

Dominating Manipulations in Voting with Partial Information

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Manipulation with full/no information

We first investigate the following two special cases: (1) the manipulator knows the profile of the other voters, and (2) the manipulator knows nothing about the preferences of the non-manipulators. The formal case corresponds to $\mathcal{I} = \{\{P\} : P \in \mathcal{F}_n\}$, and the latter case corresponds to $\mathcal{I} = \{\mathcal{F}_n\}$.

When the manipulator has full information about the non-manipulators' votes, the DOMINANT-MANIPULATION problem reduces to the standard manipulation problem. Therefore, we immediately obtain the following proposition from the Gibbard-Satterthwaite theorem (Gibbard 1973; Satterthwaite 1975).

Proposition 1. *When the manipulator has full information about the non-manipulators' votes, and $m \geq 3$, any voting rule that satisfies non-imposition and is immune to dominant-manipulation if and only if it is a dictatorship.*

It is easy to see that for any voting rule r under which computing the winner is in \mathbf{P} , DOMINATION is in \mathbf{P} ; and if the coalitional manipulation problem is NP-complete (respectively, \mathbf{P}), then DOMINANT-MANIPULATION is in NP-complete (respectively, \mathbf{P}). Therefore, the following proposition immediately follows the computational complexity of the coalitional manipulation problems for some common voting rules (Bartholdi, Tovey, and Trick 1989; Bartholdi and Orlin 1991; Conitzer, Sandholm, and Lang 2007; Zuckerman, Procaccia, and Rosenschein 2009; Xia et al. 2009).

Proposition 1. *When the manipulator has full information, computing DOMINANT-MANIPULATION is NP-complete for STV and ranked pairs; and it is in \mathbf{P} for Copeland, STV, positional scoring rules, plurality with Runoff, cup, maximin, ranked pairs, and Bucklin.*

Now we move to the case where the manipulator has no information about the non-manipulator's votes. We first prove a positive result, which states that any Condorcet consistent voting rule is immune to dominant-manipulation.

Theorem 1. *When the manipulator has no information about the non-manipulators' votes, any Condorcet consistent voting rule r is immune to dominant-manipulation.*

Proof. For the sake of contradiction, let U dominates V_M . Since $U \neq V_M$, there exist two alternatives a and b such that $a \succ_{V_M} b$ and $b \succ_U a$. We prove the theorem in the following two cases.

Case 1: n is even. For any j such that $1 \leq j \leq n/2$, we let $V_{2j-1} = [a \succ b \succ (\mathcal{C} \setminus \{a, b\})]$, where the alternatives in $\mathcal{C} \setminus \{a, b\}$ are ranked according to the ascending order of their subscripts; let $V_{2j} = [b \succ a \succ \text{Rev}(\mathcal{C} \setminus \{a, b\})]$. Here $\text{Rev}(\mathcal{C} \setminus \{a, b\})$ is the reverse of $\mathcal{C} \setminus \{a, b\}$. Let $P = (V_1, \dots, V_n)$. It follows that a is the Condorcet winner for $P \cup \{V_M\}$ and b is the Condorcet winner for $P \cup \{U\}$. Since $a \succ_{V_M} b$, V_M is not dominated by U , which contradicts the assumption.

Case 2: n is odd. For any j such that $1 \leq j \leq (n-1)/2$, we let $V_{2j-1} = [a \succ b \succ (\mathcal{C} \setminus \{a, b\})]$ and $V_{2j} = [b \succ a \succ \text{Rev}(\mathcal{C} \setminus \{a, b\})]$. Suppose $a = c_{i_1}$ and $b = c_{i_2}$. Let $V_n = \begin{cases} V_1 & \text{if } i_1 > i_2 \\ V_2 & \text{if } i_1 < i_2 \end{cases}$. Let $P = (V_1, \dots, V_n)$. It follows that a is the Condorcet winner for $P \cup \{V_M\}$ and b is the Condorcet winner for $P \cup \{U\}$, which contradicts the assumption. \square

Theorem 2. *When the manipulator has no information about the non-manipulators' votes, Borda is immune to dominant-manipulation.*

Proof. For the sake of contradiction, let U dominates V_M . Since $U \neq V_M$, there exist $i^* \leq m$ such that $\text{Alt}(V_M, i^*) \neq \text{Alt}(U, i^*)$ and for any $i < i^*$, $\text{Alt}(V_M, i) = \text{Alt}(U, i)$. That is, i^* is the first position from top where the alternatives in V_M and U are different. Let $c_{i_1} = \text{Alt}(V_M, i^*)$ and $c_{i_2} = \text{Alt}(U, i^*)$. We prove the theorem in the following three cases.

Case 1: n is even. For any $i < i' \leq m$, let $V_M^{[i, i']}$ denote the sub-linear-order of V_M that starts at the i th position of V_M and ends at the i' th position of V_M . For any j such that $1 \leq j \leq n/2$, we let $V_{2j-1} = [V_M^{[i^*, m]} \succ \text{Rev}(V_M^{[1, i^*-1]})]$ and $V_{2j} = [\text{Rev}(V_M^{[i^*, m]}) \succ \text{Rev}(V_M^{[1, i^*-1]})]$. Let $P = (V_1, \dots, V_n)$. It follows that $\text{Borda}(P \cup \{V_M\}) = c_{i_1}$ and $\text{Borda}(P \cup \{U\}) = c_{i_2}$. We note that $c_{i_1} \succ_{V_M} c_{i_2}$, which contradicts the assumption.

Case 2: n is odd and c_1 is ranked within top i^* positions in V_M . For any j such that $1 \leq j \leq (n-1)/2$, we let $V_{2j-1} = [c_1 \succ c_2 \succ \dots \succ c_m]$ and $V_{2j} = [c_m \succ c_{m-1} \succ \dots \succ c_1]$. Let $V_n = \text{Rev}(V)$ and $P = (V_1, \dots, V_n)$. It

follows that $\text{Borda}(P \cup \{V\}) = c_1$ and $\text{Borda}(P \cup \{U\}) \neq c_1$, which contradicts the assumption.

Case 3: n is odd and c_1 is not ranked within top i^* positions in V_M . Let V_1, \dots, V_{n-1} be defined the same as in Case 2. Let $V' = [V_M^{[i^*, m]} \succ V_M^{[1, i^* - 1]}]$. Let $U' = [U^{[i^*, m]} \succ U^{[1, i^* - 1]}]$. It follows that $\text{Borda}(V', V_M) = c_{i_1}$. Let $a = \text{Borda}(V', U)$. If $a \neq c_{i_1}$, then $c_{i_1} \succ_{V_M} a$. This is because the alternatives ranked in top $i^* - 1$ positions in V_M gets exactly the average score in $\{V', U\}$, which means that in order for any of them to win, the score of any alternative in $\{V', U\}$ must be the same. However, due to the tie-breaking mechanism, the winner is c_1 , which contradicts the assumption that c_1 is not ranked within top i^* positions in V_M . Let $P = (V_1, \dots, V_{n-1}, V')$, we have that $\text{Borda}(P \cup \{V_M\}) \succ_{V_M} \text{Borda}(P \cup \{U\})$, which contradicts the assumption. If $a = c_{i_1}$, then $\text{Borda}(U', V_M) = \text{Borda}(V', U) = c_{i_1}$. We have that $\text{Borda}(P' \cup \{V_M\}) = c_{i_1} \succ_{V_M} c_{i_2} = \text{Borda}(P' \cup \{U\})$, which is a contradiction.

Therefore, when $n \geq 1$, V_M is not dominated by any other vote under Borda. \square

Theorem 3. *When the manipulator has no information and $n \geq 6(m - 2)$, any positional scoring rule is immune to dominant-manipulation.*

Proof. For the sake of contradiction, let U dominates V_M . Let $c = \arg \max_{c^*} \{\bar{s}_m(V_M, c^*) : \bar{s}_m(V_M, c^*) > \bar{s}_m(U, c^*)\}$. It follows that there exists an alternative c' such that $\bar{s}_m(V_M, c') < \bar{s}_m(V_M, c)$ and $\bar{s}_m(U, c') = \bar{s}_m(V_M, c)$. We have that $\bar{s}_m(V_M, c) > \bar{s}_m(V_M, c')$ and $\bar{s}_m(U, c') = \bar{s}_m(V_M, c) > \bar{s}_m(U, c)$.

We prove the theorem for the case where $c = c_1$ and $c' = c_2$. The other cases can be proved similarly. Let M_{m-2} denote the cyclic permutation such that $c_3 \rightarrow c_4 \rightarrow \dots \rightarrow c_m \rightarrow c_3$. Let $W = [c_1 \succ c_2 \succ c_3 \succ \dots \succ c_m]$ and $W' = [c_2 \succ c_1 \succ c_3 \succ \dots \succ c_m]$. Let P_1 denote the $6(m - 2)$ -profile that is composed of three copies of $\{W, W', M_{m-2}(W), M_{m-2}(W'), \dots, M_{m-2}^{m-3}(W), M_{m-2}^{m-3}(W')\}$.

If n is even, then let P be composed of P_1 plus $n/2 - 3(m - 2)$ copies of $\{W_1, W_2\}$. If n is odd, then let W^* denote the a vote obtained from V_M by exchanging the positions of c and c' and P be composed of $\lfloor n/2 \rfloor - 3(m - 2)$ copies of $\{W_1, W_2\}$ and W^* . Because $\bar{s}_m(1) > \bar{s}_m(m)$, we have that $r(P \cup \{V_M\}) = c_1$ and $r(P \cup \{U\}) = c_2$. We note that $c_1 \succ_{V_M} c_2$. Therefore, we obtain a contradiction, which means that V_M is not dominated. \square

The next theorem states that for any voting rule that satisfies anonymity and unanimity, the true preferences is not dominated by any other vote that has a different top-ranked alternative.

Theorem 4. *When the manipulator has no information, for any voting rule r that satisfies anonymity and unanimity, any true preferences V_M of the manipulator and any vote U such that $\text{Alt}(U, 1) \neq \text{Alt}(V_M, 1)$, V_M is not dominated by U .*

Proof. For the sake of contradiction, let U be a vote that dominates V_M and $\text{Alt}(U, 1) \neq \text{Alt}(V_M, 1)$. For any $0 \leq$

$j \leq n + 1$, let P_j be the profile that consists of j copies of U and $n + 1 - j$ copies of V . Because r satisfies unanimity, $r(P_0) = \text{Alt}(V_M, 1)$ and $r(P_{n+1}) = \text{Alt}(U, 1)$. Let j^* be the smallest number such that $r(P_{j^*}) \neq \text{Alt}(V_M, 1)$. Obviously $j^* \geq 1$. Let P be a profile that consists of $j^* - 1$ copies of U and $n + 1 - j^*$ copies of V_M . It follows that $r(P \cup \{V_M\}) = r(P_{j^* - 1}) = \text{Alt}(V_M, 1) \succ_{V_M} r(P_{j^*}) = r(P \cup \{U\})$, which contradicts the assumption. \square

Theorem 1, 2, and 3 tells that when the manipulator has no information, then computing whether her true preferences is dominated is trivial (it is often undominated.) We recall that when the manipulator has full information, for many common voting rules (except STV and ranked pairs), computing whether her preferences is dominated is easy. The next section investigate the intermediate case—the manipulator has partial information about the other votes, represented by partial orders.

Information sets represented by partial orders

In this section, we focus on the cases where the manipulator's information about the non-manipulators' votes is represented by a profile of partial orders. In such cases, a signal is a profile partial order P_{po} , and the corresponding information set is $S = \{P \in \mathcal{F}_n : P \text{ extends } P_{po}\}$. In other words, in DOMINATION and DOMINANT-MANIPULATION with partial orders, the input S is represented by a profile of partial orders.

We note that the two cases discussed in the previous section (full information and no information) are special cases of manipulation with partial orders. Consequently, by Proposition 1, when the manipulator's information is represented by partial orders and $m \geq 3$, no voting rule that satisfies non-imposition and non-dictatorship is immune to dominant-manipulation. It also follows from Theorem 2 that STV and ranked pairs are resistant to dominant-manipulation.

The next theorem shows that even when the manipulator has slightly imperfect information, the DOMINATION problem with partial orders becomes NP-hard for Borda.

Theorem 5. *For any positional scoring rule r with scoring vectors $\{\bar{s}_1, \bar{s}_2, \dots\}$, suppose there exists a polynomial $f(x)$ such that for any $x \in \mathbb{N}$, there exist l and k , such that $x \leq l \leq f(x)$ and $k \leq l - 4$, and satisfy the following conditions:*

- (1) $\bar{s}_l(k) - \bar{s}_l(k + 1) = \bar{s}_l(k + 1) - \bar{s}_l(k + 2) = \bar{s}_l(k + 2) - \bar{s}_l(k + 3) > 0$,
- (2) $\bar{s}_l(k + 3) - \bar{s}_l(k + 4) > 0$,
- (3) $\bar{s}_l(2) - \bar{s}_l(l + 1 - q) > 0$,

Then, DOMINATION and DOMINANT-MANIPULATION with partial orders are both NP-hard, even when the number of undetermined pairs in each partial order is no more than 4.

Proof. The reductions is similar to the proof of the NP-hardness for the possible winner problem under positional scoring rules in (Xia and Conitzer accepted with minor revisions). We first show that if r satisfies the above three conditions, then DOMINATION is NP-hard, via a reduction from x3C.

Given an x3C instance $\mathcal{V} = \{v_1, \dots, v_q\}$, $\mathcal{S} = \{S_1, \dots, S_t\}$, we choose l and k that satisfy the conditions in the statement of the theorem, where $2q + 3 \leq l \leq f(2q + 3)$. We construct a DOMINATION instance as follows.

Alternatives: $\mathcal{C} = \{c, w, d\} \cup \mathcal{V} \cup A$, where d and $A = \{a_1, \dots, a_{l-q-3}\}$ are auxiliary alternatives. Ties are broken in the following order: $c \succ w \succ \mathcal{V} \succ A \succ d$.

Manipulator's preferences: $V_M = [w \succ c \succ A \succ d \succ \mathcal{V}]$. We are asked whether $V = V_M$ is dominated by $U = [w \succ A \succ d \succ c \succ \mathcal{V}]$.

The profile of partial orders: Let $P_{po} = P_1 \cup P_2$, defined as follows.

First part (P_1) of the profile: For each $i \leq t$, choose any $B_i \subset \mathcal{C} \setminus (S_i \cup \{w, d\})$ with $|B_i| = k - 1$. We define a partial order O_i as follows.

$O_i = [B_i \succ w \succ S_i \succ d \succ \text{Others}] \setminus [\{w\} \times (S_i \cup \{d\})]$

That is, O_i is a partial order that agrees with $B_i \succ w \succ S_i \succ d \succ \text{Others}$, except that the pairwise relations between (w, S_i) and (w, d) are not determined (and these are the only 4 undetermined relations). Let $P_1 = \{O_1, \dots, O_t\}$.

Second part (P_2) of the profile: Here we give the properties that we need P_2 to satisfy; P_2 can be constructed in polynomial-time. We omit the construction due to the space constraint. We note that all votes in P_2 are linear orders. Let $P'_1 = \{B_i \succ w \succ S_i \succ d \succ \text{Others} : i \leq t\}$. That is, P'_1 ($|P'_1| = t$) is an extension of P_1 (in fact, P'_1 is the set of linear orders that we started with to obtain P_1 , before removing some of the pairwise relations). P_2 is a set of linear orders such that the following holds for $Q = P'_1 \cup P_2 \cup \{V\}$:

(1) For any $i \leq q$, $\bar{s}_i(Q, c) - \bar{s}_i(Q, v_i) = \bar{s}_i(k) - \bar{s}_i(k + 1)$, $\bar{s}_i(Q, w) - \bar{s}_i(Q, c) = \frac{q}{3} \times (\bar{s}_i(k) - \bar{s}_i(k + 4))$.

(2) For any $i \leq q$, the scores of v_i and w, c are much higher than the scores of the other alternatives, in any extension of $P_1 \cup P_2 \cup \{V\}$.

(3) P_2 's size is polynomial in $t + q$.

We next prove that V is dominated by U if and only if c is the winner in at least one extension of $P_{po} \cup \{V\}$. We note that in U , for any alternative $v \in \mathcal{V}$, the score difference between w and v is the same as the score difference between w and v in $V_M = V$. Therefore, for any extension P^* of P_{po} , if $r(P^* \cup \{V\}) \in (\{w\} \cup \mathcal{V})$, then $r(P^* \cup \{V\}) = r(P^* \cup \{U\})$ (because d and the alternatives in A cannot win). If there exists an extension P^* of P_{po} such that $r(P^* \cup \{V\}) = c$, then we claim that the manipulator is strictly better off by casting U than casting V . Let P_1^* denote the extension of P_1 in P^* . Then, because the total score of w is no more than the total score of c , w is ranked lower than d for at least $\frac{q}{3}$ times in P_1^* . Meanwhile, for each $j \leq q$, v_j is not ranked higher than w for more than one time in P_1^* , because otherwise the total score of v_j will be higher than the total score of c . That is, the votes in P_1^* where $d \succ w$ constitutes a solutions to the x3C instance. Therefore, the only possibility for c to win is that the scores of c, w, \dots , and all alternatives in \mathcal{V} are the same (such that c wins according to the tie-breaking mechanism). Now, we have $w = r(P^* \cup \{U\})$. Because $w \succ_{V_M} c$, the manipulator is better off casting U . It follows that V is dominated by U if and only if there exists an extension of $P_{po} \cup \{V\}$ where c is the winner.

The above reasoning also shows that V is dominated by U if and only if the x3C instance has a solution. Therefore, DOMINATION MANIPULATION is NP-hard to compute. \square

We note that Theorem 5 applies to Borda (by letting $f(x) = x + 4$, $l = x + 4$, and $k = x$).

Intuitively, both DOMINATION and DOMINATING MANIPULATION problems seem to be harder than the possible winner problem under the same rule. Next, we present two theorems, which show that for any WMG-based rule, DOMINATION and DOMINATING MANIPULATION are harder than two special possible winner problems, respectively. We first define a notion that will be used in defining the two special possible winner problems. For any instance of the possible winner problem (r, P_{po}, c) , we define its WMG partition $\mathcal{R} = \{R_{c'} : c' \in \mathcal{C}\}$ as follows. For any $c' \in \mathcal{C}$, let $R_{c'} = \{\text{WMG}(P) : P \text{ extends } P_{po} \text{ and } r(P) = c'\}$. That is, $R_{c'}$ is composed of all WMGs of the extensions of P_{po} where the winner is c' . It is possible that for some $i \leq m$, $R_{c'}$ is empty. For any set $\mathcal{C}' \subseteq \mathcal{C} \setminus \{c\}$, we let $G_{\mathcal{C}'}$ denote the weighted majority graph where for each $c' \in \mathcal{C}'$, there is an edge $c' \rightarrow c$ with weight 2, and these are the only edges in the $G_{\mathcal{C}'}$.

We are ready to define the two special possible winner problems for WMG-based voting rules.

Definition 1. Let d^* be an alternative and let \mathcal{C}' be a nonempty subset of $\mathcal{C} \setminus \{c, d^*\}$. For any WMG-based voting rule r , we let $\text{PW}_1(d^*, \mathcal{C}')$ denote the set of possible winner problem instances (r, P_{po}, c) satisfying the following conditions: (R_c is the element in the WMG partition of this instance)

1. For any $G \in R_c$, $r(G + G_{\mathcal{C}'}) = d^*$.
2. For any $c' \neq c$ and any $G \in R_{c'}$, $r(G + G_{\mathcal{C}'}) = r(G)$.
3. For any $c' \in \mathcal{C}'$, $R_{c'} = \emptyset$.

Definition 2. Let d^* be an alternative and let \mathcal{C}' be a nonempty subset of $\mathcal{C} \setminus \{c, d^*\}$. For any WMG-based voting rule r , we let $\text{PW}_2(d^*, \mathcal{C}')$ denote the set of possible winner problem instances (r, P_{po}, c) satisfying:

1. For any $G \in R_c$, $r(G + G_{\mathcal{C}'}) = d^*$.
2. For any $G \in R_{d^*}$, $r(G + G_{\mathcal{C}'}) = r(G)$.
3. For any $c' \in \mathcal{C} \setminus \{c, d^*\}$, $R_{c'} = \emptyset$.

Theorem 6. Let r be a WMG-based voting rule. There is a polynomial time reduction from $\text{PW}_1(d^*, \mathcal{C}')$ to DOMINATION, both under r .

Proof. Let (r, P_{po}, c) be a $\text{PW}_1(d^*, \mathcal{C}')$ instance. We construct the following DOMINATION instance. Let the profile of partial orders be $Q_{po} = P_{po} \cup \{\text{Rev}(d^* \succ c \succ \mathcal{C}' \succ \text{Others})\}$, $V = V_M = [d^* \succ c \succ \mathcal{C}' \succ \text{Others}]$, and $U = [d^* \succ \mathcal{C}' \succ c \succ \text{Others}]$. Let P be an extension of P_{po} . It follows that $\text{WMG}(P \cup \{\text{Rev}(d^* \succ c \succ \mathcal{C}' \succ \text{Others}), V\}) = \text{WMG}(P)$, and $\text{WMG}(P \cup \{\text{Rev}(d^* \succ c \succ \mathcal{C}' \succ \text{Others}), U\}) = \text{WMG}(P) + G_{\mathcal{C}'}$. Therefore, the manipulator can change the winner if and only if $\text{WMG}(P) \in R_c$, which is equivalent to that c is a possible winner. We recall that by the definition of $\text{PW}_1(d^*, \mathcal{C}')$, for any $G \in R_c$, $r(G + G_{\mathcal{C}'}) = d^*$; for any $c' \neq c$ and any

$G \in R_{c'}$, $r(G + G_{c'}) = c'$; and $d^* \succ_V c$. It follows that $V (=V_M)$ is dominated by U if and only if the $\text{PW}_1(d^*, C')$ instance has a solution. \square

Proof. The reductions is similar to the proof of the NP-hardness for the possible winner problem under Borda (Xia and Conitzer accepted with minor revisions). We first show that if r satisfies the three conditions in the statement of the theorem, then DOMINATION problem is NP-hard via a reduction from x3C.

Given an x3C instance $\mathcal{V} = \{v_1, \dots, v_q\}$, $\mathcal{S} = \{S_1, \dots, S_t\}$, let $q+3 \leq l \leq f(q+3)$ (where q is the number of elements in the x3C instance) satisfy the two conditions in the assumption, and let $k \leq l-4$ satisfy $\bar{s}_l(k) - \bar{s}_l(k+1) = \bar{s}_l(k+1) - \bar{s}_l(k+2) = \bar{s}_l(k+2) - \bar{s}_l(k+3) > 0$, and $\bar{s}_l(k+3) - \bar{s}_l(k+4) > 0$. We construct a DOMINATION instance as follows.

Alternatives: $C = \{c, w, d\} \cup \mathcal{V} \cup A$, where d and $A = \{a_1, \dots, a_{l-q-3}\}$ are auxiliary alternatives. Ties are broken in the following order: $c \succ w \succ \mathcal{V} \succ A \succ d$.

Preferences of the manipulator: $V_M = [w \succ c \succ A \succ d \succ \mathcal{V}]$. We are asked if V is dominated by $U = [w \succ A \succ d \succ c \succ \mathcal{V}]$.

First part (P_1) of the profile: For any $i \leq t$, choose any $B_i \subset C \setminus (S_i \cup \{w, d\})$ with $|B_i| = k-1$. We define a partial order O_i as follows.

$O_i = [B_i \succ w \succ S_i \succ d \succ \text{Others}] \setminus [\{w\} \times (S_i \cup \{d\})]$

That is, O_i is a partial order that agrees with $B_i \succ w \succ S_i \succ d \succ \text{Others}$, except that the pairwise relations between (w, S_i) and (w, d) are not determined (and these are the only 4 undetermined relations). Let $P_1 = \{O_1, \dots, O_t\}$.

Second part (P_2) of the profile: We first give the properties that we need P_2 to satisfy; we will show how to construct P_2 in polynomial time later in the proof. We recall that all votes in P_2 are linear orders. Let $P'_1 = \{B_i \succ w \succ S_i \succ d \succ \text{Others} : i \leq t\}$. That is, P'_1 ($|P'_1| = t$) is an extension of P_1 (in fact, P'_1 is the set of linear orders that we started with to obtain P_1 , before removing some of the pairwise relations). P_2 is a set of linear orders such that the following holds for $Q = P'_1 \cup P_2 \cup \{V, \text{Rev}(V)\}$:

- (1) For any $i \leq q$, $\bar{s}_i(Q, c) - \bar{s}_i(Q, v_i) = \bar{s}_i(k) - \bar{s}_i(k+1)$, $\bar{s}_i(Q, w) - \bar{s}_i(Q, c) = \frac{q}{3} \times (\bar{s}_i(k) - \bar{s}_i(k+4))$.
- (2) For any $i \leq q$, the scores of v_i and w, c are higher than those of the other alternatives in any extension of $P_1 \cup P_2$.
- (3) P_2 's size is polynomial in $t + q$.

Given such a P_2 , c has the same total score as the total score of w if and only if there exists an extension P_1^* of P_1 such that w is ranked lower than c at least $\frac{q}{3}$ times, in order for the total score of w to be no more than the total score of c . Meanwhile, for any $j \leq q$, v_j is not ranked higher than w more than once in P_1^* , because otherwise the total score of v_j will be higher than the total score of c . Now due to the tie-breaking mechanism c is the winner. Since $\bar{s}_l(2) - \bar{s}_l(l+1-q) > 0$, the total score of c is strictly smaller than the total score of w in $P_1^* \cup P_2 \cup \{\text{Rev}(V_M), U\}$. We note that in U , for any alternative $v \in \mathcal{V}$, the score difference between w and v is the same as the score difference between w and v in V_M . It follows that the winner for $P_1^* \cup P_2 \cup \{\text{Rev}(V_M), V_M\}$ is different from the winner for

$P_1^* \cup P_2 \cup \{\text{Rev}(V_M), U\}$ if and only if c is the winner for $P_1^* \cup P_2 \cup \{\text{Rev}(V_M), V_M\}$, which only happens when the total scores of c, w, \mathcal{V} are the same. Therefore, if there exists an extension P_1^* of P_1 such that the total scores of c, w, \mathcal{V} are the same, then V_M is dominated by U , otherwise V_M is not dominated by U .

Given a solution to the DOMINATION instance, let I be the set of subscripts of votes in P_1^* for which w is ranked lower than c ; then, $S_I = \{S_i : i \in I\}$ is a solution to the x3C instance. Conversely, given a solution to the x3C instance, let I be the set of indices of S_i that are included in the x3C. Then, a solution to the DOMINATION instance can be obtained by ranking c ahead of w exactly in the votes with subscripts in I . Therefore, V is dominated by W if and only if there exists a solution to the x3C problem, which means that the DOMINATION problem with partial orders is NP-complete under positional scoring rules that satisfy the conditions stated in the theorem.

For the DOMINANT-MANIPULATION problem, we add to P_1 the following votes. For any $e \in \mathcal{V} \cup \{w\}$ and $i \leq l-1$, we obtain a vote $V_{e,i}$ from V_M by exchanging the alternative ranked in the $(i+1)$ th position with e and then exchanging the alternative ranked in the i th position with d ; let $O_{e,i}$ denote the partial order obtained from $V_{e,i}$ by removing $d \succ e$. Let $P_E = \{O_{e,i}, \text{Rev}(V_{e,i}) : e \in \mathcal{V} \cup \{w\}, i \leq l-1\}$. We add q copies of P_E to P_1 . For any vote W where there exists $v \in \mathcal{V}$ such that the score difference between w and v is different from the score difference between w and v in V_M , there must exists $v' \in C$ such that the score difference between w and v' in W is strictly smaller than their score difference in V_M . Then, it is easy to find an extension of P_1 such that if the manipulator cast V_M , then w wins, and if the manipulator cast W , then v' wins, which means that V_M is not dominated by W . Therefore, in such an instance, V_M can only be dominated by a vote W where the score difference between w and any alternative in \mathcal{V} is the same across V_M and W . Following the same reasoning as for the DOMINATION problem, we conclude that DOMINANT-MANIPULATION is NP-hard.

Next, we show how to construct the profile P_2 so that it satisfies the three conditions. P_2 consists of the following three parts.

The first part, P'_2 . Let M_V denote the cyclic permutation among $\mathcal{V} \cup \{c, w\}$. That is, $M_V = c \rightarrow w \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_q \rightarrow c$. For any $j \in \mathbb{N}$, and any $e \in \mathcal{V} \cup \{c, w\}$, we let $M_V^0(e) = e$, and $M_V^j(e) = M_V(M_V^{j-1}(e))$. The first part of P_2 is $P'_2 = M_V(P'_1) \cup M_V^2(P'_1) \cup \dots \cup M_V^{q+1}(P'_1)$. It follows that for any $e, e' \in \mathcal{V} \cup \{c, w\}$, $\bar{s}_i(P'_1 \cup P'_2, e) = \bar{s}_i(P'_1 \cup P'_2, e')$.

The second part, P_2^* . Choose any $B \subseteq C \setminus \{d, w, c\}$ such that $|B| = k-1$, and any $A' \subseteq C \setminus (B \cup \{d, w\})$ such that $|A'| = 3$. We define the following partial orders.

$$\begin{aligned} V_1 &= O(B, d, w, c, \text{Others}), & V'_1 &= O(B, c, w, d, \text{Others}) \\ V_2 &= O(B, d, c, w, \text{Others}), & V'_2 &= O(B, w, c, d, \text{Others}) \\ V_3 &= O(B, d, A', w, \text{Others}), & V'_3 &= O(B, w, A', d, \text{Others}) \\ V_4 &= O(B, A', d, w, \text{Others}), & V'_4 &= O(B, A', w, d, \text{Others}) \end{aligned}$$

P_2^* is defined as follows.

$$P_2^* = \{V_1', V_2', M_V(V_1), M_V(V_2), \dots, M_V^{q+1}(V_1), M_V^{q+1}(V_2)\} \\ \cup \frac{q}{3} \times \{V_3', M_V(V_3), \dots, M_V^{q+1}(V_3)\} \cup \{V_4', M_V(V_4), \dots, M_V^{q+1}(V_4)\}$$

Here $\frac{q}{3} \times \{V_3', M_V(V_3), \dots, M_V^{q+1}(V_3)\}$ represents $\frac{q}{3}$ copies of $\{V_3', M_V(V_3), \dots, M_V^{q+1}(V_3)\}$. Putting P_2' and P_2^* together, the condition (1) in the description of P_2 is satisfied.

The third part, \tilde{P}_2 . \tilde{P}_2 is defined in a way such that in \tilde{P}_2 , the total scores of any two alternatives in $\mathcal{V} \cup \{c, w\}$ are the same, and the total score of any alternative in $\mathcal{V} \cup \{c, w\}$ is significantly higher than the total score of any alternative in $\mathcal{C} \setminus (\mathcal{V} \cup \{c, w\})$. Let M_O be a cyclic permutation among $\mathcal{C} \setminus (\mathcal{V} \cup \{c, w\})$. That is, we let $M_O = d \rightarrow a_1 \rightarrow a_2 \rightarrow \dots \rightarrow a_{l-q-3} \rightarrow d$. Let $V_5 = [V \succ c \succ w \succ \text{Others}]$. We define the third part \tilde{P}_2 as follows.

$$\tilde{P}_2 = (|P_1 \cup P_2' \cup P_2^*| + 1) \times \{M_V^i(M_O^j(V_5)) : i \leq q+2, j \leq l-q-2\}$$

We note that $|P_1 \cup P_2' \cup P_2^*| + 1 = t(q+3) + 3(q+2) + q(q+2)/3$, which is polynomial in $t+q$. \square

It seems that both DOMINATION and DOMINATING-MANIPULATION problems are harder than the possible winner problem under the same rule. Next, we present two theorems, which show that for any WMG-based rule, there exist polynomial-time reductions from two restricted versions of the possible winner problems to DOMINATION and DOMINATING-MANIPULATION, respectively. Therefore, if computing such restricted possible winner problems is NP-hard, then DOMINATION and DOMINATING-MANIPULATION are also NP-hard. We first define a notion that will be used in defining the two restrictions of the possible winner problems. For any instance of the possible winner problem, we define its *WMG partition* $\mathcal{R} = \{R_1, \dots, R_m\}$ as follows.

- $\bigcup_{i=1}^m R_i$ is the set of all weighted majority graphs corresponding to the extensions of P_{po} . That is, $\bigcup_{i=1}^m R_i = \{WMG(P) : P \text{ extends } P_{po}\}$.
- For any $i_1 \neq i_2$, $R_{i_1} \cap R_{i_2} = \emptyset$, and for any $i \leq m$, any $G \in R_i$, the winner for G is c_i .

It is possible that for some $i \leq m$, R_i 's are empty. For any set $\mathcal{C}' \subseteq \mathcal{C} \setminus \{c\}$, we let $G_{\mathcal{C}'}$ denote the weighted majority graph where for each $c' \in \mathcal{C}'$, there is an edge $c' \rightarrow c$ with weight 2, and these are the only edges in the graph.

In a *possible winner* problem, we are given (r, P_{po}, c) , where r is a voting rule, P_{po} is a profile that is composed of n partial orders, and c is an alternative. We are asked whether there exists an extension P of P_{po} such that $c = r(P)$. We now define the two restricted versions of the possible winner problems for voting rules that are based on weighted majority graphs.

Definition 3. Let d^* be an alternative and let \mathcal{C}' be a nonempty subset of $\mathcal{C} \setminus \{c, d^*\}$. For any WMG-based voting

rule r , we let $PW_1(d^*, \mathcal{C}')$ denote the set of possible winner problem instances (r, P_{po}, c) that satisfies the following constraints.

1. For any $G \in R_c$, $r(G + G_{\mathcal{C}'}) = d^*$.
2. For any $c' \neq c$ and any $G \in R_{c'}$, $r(G + G_{\mathcal{C}'}) = r(G)$.
3. For any $c' \in \mathcal{C}'$, $R_{c'} = \emptyset$.

Definition 4. Let d^* be an alternative and let \mathcal{C}' be a nonempty subset of $\mathcal{C} \setminus \{c, d^*\}$. For any WMG-based voting rule r , we let $PW_2(d^*, \mathcal{C}')$ denote the set of possible winner problem instances (r, P_{po}, c) that satisfies the following constraints.

1. For any $G \in R_c$, $r(G + G_{\mathcal{C}'}) = d^*$.
2. For any $G \in R_{d^*}$, $r(G + G_{\mathcal{C}'}) = r(G)$.
3. For any $c' \in \mathcal{C} \setminus \{c, d^*\}$, $R_{c'} = \emptyset$.

Theorem 7. Let r be a WMG-based voting rule. There exists a (many-one) reduction from $PW_1(d^*, \mathcal{C}')$ to DOMINATION, both under r .

Proof. Let (r, P_{po}, c) be an instance of the $PW_1(d^*, \mathcal{C}')$ problem. We construct the following DOMINATION instance. Let the profile of partial orders be $Q_{po} = P_{po} \cup \{\text{Rev}(d^* \succ c \succ \mathcal{C}' \succ \text{Others})\}$, $V = V_M = [d^* \succ c \succ \mathcal{C}' \succ \text{Others}]$, and $U = [d^* \succ \mathcal{C}' \succ c \succ \text{Others}]$. Let P be an extension of P_{po} . It follows that $WMG(P \cup \{\text{Rev}(d^* \succ c \succ \mathcal{C}' \succ \text{Others}), V\}) = WMG(P)$, and $WMG(P \cup \{\text{Rev}(d^* \succ c \succ \mathcal{C}' \succ \text{Others}), U\}) = WMG(P) + G_{\mathcal{C}'}$. Therefore, the manipulator can change the winner if and only if $WMG(P \cup \{\text{Rev}(d^* \succ c \succ \mathcal{C}' \succ \text{Others}), V\}) = WMG(P) \in R_c$, which is equivalent to saying that c is a possible winner. We recall that by the definition of $PW_1(d^*, \mathcal{C}')$, for any $G \in R_c$, $r(G + G_{\mathcal{C}'}) = d^*$; for any $c' \neq c$ and any $G \in R_{c'}$, $r(G + G_{\mathcal{C}'}) = c'$; and $d^* \succ_V c$. It follows that $V (=V_M)$ is dominated by U if and only if the $PW_1(d^*, \mathcal{C}')$ instance has a solution. \square

We next show how to use Theorem 7 to prove that DOMINATION is NP-hard for Copeland, maximin, and voting trees, even when the number of undetermined pairs in each partial order is bounded above by a constant. It suffices to show that the $PW_1(d^*, \mathcal{C}')$ problems for these rules are NP-hard.

Corollary 1. It is NP-hard to compute DOMINATION for Copeland, maximin, and voting trees, even when the number of unknown pairs in each vote is bounded above by a constant.

Proof. Copeland: We tweak the reduction in the NP-completeness proof of PW w.r.t. Copeland (Theorem 3 (Xia and Conitzer accepted with minor revisions)) by letting $D(c, v) = 1$ for any alternative $v \in \mathcal{V}$ and use the tie-breaking mechanism where $w \succ c \succ \text{Others}$. Let $d^* = w$, $\mathcal{C}' = B$, $V = V_M = [w \succ c \succ \mathcal{C}' \succ \text{Others}]$ and $U = [w \succ \mathcal{C}' \succ c \succ \text{Others}]$. It follows that the alternatives in B never wins the elections, and if c wins the election in an extension P of P_{po} , then the Copeland score of c is $8t + 1$ and the Copeland score of $w = 8t$. However, in the weighted majority graph $WMG(P) + G_{\mathcal{C}'}$, c loses to all alternatives in \mathcal{C}' in their pairwise elections, which means

that the Copeland score of c is $t + 1$. Consequently w is the winner. On the other hand, for any extension P where c is not the winner, w is the winner, and w is also the winner in the weighted majority graph $\text{WMG}(P) + G_{C'}$. Therefore, the PW instance is a $\text{PW}_1(d^*, C')$ instance.

Maximin: We tweak the reduction in the NP-completeness proof of PW w.r.t. maximin (Theorem 5 (Xia and Conitzer accepted with minor revisions)) by letting $D(w', w) = t$. Let $d^* = w$, $C' = \{w'\}$, $V = V_M = [w \succ c \succ w' \succ \mathcal{V}]$ and $U = [w \succ w' \succ c \succ \mathcal{V}]$. We adopt the tie-breaking mechanism where $w \succ c \succ \mathcal{V} \succ w'$. It is easy to check that w' never wins the elections. If c wins the election in an extension P of P_{po} , then the minimum pairwise score of c is $-t + 2$, and the minimum pairwise score of w and the alternatives in \mathcal{V} are $-t$. We note that in the majority graph $\text{WMG}(P) + G_{C'}$, the minimum pairwise score of c is $-t$ (against w'), which means that $r(\text{WMG}(P) + G_{C'}) = w$. For any extension P of P_{po} such that $r(P) \neq c$, it is easy to check that the winner is in $\{w\} \cup \mathcal{V}$, and the minimum pairwise scores of them are the same as in the weighted majority graph $\text{WMG}(P) + G_{C'}$. Therefore, the PW instance is a $\text{PW}_1(d^*, C')$ instance.

Voting trees: We tweak the reduction in the NP-completeness proof of PW w.r.t. voting trees (Theorem 7 (Xia and Conitzer accepted with minor revisions)) by letting $D(c, d) = 1$. Let $d^* = w$, $C' = \{d\}$, $V = V_M = [w \succ c \succ d \succ \text{Others}]$ and $U = [w \succ d \succ c \succ \text{Others}]$. For any extension P of P_{po} where c wins, the winner for the weighted majority graph $\text{WMG}(P) + G_{C'}$ is w , because c loses to d in the first round, and w beats any other alternatives (except c) in their pairwise elections. For any extension P of P_{po} where c does not win, the winner is w . Therefore, the PW instance is a $\text{PW}_1(d^*, C')$ instance. \square

Theorem 8. *Let r be a WMG-based voting rule. There exists a polynomial-time (many-one) reduction from $\text{PW}_2(d^*, C')$ to DOMINATING-MANIPULATION, both under r .*

Proof. The proof is similar to the proof for Theorem 7. We note that d^* is the manipulator's top-ranked alternative. Therefore, if c is not a possible winner, then $V (= V_M)$ is not weakly dominated by any other vote; if c is a possible winner, then V is dominated by $U = [w \succ C' \succ c \succ \text{Others}]$. \square

By Theorem 7 and the proofs for Theorem 3, Theorem 6, and Theorem 7 in (Xia and Conitzer accepted with minor revisions), we immediately have the following corollary.

Corollary 2. *It is NP-hard to compute DOMINATING-MANIPULATION for Copeland, ranked pairs, and voting trees, even when the number of unknown pairs in each vote is bounded above by a constant.*

Theorem 9. *It is NP-hard to compute DOMINATING-MANIPULATION for maximin.*

Proof: We prove the hardness result by a reduction from x3C. Given an x3C instance $\mathcal{V} = \{v_1, \dots, v_q\}$, $S = \{S_1, \dots, S_t\}$, where $q = t > 3$, we construct a DOMINATING-MANIPULATION instance as follows.

Alternatives: $\mathcal{V} \cup \{c, w, w'\}$. Ties are broken in the order $w \succ \mathcal{V} \succ c \succ w'$.

First part P_1 of the profile: for each $i \leq t$, we start with the linear order $V_i = [w \succ S_i \succ c \succ (\mathcal{V} \setminus S_i) \succ w']$, and subsequently obtain a partial order O_i by removing the relations in $\{w\} \times (S_i \cup \{c\})$. For each $i \leq t$, we let O'_i be a partial order obtained from $V'_i = [w \succ v_i \succ \text{Others}]$ by removing $w \succ v_i$. We let O' be a partial order obtained from $V' = [w' \succ w \succ \text{Others}]$ by removing $w' \succ w$. Let P_1 be the profile composed of $\{O_1, \dots, O_t\}$, 2 copies of $\{O'_1, \dots, O'_t\}$, and 3 copies of O' . Let P'_1 denote the extension of P_1 that consists of V_1, \dots, V_t , 2 copies of $\{V'_1, \dots, V'_t\}$, and 3 copies of V' .

Second part P_2 of the profile: P_2 is defined to be a set of linear orders such that the pairwise score differences of $P'_1 \cup P_2 \cup \{V\}$ satisfy:

- (1) $D(w, c) = 2t + \frac{2q}{3}$, $D(w', w) = 2t + 6$, $D(w', c) = 2t$, and for all $i \leq q$, $D(w, v_i) = 2t + 4$ and $D(v_i, w') = 4(t + q)$.
- (2) $D(l, r) \leq 1$ for all other pairwise scores not defined in (1).

Preferences of the manipulator: $V_M = [w \succ \mathcal{V} \succ c \succ w']$.

We note that in any extension of $P_1 \cup P_2$, after the manipulator changes her vote from V_M to $[w \succ \mathcal{V} \succ w' \succ c]$, the only change made to the weighted majority graph is that the weight on $w \rightarrow c$ increases by 2. Since w' never wins in any extension, if c does not win when the manipulator votes for V_M , then the winner does not change after the manipulator changes her vote to $[w \succ \mathcal{V} \succ w' \succ c]$. It follows from the proof of Theorem 7, Corollary 1, and Theorem 5 in (Xia and Conitzer accepted with minor revisions) that if the x3C instance has a solution, then V_M is dominated by $U = [w \succ \mathcal{V} \succ w' \succ c]$. Suppose that the x3C instance does not have a solution, we next show that V_M is not dominated by any vote.

For the sake of contradiction, suppose the x3C instance does not have a solution and V_M is dominated by a vote U . There are following cases.

- **Case 1:** There exist $v_i \in \mathcal{V}$ such that $w \succ_V v_i$ and $v_i \succ_U w$. We let P^* be the extension of $P_1 \cup P_2$ obtained from $P'_1 \cup P_2$ as follows. (1) Let $w \succ w'$ in 3 extensions of O' (we recall that there are $q > 3$ copies of O' in P_1). (2) Let $v_i \succ w$ in 2 extensions of O'_i . It is easy to check that in P^* , the minimum pairwise score of w is $-2t$ (via w') and the minimum pairwise score of v_i is $-2t$ (via w). Therefore, due to the tie-breaking mechanism, w wins. However, if the manipulator changes her vote from V_M to U , then the minimum pairwise score of w at most $-2t$ and the minimum pairwise score of v_i is at least $-2t + 2$, which means that v_i wins. We note that $w \succ_V v_i$. This contradicts the assumption that U dominates V_M .
- **Case 2:** $w \succ_W v_i$ for each $v_i \in \mathcal{V}$. By changing her vote from V_M to U , the manipulator might reduce the minimum score of U by 2, increase the minimum score of c

by 2, or increase the minimum score of w' by 2. Therefore, by changing her vote to U , the manipulator would either make no changes, make w lose, or make c win (we note that w' is not winning anyway). In each of these three cases the manipulator is not better off, which means that U does not dominate V_M . This contradicts the assumption.

◇

Proposition 2. *For any voting rule for which computing the winner is in P , the DOMINATION problem with partial orders is in $P_{||}^{NP}$.*

For plurality and veto, we present polynomial-time algorithms for both DOMINATION and DOMINATING-MANIPULATION.

Given a DOMINATION instance, we let c_{i^*} denote the top-ranked alternative in V and let c_{j^*} denote the top-ranked alternative in U . The key idea behind the algorithm for DOMINATION for plurality is that we check for all pairs of alternatives d, d' such that (1) $d' \succ_{V_M} d$, (2) d wins if the manipulator votes for V , and (3) d' wins if the manipulator votes for U . It follows that either $d = c_{i^*}$ or $d' = c_{j^*}$ (or both hold). We will check whether there exists $0 \leq l \leq n$, $d, d' \in \mathcal{C}$ and an extension P^* of P_{po} , such that if the manipulator votes for V , then the winner is d , whose plurality score in P^* is l , and if the manipulator votes for U , then the winner is d' . Here the plurality score of an alternative in a profile is the number of times it is ranked in the top position in this profile. To this end, we will convert it to multiple maximum-flow problems. First we define the maximum-flow problems.

Let $\mathcal{C}' \subset \mathcal{C}$ denote a set of alternatives. Let $\vec{e} = (e_1, \dots, e_m) \in \mathbb{N}^m$ be an arbitrary vector composed of m natural numbers such that $\sum_{i=1}^m e_i \geq n$. Given an instance of the DOMINATION problem, we define a maximum-flow problem $F_{\mathcal{C}'}^{\vec{e}}$, that has the following property. $F_{\mathcal{C}'}^{\vec{e}}$ has a solution if and only if there exists an extension P^* of P_{po} such that for any $c_i \in \mathcal{C}'$, its plurality score is exactly e_i , and the plurality score of any $c_j \in \mathcal{C} \setminus \mathcal{C}'$ is at most e_j , both in P^* . $F_{\mathcal{C}'}^{\vec{e}}$ is defined as follows.

Vertices: $\{s, O_1, \dots, O_n, c_1, \dots, c_m, y, t\}$.

Edges:

- There is an edge from s to O_i with weight 1.
- For any O_i and c_j , there is an edge $O_i \rightarrow c_j$ with weight 1 if and only if c_j can be ranked in the top position in at least one extension of O_i .
- For any $c_i \in \mathcal{C}'$, there is an edge $c_i \rightarrow t$ with weight e_i .
- For any $c_j \in \mathcal{C} \setminus \mathcal{C}'$, there is an edge $c_j \rightarrow y$ with weight e_j .
- There is an edge $y \rightarrow t$ with weight $n - \sum_{c_i \in \mathcal{C}'} e_i$.

For example, $F_{\{c_1, c_2\}}^{\vec{e}}$ is illustrated in Figure 1.

Proposition 3. *Given an instance of the DOMINATION problem for plurality, $F_{\mathcal{C}'}^{\vec{e}}$ has a solution if and only if there exists an extension P^* of P_{po} such that the plurality score of any alternative $c_i \in \mathcal{C}'$ is e_i , and the plurality score of any alternative $c_j \in \mathcal{C} \setminus \mathcal{C}'$ is at most e_j , both in P^* .*

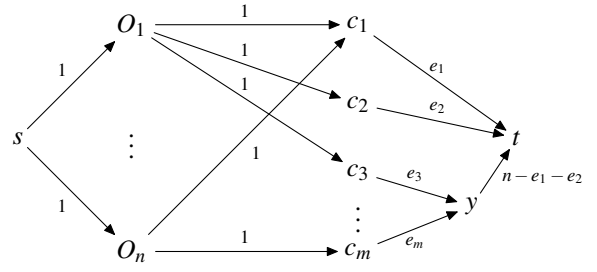


Figure 1: $F_{\{c_1, c_2\}}^{\vec{e}}$.

Now, we are ready to present the algorithm for DOMINATION for plurality. For any pair of alternatives c_i and c_j such that $i \neq j$, we define $\delta(c_i, c_j) = \begin{cases} 1 & \text{if } c_i \succ c_j \text{ in tie-breaking} \\ 0 & \text{if } c_j \succ c_i \text{ in tie-breaking} \end{cases}$. Given any instance of the DOMINATION problem, any $0 \leq l \leq n$, and any alternatives $c_{i^*}, c_{j^*}, d = c_i, d' = c_j$ such that $d' \succ_V d$, $i^* \neq j^*$, and either $d = c_{i^*}$ or $d' = c_{j^*}$, we define the set of *admissible maximum-flow problems* A_{Plu} as follows.

- If $i = i^*$ and $j \neq j^*$, then let $e_i = l$, $e_j = l + 1 - \delta(c_j, c_i)$, and $e_{j^*} = \min(l + 1 - \delta(j^*, i), e_j - 1 - \delta(j^*, j))$. For any $c_{i'} \in \mathcal{C} \setminus \{c_i, c_j, c_{j^*}\}$, we let $e_{i'} = \min(l + 1 - \delta(i', i), e_j - \delta(i', j))$. Let $A_{\text{Plu}} = \{F_{\{c_i, c_j\}}^{\vec{e}}\}$.
- If $i \neq i^*$ and $j = j^*$, then let $e_i = l$, $e_j = l - \delta(c_j, c_i)$, and $e_{i^*} = \min(l - 1 - \delta(i^*, i), e_j + 1 - \delta(i^*, j))$. For any $c_{i'} \in \mathcal{C} \setminus \{c_i, c_j, c_{i^*}\}$, we let $e_{i'} = \min(l - \delta(i', i), e_j + 1 - \delta(i', j))$. Let $A_{\text{Plu}} = \{F_{\{c_i, c_j\}}^{\vec{e}}\}$.
- If $i = i^*$ and $j = j^*$, then we define A_{Plu} as follows.
 - Let $e_i = l$, $e_j = l + 1 - 2\delta(c_j, c_i)$. For any $c_{i'} \in \mathcal{C} \setminus \{c_i, c_j\}$, we let $e_{i'} = \min(l + 1 - \delta(i', i), e_j + 1 - \delta(i', j))$.
 - Let $e'_i = e'_j = l$. For any $c_{i'} \in \mathcal{C} \setminus \{c_i, c_j\}$, we let $e'_{i'} = \min(l + 1 - \delta(i', i), e_j + 1 - \delta(i', j))$. Let $\vec{e}' = (e'_1, \dots, e'_m)$.
 - Let $A_{\text{Plu}} = \{F_{\{c_i, c_j\}}^{\vec{e}}, F_{\{c_i, c_j\}}^{\vec{e}'}\}$.

We are ready to present the algorithm that solves DOMINATION (Algorithm 1).

The algorithm for DOMINATING-MANIPULATION for plurality simply runs Algorithm 1 for $m - 1$ times, where $V = V_M$ and the top-ranked alternatives in U are $\mathcal{C} \setminus \{\text{Alt}(V, 1)\}$. If in any step V is dominated by U , then there is a dominating-manipulation; otherwise V is not dominated by any other vote.

The idea behind the algorithm for DOMINATION and DOMINATING-MANIPULATION for veto are similar. Given an instance of the DOMINATION problem, let c_{i^*}, c_{j^*} denote the alternatives that are ranked in the bottom positions in V and U , respectively. For any $\mathcal{C}' \subseteq \mathcal{C}$ and any $\vec{e} = (e_1, \dots, e_m) \in \mathbb{N}^m$, where $\sum_i e_i \leq n$, we construct a maximum-flow problem $L_{\mathcal{C}'}^{\vec{e}}$ as follows.

Vertices: $\{s, O_1, \dots, O_n, c_1, \dots, c_m, y, t\}$.

Edges:

Algorithm 1: compDominationPlurality

```
1 Let  $c_{i^*} = \text{Alt}(V, 1)$  and  $c_{j^*} = \text{Alt}(U, 1)$ .
2 for any  $0 \leq l \leq n$  and any pair of alternatives
   $d = c_i, d' = c_j$  such that  $d' \succ_{V_M} d, i^* \neq j^*$ , and either
   $d = c_{i^*}$  or  $d' = c_{j^*}$  do
3   Compute  $A_{\text{Plu}}$ .
4   for each maximum-flow problem  $F_{\mathcal{C}'}^{\vec{e}}$  in  $A_{\text{Plu}}$  do
5     if  $\sum_{c_i \in \mathcal{C}'} e_i \leq n$  and  $F_{\mathcal{C}'}^{\vec{e}}$  has a solution then
6       Output that the  $V$  is dominated by  $U$ ,
7       terminate the algorithm.
8     end
9 end
10 Output that  $V$  is not dominated by  $U$ .
```

- There is an edge from s to O_i with weight 1.
- For any O_i and c_j , there is an edge $O_i \rightarrow c_j$ with weight 1 if and only if c_j can be ranked in the bottom position in at least one extension of O_i . There is an edge from O_i to y if and only if there exists $c_j \in \mathcal{C} \setminus \mathcal{C}'$ such that c_j can be ranked in the bottom position in at least one extension of O_i .
- For any $c_i \in \mathcal{C}$, there is an edge $c_i \rightarrow t$ with weight e_i .
- There is an edge $y \rightarrow t$ with weight $n - \sum_{i=1}^m e_i$.

For example, $L_{\{c_1, c_2\}}^{\vec{e}}$ is illustrated in Figure 2.

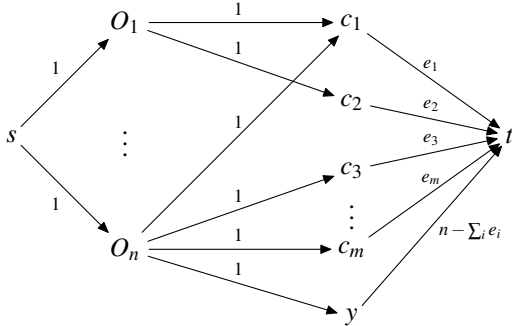


Figure 2: $L_{\{c_1, c_2\}}^{\vec{e}}$.

Proposition 4. *Given an instance of the DOMINATION problem for veto, $L_{\mathcal{C}'}^{\vec{e}}$ has a solution if and only if there exists an extension P^* of P_{po} such that the veto score of any alternative $c_i \in \mathcal{C}'$ is e_i , and the veto score (the number of times that an alternative is ranked in the bottom position in the profile) of any alternative $c_j \in \mathcal{C} \setminus \mathcal{C}'$ is at least e_j , both in P^* .*

Now, we are ready to present the algorithm for DOMINATION for veto. Given an instance of the DOMINATION problem, any $0 \leq l \leq n$, and any alternatives $c_{i^*}, c_{j^*}, d = c_i, d' = c_j$ such that $d' \succ_V d, i^* \neq j^*$, and either $d = c_{j^*}$ or $d' = c_{i^*}$, we define the set of *admissible maximum-flow problems* A_{Veto} as follows.

- If $i = j^*$ and $j \neq i^*$, then let $e_i = l, e_j = l + \delta(c_j, c_i)$, and $e_{i^*} = \max(l - 1 + \delta(i^*, i), e_j + \delta(i^*, j))$. For any $c_{i'} \in \mathcal{C} \setminus \{c_i, c_j, c_{i^*}\}$, we let $e_{i'} = \max(l + \delta(i', i), e_j + \delta(i', j))$. Let $A_{\text{Veto}} = \{L_{\{c_i, c_j\}}^{\vec{e}}\}$.
- If $i \neq j^*$ and $j = i^*$, then let $e_i = l, e_j = l - 1 + \delta(c_j, c_i)$, and $e_{j^*} = \max(l + \delta(j^*, i), e_j - 1 + \delta(j^*, j))$. For any $c_{i'} \in \mathcal{C} \setminus \{c_i, c_j, c_{j^*}\}$, we let $e_{i'} = \max(l + \delta(i', i), e_j + \delta(i', j))$. Let $A_{\text{Veto}} = \{L_{\{c_i, c_j\}}^{\vec{e}}\}$.
- If $i = j^*$ and $j = i^*$, then we define A_{Plu} as follows.
 - Let $e_i = l, e_j = l + 1 + 2\delta(c_j, c_i)$. For any $c_{i'} \in \mathcal{C} \setminus \{c_i, c_j\}$, we let $e_{i'} = \max(l + \delta(i', i), e_j + \delta(i', j))$.
 - Let $e'_i = e'_j = l$. For any $c_{i'} \in \mathcal{C} \setminus \{c_i, c_j\}$, we let $e'_{i'} = \max(l + \delta(i', i), e_j + \delta(i', j))$. Let $\vec{e}' = (e'_1, \dots, e'_m)$.
 - Let $A_{\text{Veto}} = \{L_{\{c_i, c_j\}}^{\vec{e}}, L_{\{c_i, c_j\}}^{\vec{e}'}\}$.

Now we are ready to present our algorithm (Algorithm 2).

Algorithm 2: compDominationVeto

```
1 Let  $c_{i^*} = \text{Alt}(V, m)$  and  $c_{j^*} = \text{Alt}(U, m)$ .
2 for any  $0 \leq l \leq n$  and any pair of alternatives
   $d = c_i, d' = c_j$  such that  $d' \succ_{V_M} d, i^* \neq j^*$ , and either
   $d = c_{j^*}$  or  $d' = c_{i^*}$  do
3   Compute  $A_{\text{Veto}}$ .
4   for each maximum-flow problem  $F_{\mathcal{C}'}^{\vec{e}}$  in  $A_{\text{Veto}}$  do
5     if  $\sum e_i \leq n$  and  $F_{\mathcal{C}'}^{\vec{e}}$  has a solution then
6       Output that the  $V$  is dominated by  $U$ ,
7       terminate the algorithm.
8     end
9 end
10 Output that  $V$  is not dominated by  $U$ .
```

Similarly we can obtain a polynomial-time algorithm for DOMINATING-MANIPULATION for Veto.

Table 1 summarizes our results in this section.

	DOMINATION/DOMINATING-MANIPULATION
STV	NP-hard (Proposition 1)
Ranked pairs	NP-hard (Proposition 1)
Pos. scoring	NP-hard (Theorem 5)
Copeland	NP-hard (Corollary 1/Corollary 2)
Voting trees	NP-hard (Corollary 1/Corollary 2)
Maximin	NP-hard (Corollary 1/Theorem 9)
Plurality	P (Algorithm 1)
Veto	P (Algorithm 2)

Table 1: Computational complexity of DOMINATION and DOMINATING-MANIPULATION for common voting rules.

Manipulation with Other Types of Partial Information

We consider different amounts of information that the manipulators know about the other votes. For example, the

manipulators may know the current winner (that is, the candidate who wins if we ran the election on the votes of the non-manipulators), the possible winners, the current scores of the candidates (for counting rules like plurality or scoring rules like Borda), the current elimination order (for elimination rules like STV or Nanson's rule), or the current majority graph (for rules like cup or Copeland which are based on the majority graph). However, there are many other possibilities like the top k scores, or those links in the majority graph which are already decided and the manipulators cannot change.

We suppose that the manipulators are risk averse. That is, they will only vote strategically (that is, cast a vote different to their preferences) if they can be sure that this will improve the result for them or leave it the same. They will not vote strategically if this could make the result worse for them.

We consider three properties which measure the degree to which a voting rule is manipulable. We say that a voting rule is *immune* to manipulation based on the partial information known by the manipulators if the manipulators can never safely vote strategically, *resistant* to manipulation based on the partial information if the rule is not immune but computing a safe manipulation is NP-hard, and *vulnerable* to manipulation based on the partial information if the rule is not immune and computing a safe manipulation is polynomial. For example, plurality is immune to manipulation based on the current winner, but becomes vulnerable as soon as the manipulators know the possible winners.

Theorem 10. *With weighted or unweighted votes plurality is immune to manipulation based on the current winner, but is vulnerable based on the possible winners.*

Proof: Suppose the manipulators know the current winner. There are two cases. In the first case, the current winner is the manipulators' first choice. In this case, the manipulators should vote for this candidate. If they vote for anyone else, there are situations in which someone less preferred wins. In the second case, the current winner is not the manipulators' first choice. Voting for their first choice will either result in this candidate winning or the current winner. Voting for anyone else may result in a less desirable outcome. In both cases, the manipulators do not vote strategically. Suppose, on the other hand, that the manipulators know the set of possible winners. Then they can safely vote for their most preferred candidate in this set. If the possible winners does not include their most preferred candidate overall, this vote is strategic. \diamond

Almost every voting rule is immune to manipulation based on the current winner. This means in practice that the returning officer does not have to keep the running result secret provided nothing else is revealed. There are, however, voting rules where even knowing just the current winner can enable manipulation to take place. For example, consider the 2/3rd rule used to change the result of an earlier vote in Robert's Rules of Order. A corresponding voting rule might elect a fixed default candidate unless there is a 2/3rd majority of votes for some other candidate. This rule is vulnerable to manipulation based on the current winner. Suppose the default winner is the manipulators' third choice, the current

winner is the manipulators' second choice and the manipulators have no more votes than the non-manipulators. Then the manipulators are better off voting strategically for their second choice candidate and not for their first choice. Voting for their first choice can only make the result worse for the manipulators as their first choice cannot gain a 2/3rd majority.

Results about manipulation with partial information can provide new insight into the manipulability of voting rules. For instance, it has been shown that the veto rule is NP-hard to manipulate constructively with 3 or more candidates and weighted votes (?). However, computational complexity seems only to provide a weak shield here against manipulation. In particular, the veto rule is vulnerable to manipulation when the manipulators know just who are the possible winners.

Theorem 11. *With weighted votes, the veto rule is vulnerable to manipulation based on the possible winners.*

Proof: Consider the candidate in the set of possible winners who the manipulators like least. A safe but (possibly) strategic vote is for the manipulators to veto this candidate.

\diamond

Note that computing the set of possible winners for the veto rule with weighted votes is itself NP-hard.

Conclusions

Analysis of manipulation with partial information provides insight into what needs to be kept confidential in an election. For instance, in a plurality or veto election, revealing (perhaps unintentionally) the set of possible winners may result in manipulation. It would be interesting to identify cases where voting rules are resistant to manipulation based on partial information. It would also be interesting to consider control of elections based on partial information. Finally, we could also perform game theoretic analyses of sequential voting games where participants only have partial information about the current state (unlike the complete information assumed by Xia and Conitzer).

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