1,m}^{i^\star}, -a_1)$, this contradicts Item (3) of Definition 4.4. □

By Lemma 4.5 and Lemma 4.4, we have the following proposition, which asserts that a $\hat{k}$-chain will finally contained in a Type-2 region.

Proposition 4.3. Suppose $\{a_j : j \geq 1\}$ is a $\hat{k}$-chain in A_k, set $z_0 = \bigcup_{i=1}^k a_1^{(i)}$. Then $\{a_j\}$ is eventually contained in z_0, that is there exists some $j^* \geq 1$ such that $a_j \subseteq z_0$ holds for any $j \geq j^*$.

Proof:
By Lemma 4.4, we know $\{a_j\}$ is eventually contained in z_f for some $f \in \{0, 1, \cdots, n\}$, that is $a_j \subseteq z_f$ for all bigger enough j. Now, recall Lemma 4.5 asserts that $a_j^{(l)} \neq \varnothing$ holds for any $j \geq 1$ and any $1 \leq l \leq k$. This suggests $a_j \cap z_0 \neq \varnothing$ for all j. As a result, we have $f = 0$ and $\{a_j\}$ is eventually contained in z_0. □

We now prove the main theorem of this subsection, which asserts that there is no $\hat{k}$-chain in A_k for any $1 < \hat{k} \neq k$.

Theorem 4.3. For $k, \hat{k} \geq 2$, if there is a $\hat{k}$-chain in A_k, then $k = \hat{k}$.

Proof:
Suppose $\{a_j : j \geq 1\}$ is a $\hat{k}$-chain in A_k, set $z_0 = \bigcup_{i=1}^k a_1^{(i)}$. Then by Proposition 4.3 and $a_{j+1} \ll a_j$ ($j \geq 1$), there exists $m \geq 1$ such that $a_j \ll z_0$ for any $j \geq m$.

By the definition of $\hat{k}$-chain, we have a partition of $a_1 - a_m$, say $\{b_{1,m}^l : 1 \leq l \leq \hat{k}\}$, such that (i) each $b_{1,m}^l$ is a connected region; (ii) any two are not in contact; and (iii) each $b_{1,m}^l$ is in external contact with $-a_1$.

We first prove $\hat{k} \leq k$ by showing

(1) $b_{1,m}^l \cap z_0 \neq \varnothing$ for any $1 \leq l \leq \hat{k}$; and

(2) for any $i \in \{1, 2, \cdots, k\}$ there exists a unique $l \in \{1, 2, \cdots, \hat{k}\}$ such that $a_1^{(i)} \cap b_{1,m}^l \neq \varnothing$.

Notice that if (1) and (2) hold, then there is a surjection from $\{1, 2, \cdots, k\}$ onto $\{1, 2, \cdots, \hat{k}\}$, hence $\hat{k} \leq k$.

Suppose $b_{1,m}^{l^*} \cap z_0 = \varnothing$ for some $1 \leq l^* \leq \hat{k}$. By $a_m \ll z_0$, we have $\mathbf{DC}(b_{1,m}^{l^*}, a_m)$. Recall any two regions in $\{b_{1,m}^l\}_{l=1}^{\hat{k}}$ are not in contact. We have $\mathbf{DC}(b_{1,m}^{1^*}, a_1 - b_{1,m}^{1^*})$, contradicting the connectedness of a_1. This is because $a_1 - b_{1,m}^{1^*} = a_m \cup \bigcup\{b_{1,m}^1 : l \neq l^*, 1 \leq l \leq \hat{k}\}$. Hence $b_{1,m}^l \cap z_0 \neq \varnothing$ holds for any l. Therefore (1) has been justified.

Next we show (2) holds. Suppose there exist $1 \leq i \leq k$ and $1 \leq l \neq l' \leq \hat{k}$ such that $b_{1,m}^l \cap a_1^{(i)} \neq \varnothing$ and $b_{1,m}^{l'} \cap a_1^{(i)} \neq \varnothing$. By Proposition 2.4 we have either $b_{1,m}^l \cap a_1^{(i)}$ or $b_{1,m}^{l'} \cap a_1^{(i)}$ is an ntpp of $a_1^{(i)}$. Take $b_{1,m}^l \cap a_1^{(i)} \ll a_1^{(i)}$ for example. Notice that if $b_{1,m}^l - a_1^{(i)} \neq \varnothing$, then by $\mathbf{DC}(b_{1,m}^l \cap a_1^{(i)}, -a_1^{(i)})$ we have $\mathbf{DC}(b_{1,m}^l \cap a_1^{(i)}, b_{1,m}^l - a_1^{(i)})$. This is impossible since $b_{1,m}^l$ is a connected region in A_k. So we have $b_{1,m}^l - a_1^{(i)} = \varnothing$, that is $b_{1,m}^l \subseteq a_1^{(i)}$. By $b_{1,m}^l \cap a_1^{(i)} \ll a_1^{(i)}$ we have $b_{1,m}^l \ll a_1^{(i)} \subseteq a_1$. This is also impossible since $b_{1,m}^l$ is in contact with $-a_1$.

Above we have shown $\hat{k} \leq k$, we next prove $\hat{k} \geq k$. By the definition of $\hat{k}$-chain, we have a partition of $a_m - a_{m+1}$, say $\{b_{m,m+1}^l : 1 \leq l \leq \hat{k}\}$, such that (i) each $b_{m,m+1}^l$ is a connected region; (ii) any two are not in contact; and (iii) each $b_{m,m+1}^l$ is in external contact with $-a_m$.

We prove $\hat{k} \geq k$ by showing

(3) for any $1 \le i \le k$, there exists some $1 \le l \le \hat{k}$ such that $a_1^{(i)} \cap b_{m,m+1}^l \ne \varnothing$; and

(4) for any $1 \le l \le \hat{k}$ there exists a unique $1 \le i \le k$ such that $a_1^{(i)} \cap b_{m,m+1}^l \ne \varnothing$.

Notice that if (3) and (4) hold, then there is a surjection from $\{1, 2, \cdots, \hat{k}\}$ onto $\{1, 2, \cdots, k\}$, hence $\hat{k} \ge k$.

To show (3) is right, we need only to prove that $a_1^{(i)} \cap (a_m - a_{m+1}) \ne \varnothing$ for any $1 \le i \le k$. Suppose not so. Then there exists some $1 \le i \le k$ such that $a_1^{(i)} \cap (a_m - a_{m+1}) = \varnothing$, or, equivalently, $a_1^{(i)} \cap a_m = a_1^{(i)} \cap a_{m+1}$. Note that $a_1^{(i)} \cap a_{m+1} \ne \varnothing$ since $a_{m+1}^{(i)} \ne \varnothing$ by Lemma 4.5. Note also that by $a_1^{(i)} \cap a_m = a_1^{(i)} \cap a_{m+1}$, we have $a_1^{(i)} - a_m = a_1^{(i)} - a_{m+1}$. If $a_1^{(i)} - a_{m+1} = \varnothing$, then $a_1^{(i)} \subseteq a_{m+1} \ll a_m$. Recall $a_m \ll z_0 = a_1^{(1)} \cup \cdots \cup a_1^{(k)}$, this gives a contradiction since $a_1^{(i)} \not\ll z_0$ for any i. On the other hand, if $a_1^{(i)} - a_{m+1} \ne \varnothing$, then $\mathbf{EC}(a_1^{(i)} \cap a_m, a_1^{(i)} - a_m)$ and $\mathbf{EC}(a_1^{(i)} \cap a_{m+1}, a_1^{(i)} - a_{m+1})$ since $a_1^{(i)}$ is connected in $\mathfrak{B}_\omega$. This further shows that $a_1^{(i)} - a_m = a_1^{(i)} - a_{m+1} \in A_k$ is in external contact with both a_m and a_{m+1} in $\mathfrak{B}_\omega$ therefore in A_k. This contradicts $a_{m+1} \ll a_m$.

We next show (4) is also right. For any $1 \le l \le \hat{k}$, since $b_{m,m+1}^l$ is connected in A_k and $b_{m,m+1}^l \subseteq a_m \ll z_0 = a_1^{(1)} \cup \cdots \cup a_1^{(k)}$, we know $b_{m,m+1}^l = (b_{m,m+1}^l \cap a_1^{(1)}) \cup \cdots \cup (b_{m,m+1}^l \cap a_1^{(k)})$. Recall any two regions in $\{a_1^{(i)}\}_{i=1}^k$ are not in contact. There cannot exist two different $1 \le i, j \le k$ such that both $a_1^{(i)} \cap b_{m,m+1}^l$ and $a_1^{(j)} \cap b_{m,m+1}^l$ are nonempty.

In summary, we know $\hat{k} \ge k$. Combining with the early result that $\hat{k} \le k$, we have $\hat{k} = k$. □

5. Minimal RCC model $\mathfrak{B}_\$$

So far we have constructed a sequence of countable RCC models $A_1, A_2, \cdots, A_k, \cdots$ such that

- Each A_k is a sub-model of $\mathfrak{B}_\omega$;
- $A_1 = \mathfrak{B}_\omega$ and A_{k+1} is a sub-model of $\mathfrak{B}_\omega$;
- For $k, \hat{k} \ge 2$, A_k contains a $\hat{k}$-chain if and only if $k = \hat{k}$.

Since the notion of k-chain is RCC invariant, $A_1, A_2, \cdots, A_k, \cdots$ are pairwise non-isomorphic.

Based on these A_k, we construct in this section continuum many non-isomorphic RCC models. The strategy is, given any binary ω-string $\$ = w_1 w_2 w_3 \cdots$ over $\{0, 1\}$, constructing a sub-model, $\mathfrak{B}_\$$, of $\mathfrak{B}_\omega$. Roughly speaking, $\mathfrak{B}_\$$ is obtained by replacing, for each i with $w_i = 1$, the local structure of $\mathfrak{B}_\omega$ at $x_{1^{i-1}0}$ by A_i. For $k > 1$, we can show that the sub-model $\mathfrak{B}_\$$ contains a k-chain if and only if $w_k = 1$. This suggests that no two such models are isomorphic. Since $\mathfrak{B}_\omega$ itself is a minimal RCC model, each $\mathfrak{B}_\$$ is also a minimal RCC model.

We now examine in detail the construction procedure.

To begin with, we consider the local structure of $\mathfrak{B}_\omega$ at x_s, where s is a binary string. Write $\mathfrak{B}_s = \{a \in \mathfrak{B}_\omega : a \subseteq x_s\}$. Then $\mathfrak{B}_s$ is a subalgebra of the powerset algebra 2^{x_s}. Note that there is a bijection $\eta_s : \mathfrak{B}_s \to \mathfrak{B}_\omega$, which is defined as follows

$$(\forall a \in \mathfrak{B}_s)\ \ \eta_s(a) = \bigcup\{x_{s'} \in \mathfrak{B}_\omega : x_{ss'} \subseteq a\}. \tag{5}$$

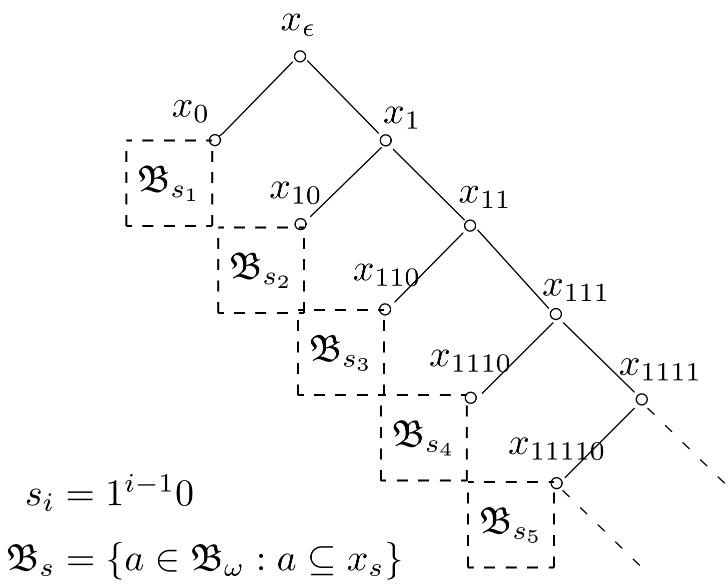


Figure 5. Local structures of $\mathfrak{B}_\omega$ at $x_{1^{i-1}0}$

It is straightforward to show that η_s is a Boolean isomorphism. Note that η_s^{-1}, the inverse mapping of η, maps x_ϵ to x_{s_i}.

Set $s_i = 1^{i-1}0$ ($i \geq 1$). We are interested in local structures at these x_{s_i}. (See Figure 5.) For any ω-string \$, we now define $\mathfrak{B}_\$$ to be the subset of $\mathfrak{B}_\omega$ obtained by replacing (when $w_i = 1$) each local structure $\mathfrak{B}_{s_i}$ by $\eta_{s_i}^{-1}(A_i)$. Recall that A_i is a subalgebra of $\mathfrak{B}_\omega$. We know $\eta_{s_i}^{-1}(A_i)$ is a subalgebra of $\mathfrak{B}_{s_i}$.

Definition 5.1. Suppose $\$ = w_1w_2w_3\cdots$ ($w_i \in \{0,1\}$) is a binary ω-string. Define a subset $\mathfrak{B}_\$$ of $\mathfrak{B}_\omega$ as follows:

$$(\forall a \in \mathfrak{B}_\omega)\ \ [a \in \mathfrak{B}_\$ \Leftrightarrow (\forall i)[w_i = 1 \rightarrow a \cap x_{1^{i-1}0} \in \eta_{s_i}^{-1}(A_i)]] \tag{6}$$

Suppose a is a sub-region of x_{s_i}. By the above definition, we have (i) if $w_i = 1$, then $a \in \mathfrak{B}_\$$ iff $a \in \eta_{s_i}^{-1}(A_i)$; and (ii) if $w_i = 0$, then a is always in $\mathfrak{B}_\$$.

Lemma 5.1. For any binary ω-string \$, $\mathfrak{B}_\$$ is a subalgebra of $\mathfrak{B}_\omega$.

Proof:
Recall that each η_{s_i} is a Boolean isomorphism. We have in particular $\eta_{s_i}(\varnothing) = \varnothing$, $\eta_{s_i}(x_{1^{i-1}0}) = x_\epsilon$. This shows $\varnothing$ and x_ϵ are elements in $\mathfrak{B}_\$$. Moreover, for any two $a, b \in \mathfrak{B}_\$$, since

$$\begin{aligned}
\eta_{s_i}(-a \cap x_{1^{i-1}0}) &= \eta_{s_i}(-(a \cap x_{1^{i-1}0}) \cap x_{1^{i-1}0}) \\
&= -\eta_{s_i}(a \cap x_{1^{i-1}0}) \\
\eta_{s_i}((a \cup b) \cap x_{1^{i-1}0}) &= \eta_{s_i}((a \cap x_{1^{i-1}0}) \cup (b \cap x_{1^{i-1}0})) \\
&= \eta_{s_i}(a \cap x_{1^{i-1}0}) \cup \eta_{s_i}(b \cap x_{1^{i-1}0})
\end{aligned}$$

we know $-a, a \cup b \in \mathfrak{B}_\$$. Therefore $\mathfrak{B}_\$$ is a subalgebra of $\mathfrak{B}_\omega$. □

For any region a in $\mathfrak{B}_\$$, there exists some $i \geq 1$ such that $a \cap x_{s_i} \neq \varnothing$. Now since both $\mathfrak{B}_{s_i}$ and $\eta_{s_i}(\mathfrak{B}_{s_i})$ are RCC models, there is a region $b \in \mathfrak{B}_\$$ such that $b \ll a \cap x_{s_i} \subseteq a$. So we know $\mathfrak{B}_\$$ is a sub-RCC model of $\mathfrak{B}_\omega$.

What remains to prove is to show no two such models are isomorphic. Suppose $\$ = 1w_2w_3\cdots$ and $\$' = 1w_2'w_3'\cdots$. If $\$ \neq \$'$, then there exists some $k > 1$ such that $w_k \neq w_k'$. Note that if we can prove that there is a k-chain in $\mathfrak{B}_\$$ iff $w_k = 1$. Then only one of $\mathfrak{B}_\$$ and $\mathfrak{B}_{\$'}$ contains a k-chain. Therefore they are non-isomorphic.

We now study in detail properties of k-chain in a sub-model $\mathfrak{B}_\$$. The following lemmas show that if $\mathfrak{B}_\$$ contains a k-chain, say $\{a_i\}_{i\geq 1}$, then $w_k = 1$ and $\{a_i\}_{i\geq 1}$ is finally contained in $x_{1^{k-1}0}$. Recall the construction of $\mathfrak{B}_\$$. We know, if $w_k = 1$, then the local structure of $\mathfrak{B}_\$$ at $x_{1^{k-1}0}$ is isomorphic to A_k. So $\mathfrak{B}_\$$ contains a k-chain if $w_k = 1$.

For any region $a \in \mathfrak{B}_\omega$, recall we define $a^{(\infty)} = \bigcup\{x_{1^l} : x_{1^l} \subseteq a, l \geq 0\}$, and say a contains a boundary of the universe if $a^{(\infty)} \neq \varnothing$. The following lemma shows that an ntpp-chain in $\mathfrak{B}_\omega$ finally contains no boundary of the universe.

Lemma 5.2. Suppose $\{a_j\}_{j\geq 1}$ is an ntpp-chain in $\mathfrak{B}_\omega$. Then there exists $m \geq 1$ such that $a_j^{(\infty)} = \varnothing$ for any $j \geq m$.

Proof:
We prove this by reduction to absurdity. Suppose all $a_j^{(\infty)}$ are nonempty. We now prove by induction that a_1 contains a boundary of x_s for each s. This suggests a_1 is in contact with each x_s, which contradicts the assumption that $a_1 \neq x_e$.

To begin with, notice that a contains a boundary of x_{s1} iff a contains a boundary of x_s. By our assumption, we know all a_j contains a boundary of x_1. Note that if $a_j \cap x_0 = \varnothing$ for some $j \geq 1$, then $\mathbf{EC}(x_0, a_j)$ and $\mathbf{EC}(x_0, a_{j+1})$ hold since both a_j and a_{j+1} contain a boundary of x_1. This contradicts the assumption that $a_{j+1} \ll a_j$. So $a_j \cap x_0 \neq \varnothing$ for any j. Furthermore, if a_m contains no boundary of x_0, then $x_0 - a_m \neq \varnothing$. By the connectedness of x_0 we have $\mathbf{EC}(a_m \cap x_0, x_0 - a_m)$, hence $\mathbf{EC}(a_m, x_0 - a_m)$. Because $a_{m+1} \ll a_m$, we have $\mathbf{DC}(a_{m+1}, x_0 - a_m)$. This is impossible, since $\mathbf{C}(x_0, x_1)$ and a_{m+1} contains a boundary of x_1, $x_0 - a_m$ contains a boundary of x_0. As a result we know all a_j contains a boundary of x_0.

In general, assuming all a_j contains a boundary of x_s, we show a_j contains a boundary of x_{s0}. Firstly all $a_j \cap x_{s0}$ are nonempty. Recall a_j also contains a boundary of x_{s1}. This follows from the same justification as for $a_j \cap x_0 \neq \varnothing$. Now if a_m contains no boundary of x_{s0}, then $x_{s0} - a_m \neq \varnothing$, hence $\mathbf{EC}(a_m, x_{s0} - a_m)$. By $a_{m+1} \ll a_m$, we have $\mathbf{DC}(a_{m+1}, x_{s0} - a_m)$. This contradicts the assumption that a_{m+1} contains a boundary of x_{s1} and that $\mathbf{C}(x_{s1}, x_{s0})$. Therefore all x_j contains a boundary of x_{s0}. □

Lemma 5.3. Given $\$$ a binary ω-string, suppose a is a connected region in $\mathfrak{B}_\$$ and $a \cap x_{1^q} = \varnothing$ for some $q \geq 1$. Then $a \subseteq x_{1^p0}$ for some $0 \leq p \leq q$.

Proof:
Note that any two regions in $\{x_{1^i0}\}_{i\geq 0}^q$ are not in contact. Hence any two regions in $\{a \cap x_{1^i0}\}_{i\geq 0}^q$ are not in contact. Recall that $a = \bigcup_{i=0}^q a \cap x_{1^i0}$ is connected. Only one of these regions, say $a \cap x_{1^p0}$, is nonempty. Hence $a \subseteq x_{1^p0}$ holds. □

If the ntpp-chain $\{a_j\}_{j\geq 1}$ is connected in $\mathfrak{B}_\$$, then the following lemma shows that the chain will be finally contained in x_{1^p0} for some $p \geq 0$. Note that by the definition of $\mathfrak{B}_\$$, all x_{1^p0} are (connected) regions in $\mathfrak{B}_\$$.

Lemma 5.4. Given a binary ω-string \$, suppose $\{a_j\}_{j\geq 1}$ is a connected ntpp-chain in $\mathfrak{B}_\$$. Then there exist $m \geq 1$ and $p \geq 0$ such that $a_j \subseteq x_{1^p0}$ for all $j \geq m$.

Proof:
By Lemma 5.2, we know there exists a_j that contains no boundary of x_1, so does each $a_{j'}$ for $j' \geq j$. Without loss of generality, we assume all a_j contains no boundary of x_1. Suppose q is the smallest number such that $a_1 = \bigcup\{x_s \subseteq a_1 : |s| = q\}$, where $|s|$ denotes the length of s. Since a_1 contains no boundary of x_1, we have $a_1 \cap x_{1^q} = \varnothing$, that is $a_1 \subseteq -x_{1^q} = \bigcup_{0\leq i<q} x_{1^i0}$. By Lemma 5.3 and the connectedness of a_1, we have a_1, hence all a_j, are included in x_{1^p0} for some $0 \leq p \leq q$. □

Moreover, if $\{a_j\}_{j\geq 1}$ is a k-chain in $\mathfrak{B}_\$$, then w_k must be 1, and the chain is finally contained in $x_{1^{k-1}0}$.

Lemma 5.5. Given a binary ω-string $\$ = w_1w_2\cdots$, suppose $\{a_j\}_{j\geq 1}$ is a k-chain ($k \geq 2$) in $\mathfrak{B}_\$$. Then $w_k = 1$ and there exists some $m \geq 1$ such that $a_j \subseteq x_{1^{k-1}0}$ for all $j \geq m$.

Proof:
By Lemma 5.4 we know there exist some $m \geq 1$ and some $p \geq 0$ such that $a_j \subseteq x_{1^p0}$ for all $j \geq m$. If $w_{p+1} = 0$, then $b \in \mathfrak{B}_\$$ for any $b \subseteq x_{1^p0}$. Moreover, for any $b \subseteq x_{1^p0}$, b is connected in $\mathfrak{B}_\$$ only if b is connected in $\mathfrak{B}_\omega$. This suggests $\{a_j\}$ is also a k-chain in $\mathfrak{B}_\omega$. Recall $\mathfrak{B}_\omega = A_1$, we know by Theorem 4.3 that $k = 1$, a contradiction. On the other hand, suppose $w_{p+1} = 1$. For any $b \in \mathfrak{B}_\$$ with $b \subseteq x_{1^p0}$, we have $\eta_{s_{p+1}}(b) \in \mathfrak{B}_{p+1}$, where $\eta_{s_{p+1}} : \mathfrak{B}_{1^p0} \to A_{p+1}$ is the isomorphism defined in Equation 5. It is also clear that if $b \in \mathfrak{B}_\$$, $b \subseteq x_{1^p0}$ is connected in $\mathfrak{B}_\$$, then $\eta_{s_{p+1}}(b)$ is connected in A_{p+1}. This shows $\{\eta_{s_{p+1}}(a_j)\}$ is a k-chain in A_{p+1}. By Theorem 4.3 again we know $p+1 = k$ and $a_j \subseteq x_{1^{k-1}0}$ for all $j \geq m$. □

Now we can prove that for any $k \geq 2$, the RCC model $\mathfrak{B}_\$$ contains a k-chain if and only if $w_k = 1$.

Theorem 5.1. Given a binary ω-string \$, suppose $k \geq 2$. Then $\mathfrak{B}_\$$ has a k-chain if and only if $w_k = 1$.

Proof:
By Lemma 5.5, we know if $\mathfrak{B}_\$$ has a k-chain, then $w_k = 1$. On the other hand, suppose $w_k = 1$, we can find a k-chain. Recall by Theorem 4.2 there is a k-chain, say $\{a_j\}$ in A_k. It is straightforward to check $\{\eta_{s_k}^{-1}(a_j)\}$ is also a k-chain in $\mathfrak{B}_\$$. □

By this theorem, we obtain our main result in this section:

Theorem 5.2. Given two ω-strings \$ and \$′ started with 1, if $\$ \neq \$'$, then $\mathfrak{B}_\$$ is not isomorphic to $\mathfrak{B}_{\$'}$.

Since there are continuum many ω-strings started with 1, we have constructed continuum many non-isomorphic RCC models. Now since each $\mathfrak{B}_\$$ is a sub-model of $\mathfrak{B}_\omega$, which is a minimal model by Li and Ying [5], we know these models are also minimal. Hence there are continuum many non-isomorphic minimal RCC models.

6. Conclusion

This paper studied minimal models of the RCC theory. We first proved that the exotic RCC model described by Gotts in [3] is isomorphic to the minimal RCC model $\mathfrak{B}_\omega$ given in [5]. Then we constructed continuum many sub-RCC models of $\mathfrak{B}_\omega$. These sub-models are also minimal models. Based on the key notion of k-chain, we showed that no two such models are isomorphic.

Recall that there is up to isomorphism only one atomless countable Boolean algebra. There are at most continuum many non-isomorphic countable RCC models. Now since each minimal model is countable, our result then suggests that there are exactly continuum many different countable RCC models as well as minimal RCC models.

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