Strategic Sequential Voting in Multi-Issue Domains and Multiple-Election Paradoxes☆

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Abstract

In many settings, a group of voters must come to a joint decision on multiple issues. In practice, this is often done by voting on the issues in sequence. We model sequential voting in multi-issue domains as a complete-information extensive-form game, in which the voters are perfectly rational and their preferences are common knowledge. In each step, the voters simultaneously vote on one issue, and the order of the issues is given exogenously before the process. We call this model *strategic sequential voting*.

We focus on domains in which all the issues are binary (yes or no), which have the advantage that strategic sequential voting always leads to a unique equilibrium outcome under a natural solution concept. We show that under some conditions on the preferences, this leads to the same outcome as truthful sequential voting, but in general it can result in very different outcomes. In particular, sometimes the order of the issues has a strong influence on the outcome. Most significantly, we exhibit several *multiple-election paradoxes* in strategic sequential voting: there exist preference profiles for which the winner under strategic sequential voting is ranked nearly at the bottom in all voters' true preferences, and the winner is Pareto-dominated by almost every other alternative. We show that changing the order of the issues cannot completely prevent such paradoxes. We also study the possibility of avoiding the paradoxes of strategic sequential voting by imposing some constraints

☆This paper is a significant extension of a previous conference version [28].

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on the preference profile, such as separability, lexicographicity or $\mathcal{O}$-legality. Finally, we investigate the existence of multiple election paradoxes for other common voting rules from a non-strategic perspective.

Keywords: Social choice, strategic voting, multi-issue domains, multiple-election paradoxes

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1. Introduction

In a traditional voting system, each voter is asked to report a linear order over the alternatives to represent her preferences. Then, a *voting rule* is applied to the resulting profile of reported preferences to select a winning alternative. In practice, the set of alternatives often has a *multi-issue* structure. That is, there are p issues $\mathcal{I} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$, and each issue can take a value in a *local domain*. In other words, the set of alternatives is the Cartesian product of the local domains. For example, in *multiple referenda*, the inhabitants of a local district are asked to vote on multiple inter-related issues [8]. Another example is committee elections in which the voters select a subset of objects [2]. Here, each object can be seen as a binary issue.

Voting in multi-issue domains has been extensively studied by economists, and more recently has attracted the attention of computer scientists. Previous work has focused on proposing a natural and compact *voting language* in which the voters can represent their preferences, as well as designing a sensible voting rule to make decisions based on preferences reported in such a language. A natural approach is to let voters vote on the issues separately, in the following way. For each issue (simultaneously, not sequentially), each voter reports her preferences for that issue, and then a *local rule* is used to select the winning value that the issue will take. This voting process is called *issue-by-issue* or *seat-by-seat* voting.

Computing the winner for issue-by-issue voting rules is easy. Nevertheless, issue-by-issue voting has major drawbacks. First, a voter may feel uncomfortable expressing her preferences over one issue independently of the values that the other issues take [18]. Issue-by-issue voting avoids this problem if the voters' preferences are *separable* (that is, for each issue i, the voter's preferences over this issue are always the same, regardless of the values chosen for the other issues) [17]. Second, *multiple-election paradoxes* arise in issue-by-issue voting [8, 17, 23, 25]. In models that do not consider strategic (game-theoretic) voting, previous works have shown several types of paradoxes by assuming that the voters are optimistic about the outcome when they vote on each issue without knowing the outcomes for the other issues: sometimes the winner is a Condorcet loser; sometimes the winner is Pareto-dominated by another alternative (that is, that alternative is preferred to the winner in all votes); and sometimes the winner is ranked in a very low position by all voters.

One way to partly escape these paradoxes consists in organizing the multiple elections *sequentially* [18]: given an order $\mathcal{O}$ over all issues (without loss of generality, we take $\mathcal{O}$ to be $\mathbf{x}_1 > \cdots > \mathbf{x}_n$), the voters first vote on issue $\mathbf{x}_1$; then, the value collectively chosen for $\mathbf{x}_1$ is determined using some voting rule and broadcast to the voters, who then vote on issue $\mathbf{x}_2$, and so on. When the issues are all binary, it is natural to use the majority rule at each stage (plus, in the case of an even number of voters, some tie-breaking mechanism). Such processes are conducted in many real-life situations. For instance, suppose there is a full-professor position and an assistant-professor position to be filled. Then, it is realistic to expect that the committee will first decide who gets the full-professor position. Another example is that at an executive meeting of the co-owners of a building, important decisions like whether a lift should be installed or not, or how much money should be spent to repair the roof, are usually taken before minor decisions. In each of these cases, it is clear that the decision made on one issue is likely to influence the preferences and votes on later issues. Thus, the order in which the issues are decided potentially has a strong influence on the final outcome.

Now, if voters are assumed to know the preferences of other voters well enough, then we can expect them to vote strategically at each step, forecasting the outcome at later steps conditional on the outcome at the current step. Let us consider the following motivating example (a similar example was briefly discussed by Lacy and Niou [17]).

Example 1 Three residents have to vote to decide whether they should build a swimming pool and/or a tennis court. There are two issues, **S** and **T**. **S** takes the value s ("the swimming pool will be built") or $\bar{s}$ ("the swimming pool will not be built"). Similarly, **T** takes its value in $\{t, \bar{t}\}$. Suppose the preferences of the three voters are as follows (voters 2 and 3, perhaps taking budget considerations into account, do not rank st as their first choice):

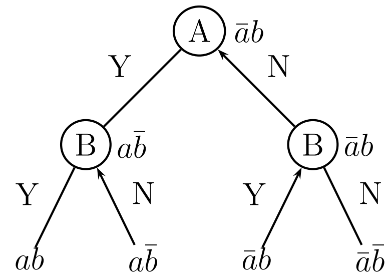

Figure 1: Backward reasoning tree for Example 1.

voter 1	$st \succ \bar{s}t \succ s\bar{t} \succ \bar{s}\bar{t}$
voter 2	$s\bar{t} \succ st \succ \bar{s}t \succ \bar{s}\bar{t}$
voter 3	$\bar{s}t \succ \bar{s}\bar{t} \succ s\bar{t} \succ st.$

Suppose the voters first vote on issue **S** and then on issue **T**. The local rule used at each step is majority (there cannot be any ties, because the number of voters is odd). Supposing voter 1 knows the true preferences of voters 2 and 3, she is likely to reason in the following way: *if the outcome of the first step is s, then voters 2 and 3 will vote for $\bar{t}$, since they both prefer $s\bar{t}$ to st, and the final outcome will be $s\bar{t}$; but if the outcome of the first step is $\bar{s}$, then voters 2 and 3 will vote for t, and the final outcome will be $\bar{s}t$; because I prefer $\bar{s}t$ to $s\bar{t}$, I am better off voting for $\bar{s}$, since either it will not make any difference, or it will lead to a final outcome of $\bar{s}t$ instead of $s\bar{t}$.* If voters 2 and 3 reason in the same way, then voter 2 will vote for s and voter 3 for $\bar{s}$; hence, the result of the first step is $\bar{s}$, and then, since two voters out of three prefer $\bar{s}t$ to $\bar{s}\bar{t}$, the final outcome will be $\bar{s}t$. This backward reasoning process is shown in Figure 1.

Note that the result is fully determined, provided that (1) it is common knowledge that voters behave strategically according to the principle we have stated informally, (2) the order in which the issues are decided, as well as the local voting rules used in all steps, are also common knowledge, and (3) voters' preferences are strict and are common knowledge. Therefore, these three assumptions allow the voters and the modeler (provided he knows as much as the voters) to predict the final outcome.

Let us take a closer look at voter 1 in Example 1. Her preferences are *separable*: she prefers s to $\bar{s}$ whatever the value of **T** is, and t to $\bar{t}$ whatever the value of **S** is. *And yet she votes strategically for $\bar{s}$*, because the outcome for **S** affects the outcome for **T**. Moreover, while voters 2 and 3 have nonseparable preferences, still, all three voters' preferences enjoy the following property: their preferences over the value of $\mathbf{S}$ are independent of the value of $\mathbf{T}$. Such a profile is called a *legal* profile with respect to the order $\mathbf{S} > \mathbf{T}$, where $\mathbf{S} > \mathbf{T}$ means that the voters vote on $\mathbf{S}$ first, then on $\mathbf{T}$. Lang and Xia [18] defined a family of sequential voting rules on multi-issue domains, restricted to $\mathcal{O}$-legal profiles for some order $\mathcal{O}$ over the issues, where at each step, each voter is expected to vote for her preferred value of the issue $\mathbf{x}_i$ under consideration given the values of all issues decided so far (the $\mathcal{O}$-legality condition ensures that this notion of "preferred value of $\mathbf{x}_i$" is meaningful); then, the value of $\mathbf{x}_i$ is chosen according to a local voting rule, and this local outcome is broadcast to the voters. For example, suppose the local rule used to decide an issue is always majority. For the profile given in Example 1, the outcome of the first step under the sequential voting rule would be s (since two voters out of three prefer s to $\bar{s}$, unconditionally), and the final outcome will be $s\bar{t}$. This outcome is different from the outcome we obtain if voters behave strategically. The reason for this discrepancy is that in the work of Lang and Xia [18], voters are not assumed to know the others' preferences and are assumed to vote truthfully.

We have seen in Example 1 that even if the voters' preferences are $\mathcal{O}$-legal and majority voting is used for each issue, voters may in fact have an incentive not to vote truthfully. Consequently, existing results on multiple-election paradoxes are not directly applicable to situations where voters vote strategically.

1.1. Our Contributions

In this paper, we analyze the complete-information game-theoretic model of sequential voting that we illustrated in Example 1. This model applies to any preferences that the voters may have (not only separable or $\mathcal{O}$-legal ones), though they must be linear orders on the set of all alternatives. That is, they express a strict preference on any pair of alternatives.

We focus on voting in multi-binary-issue domains, that is, for every $i \leq p$, $\mathbf{x}_i$ must take a value in $\{0_i, 1_i\}$. This setting has the advantage that for each issue, we can use the majority rule as the local rule for that issue. We model the sequential voting process as a p-stage complete-information game as follows. There is an order $\mathcal{O}$ over all issues (without loss of generality, let $\mathcal{O} = \mathbf{x}_1 > \mathbf{x}_2 > \cdots > \mathbf{x}_p$), indicating the order in which these issues will be voted on. For any $1 \leq i \leq p$, in stage i, the voters vote on issue $\mathbf{x}_i$ simultaneously, and the majority rule is used to choose the winning value for $\mathbf{x}_i$. We make the following assumptions: it is common knowledge that all voters are perfectly rational; the order $\mathcal{O}$ and the fact that in each step, the majority rule is used to determine the winner are common knowledge; all voters' preferences are common knowledge.

We can solve this game by a type of backward induction already illustrated in Example 1: in the last (pth) stage, only two alternatives remain (corresponding to the two possible values for the last issue), so at this point it is a weakly dominant strategy for each voter to vote for her more preferred alternative of the two. Then, in the second-to-last ($(p-1)$th) stage, there are two possible local outcomes for the $(p-1)$th issue; for each of them, the voters can predict which alternative will finally be chosen, because they can predict what will happen in the pth stage. Thus, the $(p-1)$th stage is effectively a majority election between two alternatives, and each voter will vote for her more preferred alternative; etc. We call this procedure the *strategic sequential voting procedure (SSP)*.[1]

Given exogenously the order $\mathcal{O}$ over the issues, this game-theoretic analysis maps every profile of linear orders to a unique outcome. Being a function from preference profiles to alternatives, the function mapping any preference profile to the outcome of the SSP (given $\mathcal{O}$) can be seen as a voting rule in its own right, and will be denoted by $SSP_{\mathcal{O}}$.

One of the most natural questions to ask is: How can we evaluate the equilibrium outcome of SSP? Lacy and Niou [17] showed that whenever there exists a Condorcet winner, it must be the SSP winner: SSP is Condorcet-consistent. Is there anything else positive to say about SSP? The answer we give in this paper leans to the negative side: we show that SSP is prone to all three major types of multiple-election paradoxes, and to a large extent. To better present our results, we introduce a parameter which we call the *minimax satisfaction index (MSI)* in Section 4. For an election with m alternatives and n voters, it is defined in the following way. For each profile, consider the highest position that the winner obtains across all input rankings of the alternatives (the ranking where this position is obtained corresponds to the most-satisfied voter); this is the *maximum satisfaction index* for this profile. Then, the minimax satisfaction index is obtained by taking the minimum over all profiles of the maximum satisfaction index. A low min-

[1] Lacy and Niou [17] called such a procedure *sophisticated voting* following Farquharson [12].

imax satisfaction index means that there exists a profile in which the winner is ranked in low positions in all votes, thus indicating a multiple-election paradox. Our main results are categorized into the following three settings:

The first setting (Section 5): Voters vote over multi-binary-issue domains, and their preferences are not restricted (meaning that a voter's preferences can be any complete strict order over the alternatives). In this setting, we show that unfortunately, severe multiple-election paradoxes arise under SSP. Our main theorem is the following.

Theorem 2 *For any $p \in \mathbb{N}$ and any $n \geq 2p^2+1$, the minimax satisfaction index of SSP when there are $m = 2^p$ alternatives and n voters is $\lfloor p/2 + 2 \rfloor$. Moreover, in the profile P that we use to prove the upper bound, the winner $SSP_{\mathcal{O}}(P)$ is Pareto-dominated by $2^p - (p+1)p/2$ alternatives.*

We note that an alternative c Pareto-dominates another alternative c' implies that c beats c' in their pairwise election. Therefore, Theorem 2 implies that the winner for SSP is almost a Condorcet loser. It follows from this theorem that SSP exhibits all three types of multiple-election paradoxes: the winner is ranked almost in the bottom position in every vote, the winner is almost a Condorcet loser, and the winner is Pareto-dominated by almost every other alternative. We further show a paradox (Theorem 4) that states that there exists a profile such that for *every* order $\mathcal{O}$ over the issues, and for every voter, the SSP winner w.r.t. $\mathcal{O}$ is ranked almost in the bottom position by all voters.

The second setting (Section 6): Voters vote over multi-binary-issue domains, and their preferences are restricted. We show that even when the voters' preferences share the same independence structure, which is compatible with the order in which the issues are voted on (formally, $\mathcal{O}$-legal—see Section 2.2), multiple-election paradoxes still arise. However, if we further restrict the voters' preferences to be separable or lexicographic, then some paradoxes can be avoided.

The third setting (Section 7): Voters vote over general (not necessarily multi-binary-issue) domains, and their preferences are not restricted. To see whether there are similar paradoxes for common voting rules where voters are assumed to vote truthfully, we calculate the minimax satisfaction index for some common voting rules. More precisely, we calculate MSI for dictatorships, positional scoring rules (including k-approval and Borda), plurality with runoff, Copeland, maximin, STV, Bucklin, ranked pairs, and (not necessarily balanced) voting trees. We show that k-approval with a large k, voting trees, Copeland_0, and maximin are prone to a similar paradox, but not the other rules considered.

1.2. Related Work

The setting of SSP was previously considered by Lacy and Niou [17]. But at a high level, our motivation, results, and conclusion are quite different from those of Lacy and Niou. They viewed SSP as a voting rule, and aimed at proposing solutions to aggregate non-separable profiles in combinatorial voting. They showed that SSP satisfies Condorcet consistency, but did not mention whether the other types of multiple-election paradoxes can be avoided. We, on the other hand, aim at examining the equilibrium outcomes in voting games, and show that the other three types of multiple-election paradoxes do arise in SSP.[2]

More generally, SSP is closely related to *multi-stage sophisticated voting*, studied by McKelvey and Niemi [20], Moulin [21], and Gretlein [14]. In their paper, the backward induction outcomes correspond to the truthful outcomes of voting trees. Therefore, our SSP is a special case of multi-stage sophisticated voting. However, their work focused on the characterization of the outcomes as the outcomes of *sophisticated voting* [12], and therefore did not shed much light on the quality of the equilibrium outcome. We, on the other hand, are primarily interested in the strategic outcome of the natural procedure of voting sequentially over multiple issues. Also, the relationship between sequential voting and voting trees takes a particularly natural form in the context of domains with multiple binary issues, as we will show. More importantly, we illustrate several multiple-election paradoxes for SSP, indicating that the equilibrium outcome can be extremely undesirable.

Another paper that is closely related to part of this work was written by Dutta and Sen [11]. They showed that social choice rules corresponding to binary voting trees can be implemented via backward induction via a sequential voting mechanism. This is closely related to the relationship revealed for multi-stage sophisticated voting and will also be mentioned later in this paper; that is, we will sho an equivalence between the outcome of strategic behavior in sequential voting over multiple binary issues, and a particular type of voting tree. It should be pointed out that the sequential mechanism

[2] In fact, those types of paradoxes were also discovered by Lacy and Niou in the same paper [17], but they did not discuss whether they arise in SSP.

that Dutta and Sen consider is somewhat different from sequential voting as we consider it—in particular, in the Dutta-Sen mechanism, one voter moves at a time, and a move consists not of a vote, but rather of choosing the next player to move (or in some states, choosing the winner).

It has been pointed out that typical multiple-election paradoxes partly stem from the incompleteness of information about the preferences of the voters [17]. However, the paradoxes in this paper show that assuming that voters' preferences are common knowledge does not allow one to get rid of multiple-election paradoxes. Another interpretation of these results is that we may need to move beyond sequential voting to properly address voting in multi-issue domains. However, note that approaches other than sequential voting may be extremely costly in terms of computation.

Recently, Ahn and Oliveros [1] studied simultaneous voting games over multiple binary issues where the voters have cardinal preferences (utilities) over combinations of values of the issues and incomplete information about the types of other voters. They focused on characterizing symmetric Bayesian Nash equilibrium in weakly undominated strategy. We, on the other hand, focused on the extensive-form game of sequential voting and characterizing the only backward induction outcome when voters' preferences are ordinal.

Lastly, in a recent paper [27], Xia and Conitzer studied a voting game with a different type of sequential nature: in it, the voters cast their votes one after another (strategically), and after all the voters have cast their votes, a common voting rule (not necessarily the plurality rule) is used to select the winner. This type of voting games has been studied in the literature [3, 9, 10, 26]. In the work of Xia and Conitzer [27], a strong general paradox was shown for these voting games, implying that for most common voting rules, there exists a profile such that the unique winner in all subgame-perfect equilibria is ranked within the bottom two positions in almost all the voters' true preferences. Desmedt and Elkind [10] showed similar paradoxes for such voting games with the plurality rule, where random tie-breaking is used and the voters seek to maximize their expected utility. We note that the voting games studied in this line of work [3, 9, 10, 26, 27] are quite different from the voting games studied in this paper: there, the voters move in sequence, the set of alternatives does not need to have a combinatorial structure, and a voter casts her complete vote all at once; in contrast, here in this paper, the voters move simultaneously, the set of alternatives has a combinatorial structure, and the voters vote on one issue at a time.

2. Preliminaries

2.1. Basics of Voting

Let $\mathcal{X}$ be the set of *alternatives*, with $|\mathcal{X}| = m$. A vote is a linear order (that is, a transitive, antisymmetric, and total relation) over $\mathcal{X}$. The set of all linear orders over $\mathcal{X}$ is denoted by $L(\mathcal{X})$. For any $c \in \mathcal{X}$ and $V \in L(\mathcal{X})$, we write $c \succ_V d$ if c is preferred to d in V; we let $rank_V(c)$ denote the position of c in V from the top (*i.e.*, 1 for the voter's preferred alternative, and m for her worst). For any $n \in \mathbb{N}$, an *n-profile* $P \in L(\mathcal{X})^n$ is a collection of n votes. For any $c, d \in \mathcal{X}$ and any profile P, we say that c *Pareto-dominates* d if $c \succ_V d$ for every $V \in P$. A *voting rule* $r : L(\mathcal{X}) \cup L(\mathcal{X})^2 \cup \ldots \to \mathcal{X}$ maps each profile a unique winning alternative. When there are two alternatives and n is odd, the *majority* rule selects the alternative that is preferred by the majority of voters. In this paper we focus on deterministic rules (rather than on correspondences); therefore, when ties occur, we assume there is a predefined tie-breaking mechanism, typically a priority relation over alternatives; in particular, when there are two alternatives x_1, x_2 and n is even, the majority rule with priority relation $x_1 \gg x_2$ selects x_1 if at least half of the voters prefer x_1 to x_2, and x_2 otherwise.

The *majority graph* associated with a profile P is the directed graph whose vertices are the alternatives and containing an edge from c to c' if and only if a majority of voters in P prefer c to c'.

2.2. Multi-Issue Domains

In this paper, the set of all alternatives $\mathcal{X}$ is a *multi-binary-issue domain.* That is, let $\mathcal{I} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ $(p \geq 2)$ be a set of *binary issues*, where each issue $\mathbf{x}_i$ takes a value in a binary *local domain* $D_i = \{0_i, 1_i\}$. The set of alternatives is $\mathcal{X} = D_1 \times \cdots \times D_p$, that is, an alternative is uniquely identified by its values on all issues. For any $U \subseteq \mathcal{I}$ we denote $D_U = \prod_{\mathbf{x}_i \in U} D_i$.

Conditional preferential independence originates in the literature of multiattribute decision theory [16]. Given a preference relation $\succ$ over $L(\mathcal{X})$, an issue $\mathbf{x}_i$, and a subset of issues $W \subseteq \mathcal{I} \setminus \{\mathbf{x}_i\}$, let $U = \mathcal{I} \setminus (W \cup \{\mathbf{x}_i\})$; then, $\mathbf{x}_i$ is *preferentially independent of* W *given* U *(with respect to* $\succ$*)* if for any $\vec{u} \in D_U$ and any $\vec{w}, \vec{w}' \in D_W$,

$$(\vec{u}, 0_i, \vec{w}) \succ (\vec{u}, 1_i, \vec{w}) \text{ if and only if } (\vec{u}, 0_i, \vec{w}') \succ (\vec{u}, 1_i, \vec{w}')$$

In words, if we wish to find out whether changing the value of $\mathbf{x}_i$ from 0_i to 1_i (while keeping everything else fixed) will make the voter better or worse off, we only need to know the values of the issues in U.

Let $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$. A preference relation $\succ$ is *$\mathcal{O}$-legal* [18] if for any $i \leq p$, $\mathbf{x}_i$ is preferentially independent of $\{\mathbf{x}_{i+1}, \ldots, \mathbf{x}_p\}$ given $\{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}\}$. In words, to find out whether a particular change in the value of an issue will make the voter better or worse off, we only need to know the values of earlier issues. A preference relation $\succ$ is *separable* if for any $i \leq p$, $\mathbf{x}_i$ is preferentially independent of $\mathcal{X} \setminus \{\mathbf{x}_i\}$. Note that $\succ$ is separable if and only if it is $\mathcal{O}$-legal for any $\mathcal{O}$. For example:

- $0_1 0_2 \succ 1_1 0_2 \succ 0_1 1_2 \succ 1_1 1_2$ is separable, thus $(\mathbf{x}_1 > \mathbf{x}_2)$-legal and $(\mathbf{x}_2 > \mathbf{x}_1)$-legal;

- $0_1 0_2 \succ 1_1 0_2 \succ 1_1 1_2 \succ 0_1 1_2$ is $(\mathbf{x}_2 > \mathbf{x}_1)$-legal but not $(\mathbf{x}_1 > \mathbf{x}_2)$-legal: the agent prefers 0_2 to 1_2 unconditionally, but her preference on $\mathbf{x}_1$ depends on the value of $\mathbf{x}_2$ (0_1 if $\mathbf{x}_2 = 0_2$, 1_2 if $\mathbf{x}_2 = 1_2$).

- $0_1 0_2 \succ 1_1 1_2 \succ 1_1 0_2 \succ 0_1 1_2$ is neither $(\mathbf{x}_2 > \mathbf{x}_1)$-legal nor $(\mathbf{x}_1 > \mathbf{x}_2)$-legal.

A preference relation $\succ$ is *$\mathcal{O}$-lexicographic* if for any $i \leq p$, any $\vec{u} \in D_1 \times \cdots \times D_{i-1}$, any $a_i, b_i \in D_i$, and any $\vec{d_1}, \vec{d_2}, \vec{e}_1, \vec{e}_2 \in D_{i+1} \times \cdots \times D_p$, $(\vec{u}, a_i, \vec{d_1}) \succ (\vec{u}, b_i, \vec{e}_1)$ if and only if $(\vec{u}, a_i, \vec{d_2}) \succ (\vec{u}, b_i, \vec{e}_2)$. An $\mathcal{O}$-lexicographic profil is $\mathcal{O}$-legal, with earlier issues being more important than later issues. Note that $\mathcal{O}$-lexicographicity and separability are incomparable notions. For example, $0_1 0_2 \succ 1_1 0_2 \succ 0_1 1_2 \succ 1_1 1_2$ is separable but not $(\mathbf{x}_1 > \mathbf{x}_2)$-lexicographic ($0_1 0_2 \succ 1_1 1_2$ but $1_1 0_2 \succ 0_1 1_2$), and on the other hand, $0_1 0_2 \succ 0_1 1_2 \succ 1_1 1_2 \succ 1_1 0_2$ is $(\mathbf{x}_1 > \mathbf{x}_2)$-lexicographic but not separable. A profile is separable/$\mathcal{O}$-lexicographic/$\mathcal{O}$-legal if it is composed of separable/$\mathcal{O}$-lexicographic/$\mathcal{O}$-legal preference relations.

We now recall the definition of sequential composition of local majority voting rules [18] on a domain composed of several binary issues. Given $D = D_1 \times \ldots D_p$, where $D_i = \{0_i, 1_i\}$, and, without loss of generality, the order $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$, the *sequential composition of local majority rules $Seq\mathcal{O}(maj_1, \ldots, maj_p)$* is defined for all $\mathcal{O}$-legal profiles as follows: $Seq_{\mathcal{O}}(maj_1, \ldots, maj_p)(P) = (d_1, \ldots, d_p) \in \mathcal{X}$, where for any $i \leq p$, $d_i = maj_i(P|_{\mathbf{x}_i : d_1 \cdots d_{i-1}})$, where maj_i is the majority rule on D_i (plus some tie-breaking mechanism if n is even) and $P|_{\mathbf{x}_i:d_1\cdots d_{i-1}}$ is composed of the voters' local preferences over $\mathbf{x}_i$, given that the issues preceding it take values $d_1, \ldots, d_{i-1}$. Thus, the winner is selected in p steps, one for each issue, in the following way: in step i, d_i is selected by applying the local rule maj_i to the preferences of voters over D_i, conditioned on the values $d_1, \ldots, d_{i-1}$ that have already been determined for the issues that precede $\mathbf{x}_i$. With a slight abuse of notation, we use $Seq_{\mathcal{O}}(maj, \ldots, maj)$ instead of $Seq_{\mathcal{O}}(maj_1, \ldots, maj_p)$.

3. Strategic Sequential Voting

As we have illustrated in Example 1, sequential voting in multi-issue domains can be seen as an extensive-form game where in each step, the voters decide whether to vote for or against the issue under consideration after reasoning about what will happen next. We make the following assumptions.

1. All voters act strategically (in an optimal manner that will be explained later), and this is common knowledge.
2. The order in which the issues will be voted upon, as well as the local voting rules used at the different steps (namely, majority rules), are common knowledge.
3. All voters' preferences on the set of alternatives are common knowledge.

Assumption 1 is of course standard in game theory. Assumption 2 merely means that the rule has been announced. Assumption 3 (complete information) is the most significant assumption. It may be interesting to consider more general settings with incomplete information, resulting in a Bayesian game. Nevertheless, the complete-information setting is a special case of the incomplete-information setting (where the prior distribution is degenerate), and in that sense, *all negative results obtained for the complete-information setting also apply to the incomplete-information setting.* That is, the restriction to complete information only strengthens negative results. Of course, for the incomplete-information setting in general, we need a more elaborate model to reason about voters' strategic behavior.

3.1. Formal Definition

Given these assumptions, the voting process can be modeled as a game that is composed of p stages where in each stage, the voters vote simultaneously on one issue. Let $\mathcal{O}$ be the order over the set of issues, which without loss of generality we assume to be $\mathbf{x}_1 > \cdots > \mathbf{x}_p$. Let P be the preference profile over $\mathcal{X}$. The game is defined as follows: for each $i \leq p$, in stage i the voters vote simultaneously on issue i; then, the value of $\mathbf{x}_i$ is determined by the majority rule (plus, in the case of an even number of voters, some tie-breaking mechanism), and this local outcome is broadcast to all voters.

We now show how to solve the game. Because of assumptions 1 to 3, at step i the voters vote strategically, by recursively figuring out what the final outcome will be if the local outcome for $\mathbf{x}_i$ is 0_i, and what it will be if it is 1_i. More concretely, suppose that steps 1 to $i-1$ resulted in issues $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ taking the values $d_1, \ldots, d_{i-1}$, and let $\vec{d} = (d_1, \ldots, d_{i-1})$. Suppose also that if $\mathbf{x}_i$ takes the value 0_i (respectively, 1_i), then, recursively, the remaining issues will take the tuple of values $\vec{a}$ (respectively, $\vec{b}$). Then, $\mathbf{x}_i$ is determined by a pairwise comparison between $(\vec{d}, 0_i, \vec{a})$ and $(\vec{d}, 1_i, \vec{b})$ in the following way: if the majority of voters prefer $(\vec{d}, 0_i, \vec{a})$ over $(\vec{d}, 1_i, \vec{b})$, then $\mathbf{x}_i$ takes the value 0_i; in the opposite case, $\mathbf{x}_i$ takes the value 1_i. This process, which corresponds to the strategic behavior in the sequential election, is what we call the *strategic sequential voting (SSP)* procedure, and for any profile P, the winner with respect to the order $\mathcal{O}$ is denoted by $SSP_{\mathcal{O}}(P)$.

As we shall see later, SSP can not only be thought of as the strategic outcome of sequential voting, but also as a voting rule in its own right. The following definition and two propositions merely serve to make the game-theoretic solution concept that we use precise; a reader who is not interested in this may safely skip them.

Definition 1 *Consider a finite extensive-form game which transitions among* states. *In each nonterminal state s, all players simultaneously take an action; this joint local action profile $(a_1^s, \ldots, a_n^s)$ determines the next state s'.*[3] *Terminal states t are associated with payoffs for the players (alternatively, players have ordinal preferences over the terminal states). The current state is always common knowledge among the players.*[4]

Suppose that in every final nonterminal state s (that is, every state that has only terminal states as successors), every player i has a (weakly) dominant action a_i^s. At each final nonterminal state, its local profile of dominant

[3]In the extensive-form representation of the game, each state is associated with multiple nodes, because in the extensive form only one player can move at a node.

[4]Hence, the only imperfect information in the extensive form of the game is due to simultaneous moves within states.

actions $(a_1^s, \ldots, a_n^s)$ *results in a terminal state* $t(s)$ *and associated payoffs. We then replace each final nonterminal state* s *with the terminal state* $t(s)$ *that its dominant-strategy profile leads to. Furthermore suppose that in the resulting smaller tree, again, in every final nonterminal state, every player has a (weakly) dominant strategy. Then, we can repeat this procedure,* etc. *If we can repeat this all the way to the root of the tree, then we say that the game is solvable by* within-state dominant-strategy backward induction (WSDSBI)*.*

We note that the backward induction in perfect-information extensive-form games is just the special case of WSDSBI where in each state only one player acts.

Proposition 1 *The complete-information sequential voting game with binary issues (with majority as the local rule everywhere) is solvable by WSDSBI, and the outcome is unique.*

Proof: The states correspond to the local elections in which an issue is decided. Suppose that we have managed to apply WSDSBI to solve the last k stages of the game, thereby replacing the states of the $(p-k+1)$th stage with terminal states. Then, each state in the $(p-k)$th stage is a majority election between two alternatives, where each voter has a strict preference between these two alternatives. Because the rule used is majority, it is weakly dominant for each voter to vote for her preferred alternative, so we can solve the $(p-k)$th stage as well. □

We note that SSP corresponds to a particular balanced voting tree (see Section 7 for the formal definition), as illustrated in Figure 2 for the case $p = 3$. In this voting tree, in the first round, each alternative is paired up against the alternative that differs only on the pth issue; each alternative that wins the first round is then paired up with the unique other remaining alternative that differs only on the $(p-1)$th and possibly the pth issue; etc. This bottom-up procedure corresponds exactly to the backward induction (WSDSBI) process.

Of course, there are many voting trees that do *not* correspond to an SSP election; this is easily seen by observing that there are only $p!$ different SSP elections (corresponding to the different orders of the issues), but many more voting trees. The voting tree corresponding to the order $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ is defined by the property that for any node v whose depth is i (where the root has depth 1), the alternative associated with any leaf in the left (respectively, right) subtree of v gives the value 0_i (respectively, 1_i) to $\mathbf{x}_i$.

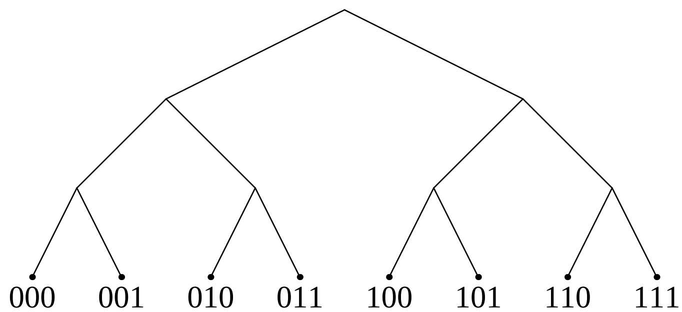

Figure 2: A voting tree that is equivalent to the strategic sequential voting procedure ($p = 3$). 000 is an abbreviation for $0_1 0_2 0_3$, etc.

3.2. Strategic Sequential Voting vs. Truthful Sequential Voting

We have seen on Example 1 that even when the profile P is $\mathcal{O}$-legal, $SSP_{\mathcal{O}}(P)$ can be different from $Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$. This means that even if the profile is $\mathcal{O}$-legal, voters may be better off voting strategically than truthfully. However, $SSP_{\mathcal{O}}(P)$ and $Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$ are guaranteed to coincide under the further restriction that P is $\mathcal{O}$-lexicographic.

Proposition 2 *For any $\mathcal{O}$-lexicographic profile P,*

$$SSP_{\mathcal{O}}(P) = Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$$

Proof: We prove the proposition by induction on p. Suppose the proposition is true over $D_{-1} = D_2 \times \cdots \times D_p$. For any $V \in P$, because V is $\mathcal{O}$-lexicographic, we have that if $0_1 \succ_{V|_{\mathbf{x}_1}} 1_1$, then for any $\vec{a}, \vec{b} \in \mathcal{X}_2$, $(0_1, \vec{a}) \succ_V (1_1, \vec{b})$, and vice versa. Therefore, in the first round, it is a dominant strategy for every voter to truthfully submit the restriction of her preferences over $\mathbf{x}_1$; hence, by its game-theoretical interpretation, under $SSP_{\mathcal{O}}$, the first issue is set to the value $maj(P|_{\mathbf{x}_1})$. Then, by the induction hypothesis, the winner over D_{-1} is the same for both SSP and Seq. Therefore, we have that $SSP_{\mathcal{O}}(P) = Seq(maj, \ldots, maj)(P)$. $\Box$

The intuition for Proposition 2 is as follows: if P is $\mathcal{O}$-lexicographic, then, as is shown in the proof of the proposition, when voters vote strategically under sequential voting (the *Seq* process), they are best off voting according to their true preferences in each round (their preferences in each round are well-defined because voters have $\mathcal{O}$-legal preferences in this case). When voters with $\mathcal{O}$-legal preferences vote truthfully in each round under sequential voting, the outcome is $Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$; when they vote strategically, the outcome is $SSP_{\mathcal{O}}(P)$; and so, these must be the same when preferences are $\mathcal{O}$-lexicographic.

The next proposition shows that there is another domain restriction under which $SSP_{\mathcal{O}}(P)$ and $Seq(maj, \ldots, maj)(P)$ coincide, namely when P is $inv(\mathcal{O})$- legal, where $inv(\mathcal{O}) = (\mathbf{x}_p > \ldots > \mathbf{x}_1)$.

Proposition 3 *Let $inv(\mathcal{O}) = \mathbf{x}_p > \ldots > \mathbf{x}_1$. For any $inv(\mathcal{O})$-legal profile P, $SSP_{\mathcal{O}}(P) = Seq_{inv(\mathcal{O})}(maj, \ldots, maj)(P)$.*

Proof: We prove the proposition by induction on p. Suppose the proposition is true over $D_{-p} = D_1 \times \cdots \times D_{p-1}$. We consider the winners in the bottom layer of the voting tree corresponding to SSP. For any $\vec{a} \in D_{-p}$, the voters are comparing $(\vec{a}, 0_p)$ and $(\vec{a}, 1_p)$. Because P is $inv(\mathcal{O})$-legal, for any $\vec{a} \in D_{-p}$ and any $j \leq n$, voter j's preferences over $\mathbf{x}_p$ are independent of the other issues. Thus, she prefers $(\vec{a}, 0_p)$ to $(\vec{a}, 1_p)$ if and only if $0_p \succ_{V_j|_{\mathbf{x}_p}} 1_p$. Therefore, the winning value for $\mathbf{x}_p$ is $d_p = maj(P|_{\mathbf{x}_p})$ *everywhere* in the first round of the voting tree, and the corresponding alternatives propagate up to the next level in the tree. For these remaining alternatives, we only need the restricted preference profile $P|_{\mathbf{x}_{-p}:d_p}$, which is $inv(\mathcal{O}_{-p})$-legal (where $inv(\mathcal{O}_{-p})$ is the order $\mathbf{x}_{p-1} > \cdots > \mathbf{x}_1$). By the induction hypothesis, the winner for the rest of the voting tree is the same as $Seq_{inv(\mathcal{O}_{-p})}(maj, \ldots, maj)(P|_{\mathbf{x}_{-p}:d_p})$. It follows that $SSP_{\mathcal{O}}(P) = Seq_{inv(\mathcal{O})}(maj, \ldots, maj)(P)$. □

As a consequence, when P is separable, it is *a fortiori* $inv(\mathcal{O})$-legal, and therefore, $SSP_{\mathcal{O}}(P) = Seq_{inv(\mathcal{O})}(maj, \ldots, maj)(P)$, which in turn is equal to $Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$ and coincides with seat-by-seat voting [4].

Corollary 1 *If P is separable, then $SSP_{\mathcal{O}}(P) = Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$.*

3.3. The Winner is Sensitive to the Order Over the Issues

In the definition of SSP, we simply fixed the order $\mathcal{O}$ to be $\mathbf{x}_1 > \mathbf{x}_2 > \cdots > \mathbf{x}_p$. A question worth addressing is, to what extent is the outcome of SSP sensitive to the variation of the order $\mathcal{O}$? More precisely, given a profile P, let $\text{PW}(P) = |\{\vec{d} \in \mathcal{X} \mid \vec{d} = SSP_{\mathcal{O}'}(P) \text{ for some order } \mathcal{O}'\}|$. In other words, $\text{PW}(P)$ is the number of different alternatives that can be made

SSP winners by choosing a particular order $\mathcal{O}'$. Then, for a given number of binary issues p, we look for the maximal value of PW(P), for all profiles P on $\mathcal{X} = D_1 \times \ldots \times D_p$; we denote this number by MW(p).

A first observation is that there are $p!$ different choices for $\mathcal{O}'$. Therefore, a trivial upper bound on MW(p) is $p!$. Since there are 2^p alternatives, the $p!$ upper bound is only interesting when $p! < 2^p$, that is, $p \leq 3$. Example 2 shows that when $p = 2$ or $p = 3$, this trivial upper bound is actually tight, i.e. MW(2) = 2! and MW(3) = 3!: there exists a profile such that by changing the order over the issues, all $p!$ different alternatives can be made winners. Due to McGarvey's theorem [19], any complete and asymmetric directed graph G over the alternatives corresponds to the majority graph of some profile. Therefore, in the example, we only show the majority graph instead of explicitly constructing the whole profile.

Example 2 *The majority graphs for $p = 2$ and $p = 3$ are shown in Figure 3. Let P (respectively, P') denote an arbitrary profile whose majority graph is the same as Figure 3(a) (respectively, Figure 3(b)). It is not hard to verify that $SSP_{\mathbf{x}_1 > \mathbf{x}_2}(P) = 00$ and $SSP_{\mathbf{x}_2 > \mathbf{x}_1}(P) = 01$. For P', the value of $SSP_{\mathcal{O}'}(P')$ for the six possible orders is shown on Table 1. Note that $2! = 2$ and $3! = 6$. It follows that when $p = 2$ or $p = 3$, there exists a profile for which the SSP winners w.r.t. different orders over the issues are all different from each other.*

The order	$\mathbf{x}_1 > \mathbf{x}_2 > \mathbf{x}_3$	$\mathbf{x}_1 > \mathbf{x}_3 > \mathbf{x}_2$	$\mathbf{x}_2 > \mathbf{x}_1 > \mathbf{x}_3$
SSP winner	010	011	001
The order	$\mathbf{x}_2 > \mathbf{x}_3 > \mathbf{x}_1$	$\mathbf{x}_3 > \mathbf{x}_1 > \mathbf{x}_2$	$\mathbf{x}_3 > \mathbf{x}_2 > \mathbf{x}_1$
SSP winner	100	110	101

Table 1: The SSP winners for P' w.r.t. different orders over the issues.

When $p \geq 4$, $p! > 2^p$. However, it is not immediately clear whether MW(p) $= 2^p$ or not, i.e., whether each of the 2^p alternatives can be made a winner by changing the order over the issues. The next theorem shows that this can actually be done, that is, MW(p) $= 2^p$.

Theorem 1 *For any $p \geq 4$ and any $n \geq 142 + 4p$, there exists an n-profile P such that for every alternative $\vec{d}$, there exists an order $\mathcal{O}'$ over $\mathcal{I}$ such that $SSP_{\mathcal{O}'}(P) = \vec{d}$.*

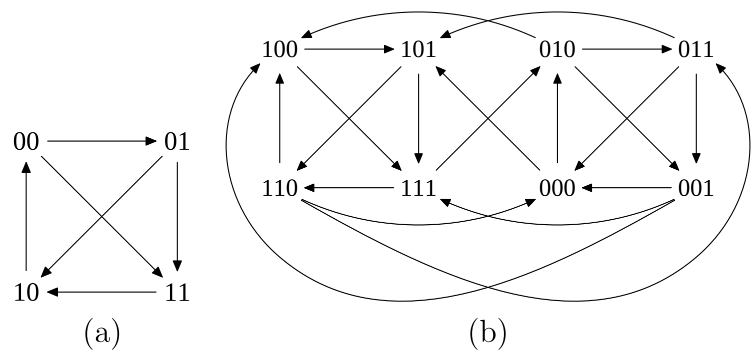

Figure 3: (a) The majority graph for $p = 2$. (b) The majority graph for $p = 3$, where four edges are not shown in the graph: $100 \to 000$, $101 \to 001$, $110 \to 010$, and $111 \to 011$. The directions of the other edges are defined arbitrarily. 000 is the abbreviation for $0_1 0_2 0_3$, etc.

Proof: We prove the theorem by induction on the number of issues p. Surprisingly, the hardest part in the inductive proof is the base case: when we first show how to construct a desirable majority graph $\mathcal{M}$ for $p = 4$, then we show how to construct an n-profile that corresponds to $\mathcal{M}$.

To define $\mathcal{M}$ when $p = 4$, we first define a majority graph $\mathcal{M}_3$ over $\mathcal{X}_3 = D_2 \times D_3 \times D_4$. Let $\mathcal{M}'$ denote the majority graph defined in Example 2 when $p = 3$. We note that $\mathcal{M}'$ is defined over $D_1 \times D_2 \times D_3$. The structure of $\mathcal{M}_3$ is exactly the same as $\mathcal{M}'$, except that $\mathcal{M}_3$ is defined over $D_2 \times D_3 \times D_4$. Formally, let $h_1 : D_1 \to D_2$ be a mapping such that $h_1(0_1) = 0_2$ and $h_1(1_1) = 1_2$; let $h_2 : D_2 \to D_3$ be a mapping such that $h_2(0_2) = 0_3$ and $h_2(1_2) = 1_3$; and let $h_3 : D_3 \to D_4$ be a mapping such that $h_3(0_3) = 0_4$ and $h_3(1_3) = 1_4$. Let $h : D_1 \times D_2 \times D_3 \to D_2 \times D_3 \times D_4$ be a mapping such that for any $(a_2, a_3, a_4) \in \{0, 1\}$, $h(a_1, a_2, a_3) = (h_1(a_1), h_2(a_2), h_3(a_3))$. For example, $h(0_1 1_2 0_3) = 0_2 1_3 0_4$. Then, we let $\mathcal{M}_3 = h(\mathcal{M}')$.

For any $\vec{a} = (a_2, a_3, a_4) \in \mathcal{X}_3$, let $f(\vec{a}) = (1_1, \vec{a})$ and let $g(\vec{a}) = (0_1, \overline{a_2}, a_3, a_4)$. That is, f concatenates 1_1 and $\vec{a}$, and g flips the first two components of $f(\vec{a})$. For example, $f(0_2 0_3 0_4) = 1_1 0_2 0_3 0_4$ and $g(0_2 0_3 0_4) = 1_1 1_2 0_3 0_4$. We define $\mathcal{M}$ as follows.

(1) The subgraph of $\mathcal{M}$ over $\{1_1\} \times \mathcal{X}_3$ is $f(\mathcal{M}_3)$. That is, for any $\vec{a}, \vec{b} \in \mathcal{X}_3$, if $\vec{a} \to \vec{b}$ in $\mathcal{M}'$, then $f(\vec{a}) \to f(\vec{b})$ in $\mathcal{M}$.

(2) The subgraph of $\mathcal{M}$ over $\{0_1\} \times \mathcal{X}_3$ is $g(\mathcal{M}_3)$.

(3) For any $\vec{a} \in \mathcal{X}_3$, we have $(1_1, \vec{a}) \to (0_1, \vec{a})$. For any $\vec{a} \in \mathcal{X}_3$ and $\vec{a} \neq 111$, we have $g(\vec{a}) \to f(\vec{a})$.

(4) We then add the following edges to $\mathcal{M}$. $0100 \to 1110$, $1000 \to 0010$, $1101 \to 0111$, $0001 \to 1011$, $1101 \to 0100$, $1000 \to 0001$, $0001 \to 1101$, $0100 \to 1000$, $1111 \to 0110$, $1100 \to 0101$, $0011 \to 1010$, $1001 \to 0000$, $1111 \to 0011$, $0011 \to 1100$, $0011 \to 1001$, $1111 \to 0000$.

(5) Any other edge that is not defined above is defined arbitrarily.

Let P be an arbitrary profile whose majority graph satisfies conditions (1) through (4) above. We make the following observations.

- If $\mathbf{x}_1$ is the first issue in $\mathcal{O}'$, then the first component of $\text{SSP}_{\mathcal{O}'}(P)$ is 1_1. Moreover, every alternative whose first component is 1_1 (except 1111 and 1000) can be made a winner by changing the order of $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.

- If $\mathbf{x}_1$ is the last issue in $\mathcal{O}'$, then the first component of $\text{SSP}_{\mathcal{O}'}(P)$ is 0_1. Moreover, every alternative whose first component is 0_1 (except 0011 and 0100) can be made to win by changing the order of $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.

- Let $\mathcal{O}' = \mathbf{x}_3 > \mathbf{x}_1 > \mathbf{x}_2 > \mathbf{x}_4$, we have $\text{SSP}_{\mathcal{O}'}(P) = 0100$; let $\mathcal{O}' = \mathbf{x}_3 > \mathbf{x}_1 > \mathbf{x}_4 > \mathbf{x}_2$, we have $\text{SSP}_{\mathcal{O}'}(P) = 1000$; let $\mathcal{O}' = \mathbf{x}_4 > \mathbf{x}_1 > \mathbf{x}_3 > \mathbf{x}_2$, we have $\text{SSP}_{\mathcal{O}'}(P) = 0011$; let $\mathcal{O}' = \mathbf{x}_2 > \mathbf{x}_4 > \mathbf{x}_1 > \mathbf{x}_3$, we have $\text{SSP}_{\mathcal{O}'}(P) = 1111$.

In summary, every alternative is a winner of SSP w.r.t. at least one order over the issues. The reader can also check out a java program online at `http://people.seas.harvard.edu/~lxia/Files/SSP.zip` to verify the correctness of such a construction. We notice that conditions (1) through (4) impose 79 constraints on pairwise comparisons. Therefore, using McGarvey's trick [19], for any $n \geq 2 \times 79 = 158$, we can construct an n-profile whose majority graph satisfies conditions (1) through (4). This means that the theorem holds for $p = 4$.

Now, suppose that the theorem holds for $p = p'$. Let $P = (V_1, \ldots, V_n)$ be an n-profile over $\mathcal{X}' = D_2 \times \ldots \times D_{p'+1}$ such that $n \geq 142 + 4p'$ and each alternative in $\mathcal{X}'$ can be made a winner in SSP by changing the order over $\mathbf{x}_2, \ldots, \mathbf{x}_{p'+1}$. Let $\mathcal{X} = D_1 \times \cdots \times D_{p'+1}$. Let $f : \mathcal{X}' \to \mathcal{X}$ be the mapping defined as follows. For any $\vec{a} \in \mathcal{X}'$, $f(\vec{a}) = (1_1, \vec{a})$. That is, for any $\vec{a} \in \mathcal{X}'$, f concatenates 1_1 and $\vec{a}$. Let $g : \mathcal{X}' \to \mathcal{X}$ be the mapping defined as follows. For any $\vec{a} = (a_2, \ldots, a_{p'+1}) \in \mathcal{X}'$, $g(\vec{a}) = (0_1, \overline{a_2}, a_3, \ldots, a_{p'+1})$. That is, for any $\vec{a} \in \mathcal{X}'$, g flips the first two components of $f(\vec{a})$. Next, we define an $(n + 4)$-profile $P' = (V'_1, \ldots, V'_{n+4})$ as follows.

For any $i \leq 2\lfloor (n-1)/2 \rfloor$, we let $V_i' = \begin{cases} f(V_i) \succ g(V_i) & \text{if } i \text{ is odd} \\ g(V_i) \succ f(V_i) & \text{if } i \text{ is even} \end{cases}$. For any $2\lfloor (n-1)/2 \rfloor + 1 \leq i \leq n$, we let $V_i' = [f(V_i) \succ g(V_i)]$. For any $j \leq 4$, we let

$$V'_{n+j} = \begin{cases} \begin{array}{c} g(0_2 \dots 0_{p+1}) \succ f(0_2 \dots 0_{p+1}) \succ g(0_2 \dots 0_p 1_{p+1}) \\ \succ f(0_2 \dots 0_p 1_{p+1}) \succ g(1_2 \dots 1_{p+1}) \succ f(1_2 \dots 1_{p+1}) \end{array} & \text{if } j \text{ is odd} \\ \begin{array}{c} g(1_2 \dots 1_{p+1}) \succ f(1_2 \dots 1_{p+1}) \succ g(1_2 \dots 1_p 0_{p+1}) \\ \succ f(1_2 \dots 1_p 0_{p+1}) \succ g(0_2 \dots 0_{p+1}) \succ f(0_2 \dots 0_{p+1}) \end{array} & \text{if } j \text{ is even} \end{cases}$$

For any pair of alternatives c, c', and any profile P^*, we let $D_{P^*}(c, c')$ denote the number of times that c is preferred to c', minus the number of times c' is preferred to c, both in the profile P^*. That is, $D_{P^*}(c, c') > 0$ if and only if c beats c' in their pairwise election. We make the following observations on P'.

- For any $\vec{a} \in \mathcal{X}'$, $D_{P'}(f(\vec{a}), g(\vec{a})) > 0$ and $D_{P'}((1_1, \vec{a}), (0_1, \vec{a})) > 0$.
- For any $\vec{a}, \vec{b} \in \mathcal{X}'$ (with $\vec{a} \neq \vec{b}$), $D_{P'}(f(\vec{a}), f(\vec{b})) > 0$ if and only if $D_P(\vec{a}, \vec{b}) > 0$; $D_{P'}(g(\vec{a}), g(\vec{b})) > 0$ if and only if $D_P(\vec{a}, \vec{b}) > 0$.

It follows that for any order $\mathcal{O}'$ over $\{\mathbf{x}_2, \dots, \mathbf{x}_{p+1}\}$, we have $\text{SSP}_{[\mathbf{x}_1 \succ \mathcal{O}']}(P') = f(SSP_{\mathcal{O}'}(P))$ (because after voting on issue $\mathbf{x}_1$, all alternatives whose first component is 0_1 are eliminated, then it reduces to SSP over $\mathcal{X}'$); we also have that $\text{SSP}_{[\mathcal{O}' \succ \mathbf{x}_1]}(P') = g(SSP_{\mathcal{O}'}(P))$ (because in the last round, the two competing alternatives considered are $f(SSP_{\mathcal{O}'}(P))$ and $g(SSP_{\mathcal{O}'}(P))$, and the majority of voters prefer the latter). We recall that each alternative in $\mathcal{X}'$ can be made a winner w.r.t. an order O' over $\{\mathbf{x}_2, \dots, \mathbf{x}_{p'+1}\}$. It follows that each alternative in $\mathcal{X}$ can also be made to win w.r.t. an order over $\{\mathbf{x}_1, \dots, \mathbf{x}_{p'+1}\}$, which means that the theorem holds for $p = p'+1$. Therefore, the theorem holds for any $p \geq 4$. □

4. Minimax Satisfaction Index

In the rest of this paper, we will show that strategic sequential voting on multi-issue domains is prone to paradoxes that are almost as severe as previously studied multiple-election paradoxes under models that are not game-theoretic but assuming that the voters are optimistic about undetermined issues [8, 17]. To facilitate the presentation of these results, we define an index that is intended to measure one aspect of the quality of a voting rule, called the *minimax satisfaction index*.

Definition 2 *For any voting rule r, the* minimax satisfaction index (MSI) *of r is defined as follows:*

$$MSI_r(m,n) = \min_{P \in L(\mathcal{X})^n} \max_{V \in P} \big(m + 1 - rank_V(r(P))\big),$$

where m is the number of alternatives, n is the number of voters, and $rank_V(r(P))$ is the position of $r(P)$ in vote V.

In words, the minimax satisfaction index of a voting rule is $\leq k$ if for every profile, the winner is ranked among the top $m+1-k$ alternatives by at least one voter. Said otherwise, the minimax index is the worst-case (over all profiles) Borda score of the happiest voter, plus one. We do not think this index has been used before to measure the quality of a voting rule.[5]

We also define the minimum satisfaction index of a voting rule *given a domain restriction.*

Definition 3 *For any voting rule r and any domain restriction $Restr \subseteq L(\mathcal{X})^n$, the* minimax satisfaction index (MSI) *of r for $Restr$ is defined as follows:*

$$MSI_{r,Restr}(m,n) = \min_{P \in Restr} \max_{V \in P} \big(m + 1 - rank_V(r(P))\big),$$

where m, n and $rank_V(r(P))$ are as in Definition 2.

We note that in this paper $m = 2^p$, where p is the number of issues. The MSI of a voting rule is not the final word on it—it only measures a particular aspect of the voting rule. A voting rule with a high MSI might be poor in other aspects. For example, the MSI for dictatorships is m, the maximum possible value, which is not to say that dictatorships are desirable. However, if the MSI of a voting rule is low, then this implies the existence

[5]On the other hand, some authors, such as Hodge and Schwallier [15] make use of the average satisfaction index, which is defined as the *sum* over all voters (instead of the maximum) of Borda scores; this index comes down to measuring the proximity of a voting rule to the Borda rule, while the minimax satisfaction index proves to be much more relevant to measuring the extent to which a rule is prone to severe paradoxes.

of a paradox for it, namely, a profile that results in a winner that makes all voters unhappy.

Many of the multiple-election paradoxes known so far implicitly refer to such an index. For example, Lacy and Niou [17] and Benoit and Kornhauser [4] showed that for multiple referenda, if voters vote on issues separately (under some assumptions on how voters vote), then there exists a profile such that in each vote, the winner is ranked near the bottom—therefore the rule has a very low MSI.

5. Multiple-Election Paradoxes for Strategic Sequential Voting

In this section, we show that over multi-binary-issue domains, for any natural number n that is sufficiently large (we will specify the number in our theorems), there exists an n-profile P such that $SSP_{\mathcal{O}}(P)$ is ranked almost in the bottom position in each vote in P. That is, the minimax satisfaction index is extremely low for the strategic sequential voting procedure.

Our main result is the determination of the MSI of $SSP_{\mathcal{O}}$. We devote most of our attention to the simpler case when n is odd, and then show how to extend the theorem to the more complicated case where n is even.

Theorem 2 *For any $p \in \mathbb{N}$ ($p \geq 2$) and any odd number n such that $n \geq 2p^2+1$, we have $MSI_{SSP_{\mathcal{O}}}(m,n) = \lfloor p/2+2 \rfloor$. Moreover, in the profile P that we use to prove the upper bound, the winner $SSP_{\mathcal{O}}(P)$ is Pareto-dominated by $2^p - (p+1)p/2$ alternatives.*

Proof of Theorem 2: The upper bound on $MSI_{SSP_{\mathcal{O}}}(m,n)$ is constructive, that is, we explicitly construct a paradox.

For any n-profile $P = (V_1, \dots, V_n)$, we define the mapping $f_P : \mathcal{X} \to \mathbb{N}^n$ as follows: for any $c \in \mathcal{X}$, $f_P(c) = (h_1, \dots, h_n)$ such that for any $i \leq n$, h_i is the number of alternatives that are ranked below c in V_i. For any $l \leq p$, we denote $\mathcal{X}_l = D_l \times \cdots \times D_p$ and $\mathcal{O}_l = \mathbf{x}_l > \mathbf{x}_{l+1} > \cdots > \mathbf{x}_p$. For any vector $\vec{h} = (h_1, \dots, h_n)$ and any $l \leq p$, we say that $\vec{h}$ is *realizable* over $\mathcal{X}_l$ (through a balanced binary tree) if there exists a profile $P_l = (V_1, \dots, V_n)$ over $\mathcal{X}_l$ such that $f_{P_l}(SSP_{\mathcal{O}_l}(P_l)) = \vec{h}$. We first prove the following lemma.

Lemma 1 *For any l such that $1 \leq l < p$,*

$$\vec{h}_* = (\underbrace{0, \dots, 0}_{\lfloor n/2 \rfloor - p + l}, \underbrace{1, \dots, 1}_{p-l+1}, \underbrace{2^{p-l+1}-1, \dots, 2^{p-l+1}-1}_{\lceil n/2 \rceil - 1})$$

is realizable over $\mathcal{X}_l$.

Proof of Lemma 1: We prove that there exists an n-profile P_l over $\mathcal{X}_l$ such that $SSP_{\mathcal{O}_l}(P_l) = 1_l \cdots 1_p$ and $\vec{h}_*$ is realized by P_l. For any $1 \leq i \leq p-l+1$, we let $\vec{b}_i = 1_l \cdots 1_{p-i} 0_{p+1-i} 1_{p+2-i} \cdots 1_p$. That is, $\vec{b}_i$ is obtained from $1_l \cdots 1_p$ by flipping the value of $\mathbf{x}_{p+1-i}$. We obtain $P_l = (V_1, \ldots, V_n)$ in the following steps.

1. Let $W_1, \ldots, W_n$ be null partial orders over $\mathcal{X}_l$. That is, for any $i \leq n$, the preference relation W_i is empty.

2. For any $j \leq \lfloor n/2 \rfloor - p + l$, we put $1_l \cdots 1_p$ in the bottom position in W_j; we put $\{\vec{b}_1, \ldots, \vec{b}_{p-l+1}\}$ in the top positions in W_j.

3. For any j with $\lfloor n/2 \rfloor + 2 \leq j \leq n$, we put $1_l \cdots 1_p$ in the top position of W_j, and we put $\{\vec{b}_1, \ldots, \vec{b}_{p-l+1}\}$ in the positions directly below the top.

4. For j with $\lfloor n/2 \rfloor - p + l + 1 \leq j \leq \lfloor n/2 \rfloor + 1$, we define preferences as follows. For any $i \leq p-l+1$, in $W_{\lfloor n/2 \rfloor - p + l + i}$, we put $\vec{b}_i$ in the bottom position, $1_l \cdots 1_p$ in the second position from the bottom, and all the remaining b_j (with $j \neq i$) at the very top.

5. Finally, we complete the profile arbitrarily: for any $j \leq n$, we let V_j be an arbitrary extension of W_j.

Let $P_l = (V_1, \ldots, V_n)$. We note that for any $i \leq p - l + 1$, $\vec{b}_i$ beats any alternative in $\mathcal{X}_l \setminus \{1_l \cdots 1_p, \vec{b}_1, \ldots, \vec{b}_{p-l+1}\}$ in pairwise elections. Therefore, for any $i \leq p - l + 1$, the ith alternative that meets $1_l \cdots 1_p$ is $\vec{b}_i$, which loses to $1_l \cdots 1_p$ (just barely). It follows that $1_l \cdots 1_p$ is the winner, and it is easy to check that $f_{P_l}(1_l \cdots 1_p) = \vec{h}_*$. This completes the proof of the lemma. $\Box$

Because the majority rule is anonymous, for any permutation π over $1, \ldots, n$ and any $l < p$, if $(h_1, \ldots, h_n)$ is realizable over $\mathcal{X}_l$, then $(h_{\pi(1)}, \ldots, h_{\pi(n)})$ is also realizable over $\mathcal{X}_l$. For any $k \in \mathbb{N}$, we define $H_k = \{\vec{h} \in \{0,1\}^n : \sum_{j \leq n} h_j \geq k\}$. That is, H_k is composed of all n-dimensional binary vectors in each of which at least k components are 1. We next show a lemma to derive a realizable vector over $\mathcal{X}_{l-1}$ from two realizable vectors over $\mathcal{X}_l$.

Lemma 2 *Let $l < p$, and let $\vec{h}_1$, $\vec{h}_2$ be vectors that are realizable over $\mathcal{X}_l$. For any $\vec{h} \in H_{\lfloor n/2 \rfloor + 1}$, $\vec{h}_1 + (\vec{h}_2 + \vec{1}) \cdot \vec{h}$ is realizable over $\mathcal{X}_{l-1}$, where $\vec{1} = (1, \ldots, 1)$, and for any $\vec{a} = (a_1, \ldots, a_n)$ and any $\vec{b} = (b_1, \ldots, b_n)$, we have $\vec{a} \cdot \vec{b} = (a_1 b_1, \ldots, a_n b_n)$.*

Proof of Lemma 2: Without loss of generality, we prove the lemma for $\vec{h} = (\underbrace{0, \ldots, 0}_{\lceil n/2 \rceil - 1}, \underbrace{1, \ldots, 1}_{\lfloor n/2 \rfloor + 1})$. Let P_1, P_2 be two profiles over $\mathcal{X}_l$, each of which is composed of n votes, such that $f(P_1) = \vec{h}_1$ and $f(P_2) = \vec{h}_2$. Let $P_1 = (V_1^1, \ldots, V_n^1)$, $P_2 = (V_1^2, \ldots, V_n^2)$, $\vec{a} = SSP_{\mathcal{O}_l}(P_1)$, $\vec{b} = SSP_{\mathcal{O}_l}(P_2)$. We define a profile $P = (V_1, \ldots, V_n)$ over $\mathcal{X}_{l-1}$ as follows.

1. Let $W_1, \ldots, W_n$ be n null partial orders over $\mathcal{X}_{l-1}$.

2. For any $j \leq n$ and any $\vec{e}_1, \vec{e}_2 \in \mathcal{X}_l$, we let $(1_{l-1}, \vec{e}_1) \succ_{W_j} (1_{l-1}, \vec{e}_2)$ if $\vec{e}_1 \succ_{V_j^1} \vec{e}_2$; and we let $(0_{l-1}, \vec{e}_1) \succ_{W_j} (0_{l-1}, \vec{e}_2)$ if $\vec{e}_1 \succ_{V_j^2} \vec{e}_2$.

3. For any $\lceil n/2 \rceil \leq j \leq n$, we let $(1_{l-1}, \vec{a}) \succ_{W_j} (0_{l-1}, \vec{b})$.

4. Finally, we complete the profile arbitrarily: for any $j \leq n$, we let V_j be an (arbitrary) extension of W_j such that $(1_{l-1}, \vec{a})$ is ranked as low as possible.

We note that $(1_{l-1}, \vec{a})$ is the winner of the subtree in which $\mathbf{x}_{l-1} = 1_{l-1}$, $(0_{l-1}, \vec{b})$ is the winner of the subtree in which $\mathbf{x}_{l-1} = 0_{l_1}$, and $(1_{l-1}, \vec{a})$ beats $(0_{l-1}, \vec{b})$ in their pairwise election (because the votes from $\lceil n/2 \rceil$ to n rank $(1_{l-1}, \vec{a})$ above $(0_{l-1}, \vec{b})$). Therefore, $SSP_{\mathcal{O}_{l-1}}(P) = (1_{l-1}, \vec{a})$.

Finally, we have that $f_P((1_{l-1}, \vec{a})) = \vec{h}_1 + (\vec{h}_2 + \vec{1}) \cdot \vec{h}$. This is because $(1_{l-1}, \vec{a})$ is ranked just as low as in the profile P_1 for voters 1 through $\lceil n/2 \rceil - 1$; for any voter j with $\lceil n/2 \rceil \leq j \leq n$, additionally, $(0_{l-1}, \vec{b})$ needs to be placed below $(1_{l-1}, \vec{a})$, which implies that also, all the alternatives $(0_{l-1}, \vec{b}')$ for which j ranked $\vec{b}'$ below $\vec{b}$ in P_2 must be below $(1_{l-1}, \vec{a})$ in j's new vote in P. This completes the proof of the lemma. □

Now we are ready to prove the main part of the theorem. It suffices to prove that for any $n \geq 2p^2 + 1$, there exists a vector $\vec{h}_p \in \mathbb{N}^n$ such that each component of $\vec{h}_p$ is no more than $\lfloor p/2 + 1 \rfloor$, and $\vec{h}_p$ is realizable over $\mathcal{X}$. We show the construction by induction in the proof of the following lemma.

Lemma 3 *Let n be odd. For any $l' < p$ (such that l' is odd),*

$$\vec{h}_{l'} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{\lceil n/2 \rceil - (l'^2+1)/2}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{\lfloor n/2 \rfloor + (l'^2+1)/2})$$

is realizable over $\mathcal{X}_{p-l'+1}$, and if $l' < p$, then

$$\vec{h}_{l'+1} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{l'+1}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{n-(l'+5)(l'+1)/2}, \underbrace{\lceil l'/2 \rceil + 1, \ldots, \lceil l'/2 \rceil + 1}_{(l'+3)(l'+1)/2})$$

is realizable over $\mathcal{X}_{p-l'}$.

Proof of Lemma 3: The base case in which $l' = 1$ corresponds to a single-issue majority election over two alternatives, where $\lceil n/2 \rceil - 1$ voters vote for one alternative, and $\lfloor n/2 \rfloor + 1$ vote for the other, so that only the latter get their preferred alternative.

Now, suppose the claim holds for some $l' \le p-2$; we next show that the claim also holds for $l'+2$. To this end, we apply Lemma 2 twice. Let $l = p - l' + 1$.

First, let $\vec{h}_* = (\underbrace{1,\ldots,1}_{l'}, \underbrace{2^{l'}-1,\ldots,2^{l'}-1}_{l'+1}, \underbrace{0,\ldots,0}_{\lfloor n/2\rfloor - l'+1}, \underbrace{2^{l'}-1,\ldots,2^{l'}-1}_{\lceil n/2\rceil - l'-2})$

By Lemma 1, $\vec{h}_*$ is realizable over $\mathcal{X}_l$ (via a permutation of the voters). Let $\vec{h} = (\underbrace{1,\ldots,1}_{l'}, \underbrace{0,\ldots,0}_{l'+1}, \underbrace{1,\ldots,1}_{\lfloor n/2\rfloor - l'+1}, \underbrace{0,\ldots,0}_{\lceil n/2\rceil - l'-2})$.

Then, by Lemma 2, $\vec{h}_{l'} + (\vec{h}_* + \vec{1}) \cdot \vec{h}$ is realizable over $\mathcal{X}_{l-1}$. We have the following calculation.

$$\begin{aligned}
&\vec{h}_{l'} + (\vec{h}_* + \vec{1}) \cdot \vec{h} \\
=&(\underbrace{\lceil l'/2\rceil + 1, \ldots, \lceil l'/2\rceil + 1}_{l'}, \\
&\underbrace{\lfloor l'/2\rfloor, \ldots, \lfloor l'/2\rfloor}_{l'+1}, \underbrace{\lceil l'/2\rceil, \ldots, \lceil l'/2\rceil}_{\lceil n/2\rceil - (l'+3)(l'+1)/2}, \\
&\underbrace{\lceil l'/2\rceil + 1, \ldots, \lceil l'/2\rceil + 1}_{(l'+1)^2/2+1}, \underbrace{\lceil l'/2\rceil, \ldots, \lceil l'/2\rceil}_{\lfloor n/2\rfloor - l'-1})
\end{aligned}$$

The partition of the set of voters into these five groups uses the fact that $n \ge 2p^2+1$ implies $\lceil n/2\rceil - (l'+3)(l'+1)/2 \ge 0$. After permuting the voters in this vector, we obtain the following vector which is realizable over $\mathcal{X}_{l-1}$:

$$\vec{h}_{l'+1} = (\underbrace{\lfloor l'/2\rfloor, \ldots, \lfloor l'/2\rfloor}_{l'+1}, \underbrace{\lceil l'/2\rceil, \ldots, \lceil l'/2\rceil}_{n-(l'+5)(l'+1)/2}, \underbrace{\lceil l'/2\rceil + 1, \ldots, \lceil l'/2\rceil + 1}_{(l'+3)(l'+1)/2})$$

We next let $\vec{h}' = (\underbrace{1,\ldots,1}_{\lfloor n/2\rfloor+1}, \underbrace{0,\ldots,0}_{\lceil n/2\rceil - 1})$ and

$$\vec{h}'_* = (\underbrace{1,\ldots,1}_{l'+1}, \underbrace{0,\ldots,0}_{\lfloor n/2\rfloor - l'}, \underbrace{2^{l'+1}-1,\ldots,2^{l'+1}-1}_{\lceil n/2\rceil - 1})$$

By Lemma 1, the latter is realizable over $\mathcal{X}_{l-1}$. Thus, by Lemma 2, $\vec{h}_{l'+1} + (\vec{h}'_* + \vec{1}) \cdot \vec{h}'$ is realizable over $\mathcal{X}_{l-2}$. Through a permutation over the voters, we obtain the desired vector:

$$\vec{h}_{l'+2} = (\underbrace{\lfloor l'/2\rfloor + 1, \ldots, \lfloor l'/2\rfloor + 1}_{\lceil n/2\rceil - (l'+2)(l'+1)/2-1}, \underbrace{\lceil l'/2\rceil + 1, \ldots, \lceil l'/2\rceil + 1}_{\lfloor n/2\rfloor + (l'+2)(l'+1)/2+1})$$

which is realizable over $\mathcal{X}_{l-2}$. Therefore, the claim holds for $l'+2$. This completes the proof of the lemma. $\square$

If p is odd, from Lemma 3 we know that the theorem is true, by setting $l'=p$. If p is even, then we first set $l'=p-1$; then, the maximum component of $\vec{h}_{l'+1}$ is $\lceil l'/2 \rceil + 1 = \lceil (p-1)/2 \rceil + 1 = p/2+1$. Thus we have proved the upper bound in the theorem.

Moreover, we note that in the step from l' to $l'+1$ (respectively, from $l'+1$ to $l'+2$), no more than l' new alternatives are ranked lower than the winner in the profile that realizes $\vec{h}_{l'+1}$ (respectively, $\vec{h}_{l'+2}$). It follows that in the profile that realizes $\vec{h}_{l'+1}$ (respectively, $\vec{h}_{l'+2}$) in Lemma 3 or Lemma 4, the number of alternatives that are ranked lower than the winner by at least one voter is no more than $(l'+1)l'/2+l'+1 = (l'+1)(l'+2)/2$ (respectively, $(l'+2)(l'+3)/2$), which equals $(p+1)p/2$ if $l'+1=p$ (respectively, $(p+1)p/2$ if $l'+2=p$). Therefore, in the profile that we use to obtain the upper bound, the winner under $SSP_{\mathcal{O}}$ is Pareto-dominated by $2^p-(p+1)p/2$ alternatives.

Finally, we show that $\lfloor p/2+2 \rfloor$ is a lower bound on $MSI_{SSP_{\mathcal{O}}}(m,n)$. Let P be an n-profile; let $SSP_{\mathcal{O}}(P)=\vec{a}$, and let $\vec{b}_1,\ldots,\vec{b}_p$ be the alternatives that $\vec{a}$ defeats in pairwise elections in rounds $1,\ldots,p$. It follows that in round j, more than half of the voters prefer $\vec{a}$ to $\vec{b}_j$ (recall that n is odd.) Therefore, summing over all votes, there are at least $p \times (\lfloor n/2 \rfloor + 1)$ occasions where $\vec{a}$ is preferred to one of $\vec{b}_1,\ldots,\vec{b}_p$. It follows that there exists some $V \in P$ in which $\vec{a}$ is ranked higher than at least $\lceil p \times (\lfloor n/2 \rfloor + 1)/n \rceil \geq \lfloor p/2+1 \rfloor$ of the alternatives $\vec{b}_1,\ldots,\vec{b}_p$. Thus $MSI_{SSP_{\mathcal{O}}}(m,n) \geq \lfloor p/2+2 \rfloor$.
(End of proof of Theorem 2.) $\square$

When n is even, we have the following lemma, whose proof is similar to the proof of Lemma 3, and therefore omitted.

Lemma 4 *Let n be even. For any $l'<p$ (such that l' is odd),*

$$\vec{h}_{l'} = (\underbrace{\lfloor l'/2 \rfloor,\ldots,\lfloor l'/2 \rfloor}_{n/2-(l'^2-l'+1)/2},\underbrace{\lceil l'/2 \rceil,\ldots,\lceil l'/2 \rceil}_{n/2+(l'^2-l'+1)/2})$$

is realizable over $\mathcal{X}_{p-l'+1}$, and if $l'+1 \leq p$, then

$$\vec{h}_{l'+1} = (\underbrace{\lfloor l'/2 \rfloor,\ldots,\lfloor l'/2 \rfloor}_{l'+1},\underbrace{\lceil l'/2 \rceil,\ldots,\lceil l'/2 \rceil}_{n-1-(l'+4)(l'+1)/2},\underbrace{\lceil l'/2 \rceil+1,\ldots,\lceil l'/2 \rceil+1}_{(l'+2)(l'+1)/2+1})$$

is realizable over $\mathcal{X}_{p-l'}$.

The upper bound in the theorem when n is even follows from Lemma 4. Therefore, for n even we get the following counterpart of Theorem 2.

Theorem 3 *For any $p \in \mathbb{N}$ ($p \geq 2$) and any even number n such that $n \geq 2p^2+1$, we have $MSI_{SSP_{\mathcal{O}}}(m,n) = \lfloor p/2+2 \rfloor$, where $MSI_{SSP_{\mathcal{O}}}(m,n) \geq \lfloor p/2+2 \rfloor$ is computed by restricting to profiles where ties never occur. Moreover, in the profile P that we use to prove the upper bound, the winner $SSP_{\mathcal{O}}(P)$ is Pareto-dominated by $2^p - (p+1)p/2$ alternatives.*

The proof (omitted) is similar to the proof of Theorem 2; the only points where they differ are on the use of Lemma 4 instead of 3, and when concluding that "It follows that in round j, more than half of the voters prefer $\vec{a}$ to $\vec{b}_j$", we use the assumption that ties do not occur.

We note that the number of alternatives is $m = 2^p$. Therefore, $\lfloor p/2+2 \rfloor$ is exponentially smaller than the number of alternatives, which means that there exists a profile for which every voter ranks the winner very close to the bottom. Moreover, $(p+1)p/2$ is still exponentially smaller than 2^p, which means that the winner is Pareto-dominated by almost every other alternative.

Naturally, we wish to avoid such paradoxes. One may wonder whether the paradox occurs only if the ordering of the issues is particularly unfortunate with respect to the preferences of the voters. If not, then, for example, perhaps a good approach is to randomly choose the ordering of the issues.[6] Unfortunately, our next result shows that we can construct a single profile that results in a paradox for *all* orderings of the issues. While it works for all orders, the result is otherwise somewhat weaker than Theorem 2: it does not show a Pareto-dominance result, it requires a number of voters that is at least twice the number of alternatives, the upper bound shown on the MSI is slightly higher than in Theorem 2, and unlike Theorem 2, no matching lower bound is shown.

Theorem 4 *For any $p, n \in \mathbb{N}$ (with $p \geq 2$ and $n \geq 2^{p+1}$), there exists an n-profile P such that for any order $\mathcal{O}$ over $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$, $SSP_{\mathcal{O}}(P) = 1_1 \cdots 1_p$, and any $V \in P$ ranks $1_1 \cdots 1_p$ somewhere in the bottom $p+2$ positions.*

Proof of Theorem 4: We first prove the following lemma.

[6] Of course, for any ordering of the issues, there exists a profile that results in the paradoxes in Theorem 2; but this does not directly imply that there exists a single profile that works for all orderings over the issues.

Lemma 5 *For any $c \in \mathcal{X}$, $\mathcal{C} \subset \mathcal{X}$ such that $c \notin \mathcal{C}$, and any $n \in \mathbb{N}$ ($n \geq 2m = 2^{p+1}$), there exists an n-profile that satisfies the following conditions. Let $F = \mathcal{X} \setminus (\mathcal{C} \cup \{c\})$.*

- *For any $c' \in \mathcal{C}$, c defeats c' in their pairwise election.*
- *For any $c' \in \mathcal{C}$ and $d \in F$, c' defeats d in their pairwise election.*
- *For any $V \in P$, c is ranked somewhere in the bottom $|\mathcal{C}| + 2$ positions.*

Proof of Lemma 5: We let $P = (V_1, \ldots, V_n)$ be the profile defined as follows. Let $F_1, \ldots, F_{\lfloor n/2 \rfloor+1}$ be a partition of F such that for any $j \leq \lfloor n/2 \rfloor + 1$, $|F_j| \leq \lceil 2m/n \rceil = 1$. For any $j \leq \lfloor n/2 \rfloor + 1$, we let $V_j = [(F \setminus F_j) \succ c \succ \mathcal{C} \succ F_j]$. For any $\lfloor n/2 \rfloor + 2 \leq j \leq n$, we let $V_j = [\mathcal{C} \succ F \succ c]$. It is easy to check that P satisfies all conditions in the lemma. □

Now, let $c = 1_1 \cdots 1_p$ and $\mathcal{C} = \{0_1 1_2 \cdots 1_p, 1_1 0_2 1_3 \cdots 1_p, \ldots, 1_1 \cdots 1_{p-1} 0_p\}$. By Lemma 5, there exists a profile P such that c beats any alternative in $\mathcal{C}$ in pairwise elections, any alternative in $\mathcal{C}$ beats any alternative in $\mathcal{X} \setminus (\mathcal{C} \cup \{c\})$ in pairwise elections, and c is ranked somewhere in the bottom $p+2$ positions. This is the profile that we will use to prove the paradox.

Without loss of generality , we assume that $\mathcal{O} = \mathbf{x}_1 > \mathbf{x}_2 \cdots > \mathbf{x}_p$. (This is without loss of generality because all issues have been treated symmetrically so far.) c beats $1_1 \cdots 1_{p-1} 0_p$ in the first round; c will meet $1_1 \cdots 1_{p-2} 0_{p-1} 1_p$ in the next pairwise election, because $1_1 \cdots 1_{p-2} 0_{p-1} 1_p$ beats every other alternative in that branch (they are all in F), and c will win; and so on. It follows that $c = SSP_{\mathcal{O}}(P)$. Moreover, all voters rank c in the bottom $p + 2$ positions.

(End of proof for Theorem 4.) □

6. Multiple-election Paradoxes for SSP with Restrictions on Preferences

The paradoxes exhibited so far placed no restriction on the voters' preferences. While SSP is perfectly well defined for any preferences that the voters may have over the alternatives, we may yet wonder what happens if the voters' preferences over alternatives are restricted in a way that is natural with respect to the multi-issue structure of the setting. In particular, we may wonder if paradoxes are avoided by such restrictions. It is well known that natural restrictions on preferences sometimes lead to much more positive results in social choice and mechanism design—for example, single-peaked preferences allow for good strategy-proof mechanisms [5, 22].

In this section, we study the MSI for $SSP_{\mathcal{O}}$ for the following three cases: (1) voters' preferences are separable; (2) voters' preferences are $\mathcal{O}$-lexicographic; and (3) voters' preferences are $\mathcal{O}$-legal. For case (1), we show a mild paradox (and that this is effectively the strongest paradox that can be obtained); for case (2), we show a positive result; for case (3), we show a paradox that is nearly as bad as the unrestricted case. All proofs as well as the description of a tool we use (called *CP-nets* [6]) are relegated to the appendix.

Let *Separable* be the set of separable profiles, $\mathcal{O}-lex$ the set of $\mathcal{O}$-lexicographic profiles and $\mathcal{O}-legal$ the set of $\mathcal{O}$-legal profiles.

Theorem 5 *For any* $n \geq 2p$, $2^{\lceil p/2 \rceil} \leq MSI_{SSP_{\mathcal{O}},Separable}(m,n) \leq 2^{\lfloor p/2 \rfloor + 1}$.

That is, the MSI of $SSP_{\mathcal{O}}$ when votes are separable is $\Theta(\sqrt{m})$.[7] Since $\lim_{m\to\infty} \Theta(\sqrt{m})/m = 0$, in that sense this is still a paradox. However, its convergence rate to 0 is much slower than for $\Theta(\log m)/m$, which corresponds to the convergence rate for the earlier paradoxes.

Theorem 6 *For any* $p \in \mathbb{N}$ *(*$p \geq 2$*) and any* $n \geq 5$, $MSI_{SSP_{\mathcal{O}},\mathcal{O}-lex}(m,n) = 3 \cdot 2^{p-2} + 1$. *Moreover,* $SSP_{\mathcal{O}}(P)$ *is ranked somewhere in the top* 2^{p-1} *positions in at least* $n/2$ *votes.*

Since $m = 2^p$, $3 \cdot 2^{p-2} + 1 = \frac{3m}{4} + 1$, and $\lim_{m\to\infty}(3m/4 + 1)/m = 3/4$, so in that sense when votes are $\mathcal{O}$-lexicographic, there is no paradox that is similar to the multiple-election paradoxes.

Under the previous two restrictions (separability and $\mathcal{O}$-lexicographicity), $SSP_{\mathcal{O}}$ coincides with $Seq(maj, \ldots, maj)$ (by Corollary 1 and Proposition 2, respectively). Therefore, Theorems 5 and 6 also apply to sequential voting rules as defined in the work of Lang and Xia [18]; furthermore, Theorem 5 also applies to seat-by-seat voting [4].

Finally, we study the MSI for $SSP_{\mathcal{O}}$ when the profile is $\mathcal{O}$-legal. Theorem 8 shows that it is nearly as bad as the unrestricted case (Theorem 2). The proof of Theorem 8 is the most involved proof in the paper and can be found in the appendix. The idea of the proof is similar to that of the

[7] $\Theta(\sqrt{m})$ refers to a function $f(m)$ such that there exist two positive numbers $k_1 < k_2$ and for all $m \in \mathbb{N}$, $k_1\sqrt{m} < f(m) < k_2\sqrt{m}$.

proof of Theorem 2, but now we cannot apply Lemma 2, because $\mathcal{O}$-legality must be preserved. We start with a simpler result for a specific tie-breaking mechanism.

Theorem 7 *There exists a tie-breaking mechanism T such that the rule $SSP'_{\mathcal{O}}$ corresponding to $SSP_{\mathcal{O}}$ plus T satisfies the following: for any $p \in \mathbb{N}$, there exists an $\mathcal{O}$-legal profile that consists of two votes, such that in one of the two votes, no more than $\lceil p/2 \rceil$ alternatives are ranked lower than the winner $SSP'_{\mathcal{O}}(P)$; and in the other vote, no more than $\lfloor p/2 \rfloor$ alternatives are ranked lower than $SSP'_{\mathcal{O}}(P)$.*

We emphasize that, unlike all of our other results, Theorem 7 is based on a specific tie-breaking mechanism. The next theorem studies the more general and complicated case in which n can be either odd or even, and the winner does not depend on the tie-breaking mechanism. That is, there are no ties in the election. The situation is almost the same as in Theorem 2.

Theorem 8 *For any $p, n \in \mathbb{N}$ with $n \geq 2p^2 + 2p + 1$, there exists an $\mathcal{O}$-legal profile such that in each vote, no more than $\lceil p/2 \rceil + 4$ alternatives are ranked lower than $SSP_{\mathcal{O}}(P)$. Moreover, $SSP_{\mathcal{O}}(P)$ is Pareto-dominated by at least $2^p - 4p^2$ alternatives.*

Of course, the lower bound on the MSI from Theorem 2 still applies when the profile is $\mathcal{O}$-legal, so together with Theorem 8 this proves that the MSI for $SSP_{\mathcal{O}}$ when the profile is $\mathcal{O}$-legal is $\Theta(\log m)$, just as in the unrestricted case.

7. Minimax Satisfaction Index of Common Voting Rules

So far, we have focused strictly on strategic sequential voting (SSP) in multi-issue domains (and voting trees, but only in the sense of their equivalence to strategic sequential voting). Hence, at this point, it may not be clear whether the paradoxes (or, in some cases, lack of paradoxes) that we have shown are due to the sequential, multi-issue nature of the process, or whether they are due to the strategic behavior of the voters, or whether such paradoxes are prevalent throughout voting settings.

First, let us address the question of to what extent they are due to strategic behavior. To answer this, it is most natural to compare to $Seq_{\mathcal{O}}(maj, \dots, maj)$ ("truthful" sequential voting), which is only well defined when the profile is

$\mathcal{O}$-legal. In fact, as we have already pointed out, our results for separable and $\mathcal{O}$-lexicographic profiles apply just as well to truthful sequential voting, because by Corollary 1 and Proposition 2, the strategic aspect makes no difference here. This only leaves the question of whether there is a paradox under $Seq_{\mathcal{O}}(maj,\ldots,maj)$ when the profile is $\mathcal{O}$-legal but not otherwise restricted; in this case, $Seq_{\mathcal{O}}(maj,\ldots,maj)$ is truly different from $SSP_{\mathcal{O}}$, as illustrated by Example 1. We answer this question by the following Proposition, which shows a much milder paradox.

Proposition 4 *For any $n \geq 2p$, when the profile is $\mathcal{O}$-legal, the MSI for $Seq_{\mathcal{O}}(maj,\ldots,maj)$ is between $2^{\lceil p/2 \rceil}$ and $2^{\lfloor p/2 \rfloor+1}$.*

Proof: Let $P = (V_1,\ldots,V_n)$ be an $\mathcal{O}$-legal profile. Without loss of generality $Seq_{\mathcal{O}}(maj,\ldots,maj)(P) = (1_1,\ldots,1_p)$.

First, we prove the lower bound. Because $1_1 \cdots 1_p$ is the winner, for any $i \leq p$, at least half of the voters prefer 1_i to 0_i, given that $\mathbf{x}_1,\ldots,\mathbf{x}_{i-1}$ all take value 1. Therefore, by a simple counting argument as in the proof of Theorem 5, there exist $j \leq n$ and a set of issues $\mathcal{I}' \subseteq \mathcal{I}$ that satisfy the following two conditions. (1) $|\mathcal{I}'| \geq p/2$, and (2) for any $\mathbf{x}_i \in \mathcal{I}'$, $1_i \succ_{V_j|\mathbf{x}_i:1_1\cdots 1_{i-1}} 0_i$, that is, voter j's preference over $\mathbf{x}_i$ is $1_i \succ 0_i$, given that $\mathbf{x}_1,\ldots,\mathbf{x}_{i-1}$ all take value 1. For any $\vec{d} = (d_1,\ldots,d_p) \in \mathcal{X}$ such that $\vec{d}$ only takes value 1 for issues outside $\mathcal{I}'$ (and $\vec{d}$ takes value 0 for at least one issue in $\mathcal{I}'$), we next prove that $(1_1,\ldots,1_p) \succ_{V_j} \vec{d}$. Let $\vec{d}$ be such an alternative, and $\mathbf{x}_{i_1},\ldots,\mathbf{x}_{i_k}$ be the issues for which $\vec{d}$ takes value 0 (with $k \geq 1$, $\{\mathbf{x}_{i_1},\ldots,\mathbf{x}_{i_k}\} \subseteq \mathcal{I}'$, and $i_1 < i_2 < \cdots < i_k$); that is, for any $l \leq k$, we have $d_{i_l} = 0_l$, and for any $\mathbf{x}_i \in \mathcal{I} \setminus \{\mathbf{x}_{i_1},\ldots,\mathbf{x}_{i_k}\}$, we have $d_i = 1_i$. We recall that for any $\mathbf{x}_i \in \mathcal{I}'$, $1_i \succ_{V_j|\mathbf{x}_i:1_1\cdots 1_{i-1}} 0_i$. Therefore, for any $l \leq k$, we have the following preference relationship.

$$\begin{aligned}
&(1_1,\ldots,1_{i_l-1},1_{i_l},d_{i_l+1},\ldots,d_p)\\
\succ_{V_j}&(1_1,\ldots,1_{i_l-1},0_{i_l},d_{i_l+1},\ldots,d_p)\\
=&(1_1,\ldots,1_{i_l-1},d_{i_l},d_{i_l+1},\ldots,d_p)
\end{aligned}$$

We obtain the following preference relationship by chaining the above preference relationships.

$$\begin{aligned}
&(1_1, \dots, 1_p)\\
\succ_{V_j} &(1_1, \dots, 1_{i_k-1}, 0_{i_k}, 1_{i_k+1}, \dots, 1_p)\\
&= (1_1, \dots, 1_{i_k-1}, d_{i_k}, d_{i_k+1}, \dots, d_p)\\
\succ_{V_j} &(1_1, \dots, 1_{i_{k-1}-1}, 0_{i_{k-1}}, 1_{i_{k-1}+1}, \dots, 1_{i_k-1}, d_{i_k}, \dots, d_p)\\
&= (1_1, \dots, 1_{i_{k-1}-1}, d_{i_{k-1}}, d_{i_{k-1}+1}, \dots, d_p)\\
&\vdots\\
\succ_{V_j} &(1_1, \dots, 1_{i_1-1}, 0_{i_1}, 1_{i_1+1}, \dots, 1_{i_2-1}, d_{i_2}, \dots, d_p)\\
&= (d_1, \dots, d_p) = \vec{d}
\end{aligned}$$

Because $|\mathcal{I}'| \geq \lceil p/2 \rceil$, the number of such $\vec{d}$'s is at least $2^{\lceil p/2 \rceil} - 1$. It follows that the minimax satisfaction index is at least $2^{\lceil p/2 \rceil}$.

The upper bound follows directly from Theorem 5 and Proposition 3 (when the profile is separable, the SSP winner is the same as the *Seq* winner, and any separable profile must be $\mathcal{O}$-legal for any $\mathcal{O}$). □

Having settled the effect of the strategic behavior, we next investigate the effect of the multi-issue nature of the setting. We do this by studying the MSI of common voting rules in non-combinatorial settings, where there is a single issue (but one that can take more than two values). In this context, studying strategic behavior seems intractable. By the Gibbard-Satterthwaite theorem [13, 24], without restrictions on preferences, no strategy-proof rules exist other than dictatorships and rules that exclude certain alternatives *ex ante*. Moreover, even with complete information, common voting rules have many different equilibria. Hence, we focus on studying the extent to which paradoxes occur when voters vote truthfully.

Specifically, we investigate the minimax satisfaction indices of positional scoring rules (including k-approval and Borda), plurality with runoff (*Pluo*), Copeland$_\alpha$, maximin, ranked pairs, Bucklin, STV, and (not necessarily balanced) voting trees. Of course, these rules can be applied to multi-issue domains as well as to any other domains, but they do not make use of multi-issue structure; in general, we just have a set of alternatives $\mathcal{C} = \{c_1, \dots, c_m\}$. Throughout the remainder of this section, we assume that $m \geq 3$, and that ties are broken in the order $c_1 \succ c_2 \succ \cdots \succ c_m$.

We list here the formal definition of voting rules that we will study.

- *Dictatorships:* for every $n \in \mathbb{N}$ there exists a voter $j \leq n$ such that the winner is always the alternative that is ranked in the top position in V_j.

- *(Positional) scoring rules*: Given a *scoring vector* $\vec{v} = (v(1), \ldots, v(m))$, for any vote $V \in L(\mathcal{X})$ and any $c \in \mathcal{X}$, let $s(V, c) = v(i)$, where i is the rank of c in V. For any profile $P = (V_1, \ldots, V_n)$, let $s(P, c) = \sum_{j=1}^{n} s(V_j, c)$. The rule will select $c \in \mathcal{X}$ so that $s(P, c)$ is maximized. Some examples of positional scoring rules are *Borda*, for which the scoring vector is $(m-1, m-2, \ldots, 0)$, k-approval (App_k, with $k \leq m$), for which the scoring vector is $(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$, *plurality*, for which the scoring vector is $(1, 0, \ldots, 0)$, and *veto*, for which the scoring vector is $(1, \ldots, 1, 0)$.

- *Copeland*$_\alpha$ ($0 \leq \alpha \leq 1$): For any two alternatives c and d, we can simulate a *pairwise election* between them, by seeing how many votes prefer c to d, and how many prefer d to c; the winner of the pairwise election is the one preferred more often. Then, an alternative receives one point for each win in a pairwise election, α points for each draw, and 0 point for each loss. The winner is the alternative that has the highest total score.

- *Plurality with runoff* (*Pluo*): The election has two rounds. In the first round, the alternatives are ranked from high to low according to the number of times they are ranked in the top position in the votes of the profile (that is, according to their plurality scores). Only the top two alternatives enter the second (runoff) round. In the runoff, we simulate a pairwise election between these two alternatives, and the alternative that wins the pairwise election is the winner.

- *Maximin*: Let $N(c, d)$ denote the number of votes that rank c ahead of d. The winner is the alternative c that maximizes $min\{N(c, c') : c' \in \mathcal{X}, c' \neq c\}$.

- *STV*: The election has $m-1$ rounds. In each round, we count for each remaining alternative how many votes rank it highest among the remaining alternatives; then, the alternative with the lowest count drops out. The last remaining alternative is the winner.

- *Bucklin*: An alternative c's Bucklin score is the smallest number k such that more than half of the voters rank c among their top k alternatives. The winner is the alternative that has the lowest Bucklin score. If multiple alternatives have the lowest score k, then ties are broken by the number of voters who rank an alternative among their top k alternatives.

- *Ranked pairs*: This rule first creates an entire ranking of all the alternatives. $N(c, d)$ is defined as for the maximin rule. In each step, we will consider a pair of alternatives c, d that we have not previously considered; specifically, we choose the remaining pair with the highest $N(c, d)$. We then fix the ordering $c \succ d$, unless it contradicts orderings that we fixed previously (that is, it violates transitivity). We continue until all pairs of alternatives are considered (hence we end up with a full ranking). The alternative at the top of the ranking wins.

- *Voting trees*: A voting tree is a binary tree with m leaves, where each leaf is associated with an alternative. In each round, there is a pairwise election between an alternative c and its sibling d: if the majority of voters prefer c to d, then d is eliminated, and c is associated with the parent of these two nodes; similarly, if the majority of voters prefer d to c, then c is eliminated, and d is associated with the parent of these two nodes. The alternative that is associated with the root of the tree (wins all its rounds) is the winner.

Proposition 5 *Let $m, n \in \mathbb{N}$.*

- $MSI_{Dict}(m, n) = m$;
- *for any $k \leq m$, $MSI_{App_k}(m, n) = m + 1 - k$ (and in particular, $MSI_{plurality}(m, n) = m$ and $MSI_{veto}(m, n) = 2$.*
- $MSI_{Pluo}(m, n) = m$;
- $MSI_{STV}(m, n) = m$;
- $MSI_{Bucklin}(m, n) \geq m/2$.

Proof:

• *Dictatorship.* The dictator always gets her most preferred alternative, and is hence maximally satisfied.

• *k-approval.* Let P be an arbitrary profile in which c_1 is ranked in the $(m + 1 - k)$th position from the bottom. It follows that the total score of c_1 is n. Therefore, $App_k(P) = c_1$ (we remember that ties are broken in favor of c_1).

Let P be a profile such that $App_k(P)$ is ranked within the $m-k$ positions from the bottom. It follows that the total score of $App_k(P)$ is 0, and there exists $c \in \mathcal{X}$ such that the total score of c is positive. This contradicts the assumption that $App_k(P)$ is the winner.

• *Plurality with runoff.* We prove that the winner must be ranked in the top position in at least one vote. Suppose, for the sake of contradiction, that there exists a profile P such that $Pluo(P)$ is ranked in the top position in none of the votes in P. Because $Pluo(P)$ enters the second round, it must be the case that all the other alternatives are never ranked in the top position, with exactly one exception, denoted by c. However, c beats $Pluo(P)$ in the second round, which is a contradiction.

• *STV.* Suppose for the sake of contradiction, there exists a profile P such that $c = STV(P)$ and c is not ranked in the top position in any vote. Then, in any round, any alternative that is ranked in the first position in some vote will not be eliminated (because c is ranked in the top position in no vote). It follows that c is not the winner, which contradicts the assumption.

• *Bucklin.* Suppose, for the sake of contradiction, that there exists a profile P such that $c = Bucklin(P)$ and c is ranked within $\lfloor m/2 \rfloor$ positions from the bottom. Then, in each vote, there are $\lceil m/2 \rceil$ alternatives ranked in top $\lceil m/2 \rceil$ positions, which means that there exists an alternative that is ranked within top $\lceil m/2 \rceil$ positions in at least $\lceil m/2 \rceil \times n/(m-1) > n/2$ votes. It follows that the Bucklin score of that alternative is no more than $\lceil m/2 \rceil$. We note that the Bucklin score of c is $\lceil m/2 \rceil + 1$. This contradicts the assumption that c is the winner under Bucklin. □

We also obtain bounds on MSI for other common voting rules mentioned in this paper.

Proposition 6 (Borda) *Let $m \in \mathbb{N}$. For any $n \in \mathbb{N}$ such that n is even, $MSI_{Borda}(m,n) = \lfloor m/2+1 \rfloor$; for any $n \in \mathbb{N}$ such that $n \geq m$, and n is odd, $MSI_{Borda}(m,n) = \lceil m/2+1 \rceil$.*

Proof: For any $n \in \mathbb{N}$ such that n is even, we let P be the profile in which $n/2$ votes are $c_2 \succ c_3 \succ \cdots \succ c_{\lfloor m/2 \rfloor} \succ c_1 \succ c_{\lfloor m/2+1 \rfloor} \cdots \succ c_m$, and the other $n/2$ votes are in the reversed order, that is, they are $c_m \succ c_{m-1} \succ \cdots \succ c_{\lfloor m/2+1 \rfloor} \succ c_1 \succ_{\lfloor m/2 \rfloor} \succ \cdots \succ c_2$. If m is odd, then the total score of any alternative is $n(m-1)/2$, thus c_1 is the winner. If m is even, then the total score of c_1 is $nm/2$, and the total score of any other alternative is $n(m-1)/2$, thus c_1 is the winner. It follows that $MSI_{Borda}(m,n) \leq \lfloor m/2+1 \rfloor$.

We next show that $MSI_{Borda}(m,n) \geq \lfloor m/2+1 \rfloor$. Suppose for the sake of contradiction, there exists a profile P that is composed of n voters, such that $Borda(P)$ is ranked below the $\lfloor m/2+1 \rfloor$th position from the bottom. Then, the total score of $Borda(P)$ is at most $\lfloor m/2-1 \rfloor n$. However, the average total score of all alternatives is $(m-1)n/2$, which means that there exists an alternative whose total score is at least $(m-1)n/2 > \lfloor m/2-1 \rfloor n$. This contradicts with the assumption that $Borda(P)$ is the winner. It follows that $MSI_{Borda}(m,n) = \lfloor m/2+1 \rfloor$.

Similarly we can prove that for any $n \in \mathbb{N}$ such that $n \geq m$ and n is odd, $MSI_{Borda}(m,n) = \lceil m/2+1 \rceil$. □

Proposition 7 (Copeland) *Let $m, n \in \mathbb{N}$. If either $0 < \alpha \leq 1$, or n is odd and $\alpha = 0$, then $MSI_{Copeland_\alpha}(m,n) \geq \alpha m/4$. For any $n \geq 2m$ such that n is even, $MSI_{Copeland_0}(m,n) = 2$.*

Proof: We first prove the proposition for the case in which $0 < \alpha \leq 1$ or n is even. Let P be an n-profile and $\text{Copeland}_\alpha(P) = c$. The sum of Copeland scores of the alternatives in $\mathcal{X} \setminus \{c\}$ is at least $\alpha(m-1)(m-2)/2$, which means that the Copeland score of c is at least $(\alpha(m-1)(m-2)/2)/(m-1) = \alpha(m-2)/2$. It follows that the number of draws and wins for c in the pairwise elections is at least $\alpha(m-2)/2$. Therefore, $MSI_{\text{Copeland}_\alpha}(m,n) \geq (\alpha(m-2)/2)/2 + 1 \geq \alpha m/4$.

Next, we prove the proposition for the case in which $\alpha = 0$, $n \geq 2m$, and n is even. Let P be an n-profile, defined as follows.

- For any $3 \leq i \leq n/2+2$, we let $V_{i-2} = [c_i \succ c_{i+1} \succ \cdots \succ c_{i+m-4} \succ c_1 \succ c_2 \succ c_{i+m-3}]$, where for any $j \in \mathbb{N}$, $c_j = c_{j+m-2}$.
- For any $3 \leq i \leq n/2+1$, we let $V_{n/2+i-2} = [c_2 \succ c_{i+m-3} \succ c_{i+m-4} \succ \cdots \succ c_{i+1} \succ c_i \succ c_1]$.
- $V_n = [c_{n/2+1} \succ c_{n/2} \succ \cdots \succ c_{n/2+3} \succ c_{n/2+2} \succ c_1 \succ c_2]$.

We observe that in P, c_1 beats c_2 in pairwise election; for any $3 \leq j \leq m$, c_j beats c_1 in pairwise election; for any $2 \leq j_1, j_2 \leq m$ with $j_1 \neq j_2$, c_{j_1} and c_{j_2} draw in pairwise election. It follows that the Copeland score of c_2 is 0, and the Copeland score of any other alternative is 1. Therefore $\text{Copeland}_0(P) = c_1$.
□

Proposition 8 (Maximin) *Let* $m, n \in \mathbb{N}$ *with* $n \geq m-1$. $MSI_{maximin}(m, n) \leq 3$.

Proof: Let M be the cyclic permutation on $\{c_2, \ldots, c_m\}$ defined as follows. For any $2 \leq i \leq m$, $M(c_i) = c_{i+1}$, where $c_i = c_{i+m-1}$. For any $j \leq n$, we let $V_j = [M^j(c_2) \succ M^j(c_3) \succ \cdots M^j(c_{m-2}) \succ c_1 \succ M^j(c_{m-1}) \succ M^j(c_m)]$. We have that $\min\{N(c_1, c_i) : i \neq 1\} \geq 2\lfloor n/(m-1) \rfloor$, and for any $2 \leq i \leq m$, $\min\{N(c_i, c_{i'}) : i' \neq i\} \leq \lceil n/(m-1) \rceil$. Because $n \geq m-1$, $2\lfloor n/(m-1) \rfloor \geq \lceil n/(m-1) \rceil$, which means that $Maximin(P) = c_1$. □

Proposition 9 (Ranked pairs) *Let* $m, n \in \mathbb{N}$ *with* $n \geq \sqrt{m}$. $MSI_{rp}(m, n) \geq \sqrt{m}$.

Proof: Suppose for the sake of contradiction, there exists a profile P such that $c = RankedPairs(P)$ and c is ranked lower than the $\lfloor \sqrt{m} \rfloor$th position from the bottom. It follows that there exists an alternative c' such that c' is ranked above c in at least $(m - \sqrt{m})n/m = (1 - 1/\sqrt{m})n$ votes. Because c is the winner, there exists a sequence of alternatives $d_1, \ldots, d_k$ such that c is ranked above d_1 in at least $(1 - 1/\sqrt{m})n$ votes, d_k is ranked above c' in at least $(1 - 1/\sqrt{m})n$ votes, and for any $i \leq k-1$, d_i is ranked above d_{i+1} in at least $(1 - 1/\sqrt{m})n$ votes. We let d_0 and d_{k+1} denote c and c', respectively. We first prove the following lemma by induction.

Lemma 6 *For any* $1 \leq i \leq k+1$, *we have* $d_0 \succ d_1 \succ \cdots \succ d_i$ *in at least* $n(1 - i/\sqrt{m})$ *votes.*

Proof: The $i = 1$ case is trivial. Suppose that the lemma holds for some $i \leq k$, we next show that the lemma also holds for $i+1$. Because at most in $n/\sqrt{m}$ votes d_{i+1} is ranked above d_i, at least in $n(1 - i/\sqrt{m}) - n/\sqrt{m}$ votes we have $d_0 \succ d_1 \succ \cdots \succ d_i \succ d_{i+1}$. Therefore the lemma holds for $i+1$. It follows that the lemma holds for any $i \leq k+1$. □

By Lemma 6, if $k + 1 < \sqrt{m} - 1$, then in at least $n(1 - (k+1)/\sqrt{m}) > n/\sqrt{m}$ votes we have $c \succ c'$, which contradicts the assumption that c' is ranked above c in at least $(1 - 1/\sqrt{m})n$ votes. Therefore, we must have that $k + 1 \geq \sqrt{m} - 1$, which means that $d_0 \succ d_1 \succ \cdots \succ d_{\lfloor \sqrt{m} \rfloor - 1}$ in at least $(1 - (\lfloor \sqrt{m} \rfloor - 1)/\sqrt{m})n \geq 1$ votes. This contradicts the assumption that c is ranked lower than the $\sqrt{m}$th position from the bottom. □

Proposition 10 (Voting trees) *Let T be a voting tree; let c be the alternative whose corresponding leaf is closest to the root among all leaves in T, and let its distance to the root be denoted l. If $l = 1$, then for any $n \geq 2m$, $MSI_{r_T}(m,n) = 3$; if $l \geq 2$, then for any $n \geq 2m$, $MSI_{r_T}(m,n) = \lfloor l/2 + 2 \rfloor$.*

Proof: We first prove the following lemma.

Lemma 7 *Let T' be a voting tree, $\mathcal{X}'$ be the set of alternatives (leaf nodes) in T', $|\mathcal{X}'| = m'$.*

$$\vec{h}_* = (\underbrace{0,\ldots,0}_{\lfloor n/2 \rfloor - m' + 1}, \underbrace{1,\ldots,1}_{m'}, \underbrace{m'-1,\ldots,m'-1}_{\lceil n/2 \rceil - 1})$$

is realizable over $\mathcal{X}'$ through T'.

Proof: The proof is constructive. Without loss of generality, we let $\mathcal{X}' = \{c_1, \ldots, c_{m'}\}$. Let P be the profile that is defined as follows.

- For any $i \leq \lfloor n/2 \rfloor - m' + 1$, let $V_i = [c_{m'} \succ c_{m'-1} \succ \cdots \succ c_1]$.
- For any $i \leq m'$, let $V_{\lfloor n/2 \rfloor - m' + 1 + i} = [\mathcal{X}' \setminus (\{c_1, c_i\})] \succ c_1 \succ c_i$.
- For any $\lfloor n/2 \rfloor \leq i \leq n$, let $V_i = [c_1 \succ c_2 \succ \cdots \succ c_{m'}]$.

It is easy to check that $f_P(r_T(P)) = \vec{h}_*$. □

Then, following a similar construction as in the proof for Theorem 2, we can prove that if $l \geq 2$, then $MSI_{r_T}(m,n) = \lfloor l/2 + 2 \rfloor$; if $l = 1$, then $MSI_{r_T}(m,n) = 3$. □

Proposition 10 implies that among all voting trees for m alternatives, balanced voting trees have the highest MSI, which in some sense implies that balanced voting trees are the voting trees that are most resistant to multiple election paradoxes.

8. Conclusion and Future Work

In practice, common decisions on multiple issues are often reached by voting on the issues sequentially. In this paper, we considered a complete-information game-theoretic analysis of sequential voting on multiple binary issues, which we called strategic sequential voting. Specifically, given that voters have complete information about each other's preferences and their

Voting rule	**MSI**	
Dictatorships	m	(Proposition 5)
Plu w/ runoff	m	(Proposition 5)
STV	m	(Proposition 5)
Copeland$_\alpha$ ($0 < \alpha \leq 1$)	$\Theta(m)$	(Proposition 7)
Borda ($n \geq m$)	$\Theta(m)$	(Proposition 6)
Bucklin	$\Theta(m)$	(Proposition 5)
$Seq_{\mathcal{O}}(maj, \ldots, maj)$ ($\mathcal{O}$-lexico profiles)	$3m/4 + 1$	(Theorem 6)
$SSP_{\mathcal{O}}$ ($\mathcal{O}$-lexico profiles)	$3m/4 + 1$	(Theorem 6)
k-Approval (incl. Plurality and Veto)	$m + 1 - k$	(Proposition 5)
Ranked pairs ($n \geq \sqrt{m}$)	$\Omega(\sqrt{m})$	(Proposition 9)
$Seq_{\mathcal{O}}(maj, \ldots, maj)$ (separable profiles)	$\Theta(\sqrt{m})$	(Theorem 5)
$SSP_{\mathcal{O}}$ (separable profiles)	$\Theta(\sqrt{m})$	(Theorem 5)
$Seq_{\mathcal{O}}(maj, \ldots, maj)$ ($\mathcal{O}$-legal profiles)	$\Theta(\sqrt{m})$	(Proposition 4)
$SSP_{\mathcal{O}}$ ($\mathcal{O}$-legal profiles)	between $\lfloor \log m/2 + 2 \rfloor$ and $\lfloor \log m/2 + 5 \rfloor$	(Theorem 8)
$SSP_{\mathcal{O}}$	$\lfloor \log m/2 + 2 \rfloor$	(Theorem 2)
Voting tree ($n \geq 2m$)	$\lfloor l/2 + 2 \rfloor$	(Proposition 10)
Maximin ($n \geq m - 1$)	≤ 3	(Proposition 8)
Copeland$_0$ (n is even)	2	(Proposition 7)

Table 2: The minimax satisfaction index for strategic sequential voting (SSP), truthful sequential voting (Seq), and common voting rules, ranked roughly from high to low. ("Roughly" because, for example, k-approval is really a family of voting rules, and plurality (namely, 1-approval) has a high MSI of m, whereas veto (namely, $(m-1)$-approval) has a low MSI of 2.) For multi-issue domains, $m = 2^p$, $p \in \mathbb{N}$. A low MSI implies the existence of a paradox for the rule. Results for SSP (Seq) are highlighted in dark grey (light grey).

preferences are strict, the game can be solved by a natural backward induction process (WSDSBI), which leads to a unique solution. We showed that under some conditions on the preferences, this process leads to the same outcome as the truthful sequential voting, but in general it can result in very different outcomes. We analyzed the effect of changing the order over the issues that voters vote on and showed that, in some elections, every alternative can be made a winner by voting according to an appropriate order over the issues.

Most significantly, we showed that strategic sequential voting is prone to multiple-election paradoxes; to do so, we introduced a concept called the minimax satisfaction index, which measures the degree to which at least one voter is made happy by the outcome of the election. We showed that the minimax satisfaction index for strategic sequential voting is exponentially small, which means that there exists a profile for which the winner is ranked almost in the bottom positions in all votes; even worse, the winner is Pareto-dominated by almost every other alternative. We showed that changing the order of the issues in sequential voting cannot completely avoid the paradoxes. These negative results indicate that the solution of the sequential game can be extremely undesirable for every voter. We also showed that multiple-election paradoxes can be avoided to some extent by restricting voters' preferences to be separable or lexicographic, but the paradoxes still exist when the voters' preferences are $\mathcal{O}$-legal.

For the sake of benchmarking our results, we also studied the minimax satisfaction index for some common voting rules (assuming truthful voting). The results are summarized in Table 2. For a voting rule with a low (high) MSI, we can (cannot) find a paradox that is similar to the first type of multiple-election paradoxes—that is, a profile for which the winner is ranked in extremely low positions in all votes.

From this table, we may conclude that: (1) in sequential voting, the paradoxes are stronger when voting is strategic than when it is truthful, though of course this is no longer true if we are in a restricted setting where truthful and strategic voting lead to identical results (that is, when the profile is separable or lexicographic); (2) the strength of the paradoxes for sequential voting ranks somewhere in the middle, though perhaps somewhat more on the strong side, among standard voting rules (when voters are assumed to vote truthfully).

There are many topics for future research. For example, given a profile, can we characterize the set of alternatives that win for some order over the issues? Is there any good criterion for selecting a good order over the issues? More generally, how can we get around the multiple-election paradoxes in sequential voting games? For example, Theorem 6 shows that if the voters' preferences are lexicographic, then we can avoid the paradoxes. It is not clear if there are other ways to avoid the paradoxes (paradoxes occur even if we restrict voters' preferences to be separable or $\mathcal{O}$-legal, as shown in Theorem 5 and Theorem 8). Another approach is to consider other, non-sequential voting procedures for multi-issue domains. What are good examples of such procedures? Will these avoid paradoxes? What is the effect of strategic behavior for such procedures? How should we even define "strategic behavior" for such procedures, or for sequential voting with non-binary issues, or for voting rules in general? How can we extend these results to incomplete-information settings?[8] Also, beyond proving paradoxes for individual rules, is it possible to show a general impossibility result that shows that under certain minimal conditions, paradoxes cannot be avoided?[9]

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[8] Of course, to the extent that the complete-information setting is a special case of incomplete-information settings, things can only get worse in the latter.

[9] This may require quite restrictive conditions or a different notion of a paradox—for example, we have already shown that several natural voting rules have MSI m, albeit under truthful voting.

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Appendix A. Proofs of Theorems in Section 6

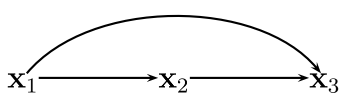
We use *Conditional preference networks (CP-nets)* [6], to better present our proofs. CP-nets are a popular language for specifying preferences developed in the artificial intelligence community. The language is based on the notion of conditional preferential independence, thus allowing for eliciting preferences and for storing them as compactly as possible.

Definition 4 (CP-nets [6]) *A* CP-net $\mathcal{N}$ *over* $\mathcal{X}$ *is a pair consisting of the following two parts:*

- *A* directed acyclic[10] graph $G = (\mathcal{I}, E)$ *over the set of issues* $\mathcal{I}$. *For any variable* $\mathbf{x}_i$, *let* $Par_G(\mathbf{x}_i)$ *denote the set of parents of* $\mathbf{x}_i$ *in* G.
- *A collection of* conditional preference tables $CPT(\mathbf{x}_i)$ *for each* $\mathbf{x}_i \in \mathcal{I}$, *defined as follows: each conditional preference table* $CPT(\mathbf{x}_i)$ *associates a total order* $\succ^i_{\vec{u}}$ *on* D_i *with each instantiation* $\vec{u}$ *of* $\mathbf{x}_i$*'s parents in* G.

Intuitively, the edges of G represent preferential dependencies: for every i, $\mathbf{x}_i$ is preferentially independent from its "non-parents" given its parents.

Example 3 *Let* $\mathcal{N}$ *be the following CP-net, whose graph* G *is depicted in the left, and the conditional preference tables are in the right.*



$CPT(\mathbf{x}_1)$	$CPT(\mathbf{x}_2)$	$CPT(\mathbf{x}_3)$
$0_1 \succ 1_1$	$0_1 : 0_2 \succ 1_2$	$0_1 0_2 : 0_3 \succ 1_3$
	$1_1 : 1_2 \succ 0_2$	$0_1 1_2 : 0_3 \succ 1_3$
		$1_1 0_2 : 0_3 \succ 1_3$
		$1_1 1_2 : 1_3 \succ 0_3$

Given a voter's CP-net, if we take an alternative and change the value of a single issue to obtain a new alternative, we can determine whether the voter prefers the old or the new alternative, based on the CPT for that issue. We can then make further inferences about the voter's preferences based on transitivity, although we will in general not be able to infer the voter's entire linear order over all alternatives. More precisely, a CP-net $\mathcal{N}$

[10] The original definition of CP-nets [6] allows G to contain cycles. However, in this paper we do not need to refer to this more general framework. Note that the assumption that G is acyclic is usual (see [7, 6]).

induces the partial order $\succ_{\mathcal{N}}$, defined as the transitive closure of $\{(a_i, \vec{u}, \vec{z}) \succ (b_i, \vec{u}, \vec{z})) \mid i \leq p; \vec{u} \in D_{\text{Par}_G(\mathbf{x}_i)}; a_i, b_i \in D_i \text{ s.t. } a_i \succ^i_{\vec{u}} b_i; \vec{z} \in D_{\mathcal{I} \setminus (\text{Par}_G(\mathbf{x}_i) \cup \{\mathbf{x}_i\})}\}$. It is known [6] that if $\mathcal{N}$ is acyclic, then $\succ_{\mathcal{N}}$ is transitive and asymmetric, that is, a strict partial order. (This is not necessarily the case if $\mathcal{N}$ is not acyclic.)

Example 4 *The partial order $\succ_{\mathcal{N}}$ of the CP-net $\mathcal{N}$ defined in Example 3 is depicted below, where* 000 *is the abbreviation for* $0_1 0_2 0_3$*, etc.*

$$\succ_{\mathcal{N}}: \quad 000 \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \begin{array}{c} 010 \\ \\ 001 \end{array} \begin{array}{c} \searrow \\ \\ \nearrow \end{array} 011 \rightarrow 111 \rightarrow 110 \rightarrow 100 \rightarrow 101$$

A linear order V *extends* a CP-net $\mathcal{N}$, denoted by $V \sim \mathcal{N}$, if it extends $\succ_{\mathcal{N}}$—that is, it is consistent with the preferences implied by the CP-net. Separability, $\mathcal{O}$-legality, and $\mathcal{O}$-lexicographicity (defined in Section 2.2) can be defined equivalently by using CP-nets as follows. A linear order V is separable if it extends a CP-net without any dependencies, *i.e.*, for which $E = \emptyset$; V is $\mathcal{O}$-legal if it extends a CP-net whose graph G is consistent with $\mathcal{O}$, that is, for any $i, j \leq p$, $\mathbf{x}_i$ is a parent of $\mathbf{x}_j$ if and only if $\mathbf{x}_i$ is ranked higher than $\mathbf{x}_j$ in $\mathcal{O}$; V is $\mathcal{O}$-lexicographic if it is lexicographic and $\mathcal{O}$-legal. We recall that in this paper, we assume that $\mathcal{O} = \mathbf{x}_1 > \mathbf{x}_2 > \cdots > \mathbf{x}_p$[11]. For any valuation $\vec{u}$ of $\text{Par}_{GG}(\mathbf{x}_i)$, let $V|_{\mathbf{x}_i:\vec{u}}$ and $\mathcal{N}|_{\mathbf{x}_i:\vec{u}}$ denote the restriction of V (or equivalently, $\mathcal{N}$) to $\mathbf{x}_i$, given $\vec{u}$. That is, $V|_{\mathbf{x}_i:\vec{u}}$ (or $\mathcal{N}|_{\mathbf{x}_i:\vec{u}}$) is the linear order $\succ^i_{\vec{u}}$.

Proof of Theorem 5 (the profile is separable): Let $P = (V_1, \ldots, V_n)$. For any $i \leq p$, we let $d_i = maj(P|_{\mathbf{x}_i})$. That is, d_i is the majority winner for the projection of the profile to the ith issue. Because any separable profile is compatible with any order over the issues, P is an $\mathcal{O}^{-1}$-legal profile. It follows from Corollary 1 that $SSP_{\mathcal{O}}(P) = (d_1, \ldots, d_p)$. Without loss of generality $(d_1, \ldots, d_p) = (1_1, \ldots, 1_p)$.

First, we prove the lower bound. Because for any $i \leq p$, at least half of the voters prefer 1_i to 0_i, the total number of times that a voter prefers 1 to 0 for an issue, counted across all voters and issues, is at least $p \cdot (n/2)$.

[11] ?

Therefore, there exists $j \leq n$ such that voter j prefers 1 to 0 on at least $p/2$ issues, otherwise the total number of times that a voter prefers 1 to 0 for an issue, counted across all voters and issues, is no more than $n \cdot (p/2) - 1 < p \cdot (n/2)$, which is a contradiction. Formally put, there exists $j \leq n$ such that $|\{i \leq p : 1_i \succ_{V_j} 0_i\}| \geq p/2$. Without loss of generality for every $i \leq \lceil p/2 \rceil$, $1_i \succ_{V_j} 0_i$. It follows that for any $\vec{a} \in D_1 \times \cdots D_{\lceil p/2 \rceil}$, we have that $(1_1, \ldots, 1_p) \succ_{V_j} (\vec{a}, 1_{\lceil p/2 \rceil+1}, \ldots, 1_p)$. Therefore, the minimax satisfaction index is at least $2^{\lceil p/2 \rceil}$.

Next, we prove the upper bound. We first show that there exists a set of n CP-nets $\mathcal{N}_1, \ldots, \mathcal{N}_n$ that satisfies the following two conditions.

1. For each $j \leq n$, the number of issues on which $\mathcal{N}_j$ prefers 1 to 0 is exactly $\lfloor p/2 \rfloor + 1$.
2. For each $i \leq p$, $maj(\mathcal{N}_1|_{\mathbf{x}_i}, \ldots, \mathcal{N}_n|_{\mathbf{x}_i}) = 1_i$.

The proof is by explicitly constructing the profile through the following n-step process. Informally, we will allocate $p(\lfloor n/2 \rfloor + 1)$ CPT entries "1 is preferred to 0", $\lfloor n/2 \rfloor + 1$ entries per issue, to n CP-nets as even as possible. Let $k_1 = \cdots = k_p = \lfloor n/2 \rfloor + 1$. In the jth step, we let $I_j = \{i_1, \ldots, i_{\lfloor p/2 \rfloor+1}\}$ be the set of indices of the highest k's. Then, for any $i \in I_j$, we let $\mathcal{N}_j|_{\mathbf{x}_i} = [1_i \succ 0_i]$ and $k_i \leftarrow k_i - 1$; for any $i \notin I_j$, we let $\mathcal{N}_j|_{\mathbf{x}_i} = [0_i \succ 1_i]$. Because of the assumption that $n \geq 2p$, we have that $n(\lfloor p/2 \rfloor + 1) \geq p(\lfloor n/2 \rfloor + 1)$, which means that after n steps, for all $i \leq p$, $k_i \leq 0$.

It left to show that there exists extension of $\mathcal{N}_1, \ldots, \mathcal{N}_n$ such that in each of these extensions, $1_1 \cdots 1_p$ is ranked within bottom $2^{\lfloor p/2 \rfloor+1}$ positions. To show this, we use the following lemma.

Lemma 8 *For any partial order W and any alternative c, we let $|Down_W(c)| = \{c' : c \succeq_W c'\}$, that is, $|Down_W(c)|$ is the set of all alternatives (including c) that are less preferred to c in W. There exists a linear order V such that V extends W and c is ranked in the $|Down_W(c)|$th position from the bottom.*

The proof for Lemma 8 is quite straightforward: for every alternative d such that $d \notin \text{Down}_W(c)$, we put $d \succ c$ in the partial order. This does not violates transitivity, which means that the ordering relation obtained in this way is a partial order, denoted by W'. Then, let V be an arbitrary linear order that extends W'. It follows that c is ranked at the $\text{Down}_W(c)$th position from the bottom in V.

We note that because for any $j \leq n$, the number of entries in $\mathcal{N}_j$ where $1 \succ 0$ is no more than $\lfloor p/2+1 \rfloor$. Therefore, for any $j \leq n$, $|\text{Down}_{\succ_{\mathcal{N}_j}}(1_1 \cdots 1_p)| \leq$

$2^{\lfloor p/2+1 \rfloor}$ (we recall that $\succ_{\mathcal{N}_j}$ is the partial order that $\mathcal{N}_j$ encodes). Let $V_1, \ldots, V_n$ be extensions of $\mathcal{N}_1, \ldots, \mathcal{N}_n$, respectively, where for all $j \leq n$, $1_1 \cdots 1_p$ is ranked as low as possible in any V_j. It follows from Lemma 8 that for any $j \leq n$, $1_1 \cdots 1_p$ is ranked in the $2^{\lfloor p/2+1 \rfloor}$th position from the bottom in V_j. This proves the upper bound.

(End of proof of Theorem 5.) □

Proof of Theorem 6 (the profile is $\mathcal{O}$-lexicographic): The proof is for profiles without ties. Without loss of generality $SSP_{\mathcal{O}}(P) = 1_1 \cdots 1_p$ and for every $j \leq \lfloor n/2 \rfloor + 1$, $1_1 \succ_{V_j|_{\mathbf{x}_1}} 0_1$. It follows that in $V_1, \ldots, V_{\lfloor n/2 \rfloor + 1}$, $1_1 \cdots 1_p$ is ranked within top $2^{p-1} = m/2$ positions. Because in at least $\lfloor n/2 \rfloor + 1$ votes $1_1 : 1_2 \succ 0_2$, there exists a vote $V \in P$ such that $1_1 \succ_{V|_{\mathbf{x}_1}} 0_1$ and $1_1 : 1_2 \succ_{V|_{\mathbf{x}_2:1_1}} 0_2$. It follows that $1_1 \cdots 1_p$ is ranked in the $(3 \cdot 2^{p-2} + 1)$th position from the bottom. This proves that when the profile is $\mathcal{O}$-lexicographic, $MSI(SSP_{\mathcal{O}}) \geq 3 \cdot 2^{p-2} + 1$.

We next prove that $3 \cdot 2^{p-2} + 1$ is also an upper bound. Consider the profile $P = (V_1, \ldots, V_n)$ defined as follows. For any $j \leq \lfloor n/2 \rfloor + 1$, $1_1 \succ_{V_j|_{\mathbf{x}_1}} 0_1$; for any $\lfloor n/2 \rfloor + 2 \leq j \leq n$, $1_2 \succ_{V_j|_{\mathbf{x}_2:1_1}} 0_2$; for $j = 1, 2$, $1_2 \succ_{V_j|_{\mathbf{x}_2:1_1}} 0_2$; for any $3 \leq j \leq n$ and any $3 \leq i \leq p$, $1_i \succ_{V_j|_{\mathbf{x}_i:1_1 \cdots 1_{i-1}}} 0_j$; for any local preferences of any voter that is not defined above, let 0 be preferred to 1.

We note that for any $i \leq p$, more than $n/2$ votes in $P|_{\mathbf{x}_i:1_1 \cdots 1_{i-1}}$ prefer 1_i to 0_i, which means that $SSP_{\mathcal{O}}(P) = 1_1 \cdots 1_p$. It is easy to check that in any vote, $1_1 \cdots 1_p$ is ranked somewhere within bottom $3 \cdot 2^{p-2} + 1$ positions.

(End of proof of Theorem 6.) □

Proof of Theorem 7 (the profile is $\mathcal{O}$-legal, for a specific tie-breaking mechanism): The proof is by induction on p. When $p = 2$, let the CPT of $\mathcal{N}_1$ be $0_1 \succ 1_1, 0_1 : 1_2 \succ 0_2, 1_1 : 1_2 \succ 0_2$; let the CPT of $\mathcal{N}_2$ be $1_1 \succ 0_1, 0_1 : 0_2 \succ 1_2, 1_1 : 0_2 \succ 1_2$; $V_1 = [0_1 1_2 \succ 0_1 0_2 \succ 1_1 1_2 \succ 1_1 0_2]$; $V_2 = [1_1 0_2 \succ 0_1 0_2 \succ 1_1 1_2 \succ 0_1 1_2]$. In the first step, ties are broken in favor of $1_1 1_2$. Given 1_1, ties are broken in favor of 1_2; given 0_1, ties are broken in favor of 1_2.

Suppose the claim is true for $p = l$. Next we construct $\mathcal{N}_1$ and $\mathcal{N}_2$ for $p = l+1$. Let $\mathcal{N}'_1, \mathcal{N}'_2, V'_1, V'_2$ be the CP-nets and the votes for the case of $p = l$, where the multi-issue domain is $D_2 \times \cdots \times D_{l+1}$. Without loss of generality $|\text{Down}_{V_1}(1_2 \cdots 1_{l+1})| \leq \lceil l/2 \rceil$ and $|\text{Down}_{V_2}(1_2 \cdots 1_{l+1})| \leq \lfloor l/2 \rfloor$. We recall that for any vote V and any alternative c, $\text{Down}_V(c)$ (defined in Lemma 8) is the set of all alternatives that are ranked below c in V, including c. Let $\vec{e} \in D_2 \times \cdots \times D_{l+1}$ be an arbitrary alternative such that $1_2 \cdots 1_{l+1} \succ_{V_2} \vec{e}$. Such $\vec{e}$ always exists, because if on the contrary $1_2 \cdots 1_{l+1}$ is in the bottom of V_2, it must be ranked higher than at least l other alternatives in V_1 to win the election, which contradicts the assumption that $|\text{Down}_{V_1}(1_2 \cdots 1_{l+1})| \leq \lceil l/2 \rceil$. We will explain later why we choose $\vec{e}$ in such a way.

Let $\mathcal{N}_1^*$ (respectively, $\mathcal{N}_2^*$) be the separable CP-net (we recall that a CP-net is separable if its graph has no edges) $D_2 \times \cdots \times D_{l+1}$ in which $\vec{e}$ is in the top (respectively, bottom) position. For $i = 1, 2$, we let $\mathcal{N}_i$ be a CP-net over $D_1 \times \cdots \times D_{l+1}$, defined as follows:

- $0_1 \succ_{\mathcal{N}_i} 1_1$.
- The sub-CP-net of $\mathcal{N}_i$ restricted on $\mathbf{x}_1 = 1_1$ is $\mathcal{N}_i'$;
- The sub-CP-net of $\mathcal{N}_i$ restricted on $\mathbf{x}_1 = 0_1$ is $\mathcal{N}_i^*$;

Let V_1, V_2 be the extension of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, that satisfy the following conditions:

- For any $\vec{b}, \vec{d} \in D_2 \times \cdots \times D_{l+1}$ such that $\vec{b} \neq \vec{e}$, and any $i = 1, 2$, we have $(0_1, \vec{b}) \succ_{V_i} (1_1, \vec{d})$. This condition can be satisfied, because we have $0_1 \succ_{\mathcal{N}_i} 1_1$.
- For any $\vec{b}, \vec{d} \in D_2 \times \cdots \times D_{l+1}$, and any $i = 1, 2$, we have that $(1_1, \vec{b}) \succ_{V_i} (1_1, \vec{d})$ if and only if $\vec{b} \succ_{V_i'} \vec{d}$. This condition says that if we focus on the order of the alternatives whose $\mathbf{x}_1$ component is 1_1 in V_i, then it is the same as in V_i'.
- For any $\vec{d} \in D_2 \times \cdots \times D_{l+1}$, we have that $(0_1, \vec{e}) \succ_{V_1} (1_1, \vec{d})$.
- $(1_1, \ldots, 1_{l+1}) \succ_{V_2} (0_1, \vec{e}) \succ_{V_2} (1_1, \vec{e})$.

We let the tie-breaking mechanism be defined as follows: in the first step, ties are broken in favor of 1_1; in the subgame in which $\mathbf{x}_1 = 1_1$, ties are broken in the same way as for the profile (V_1', V_2') (such that $1_2 \cdots 1_{l_1}$ is the winner for the profile); in the subgame in which $\mathbf{x}_1 = 0_1$, ties are broken in such a way that $\vec{e}$ is the winner (because $\vec{e}$ is ranked in the top position in one vote, and in the bottom position in the other, there exists a tie-breaking mechanism under which $\vec{e}$ is the winner).

We note that $1_1 \cdots 1_p \succ_{V_1} \vec{d}$ if and only if $\vec{d} = (1_1, \vec{d'})$ for some $\vec{d'} \in D_2 \times \cdots \times D_{l+1}$ such that $1_2 \cdots 1_p \succ_{V_1'} \vec{d'}$. It follows that $|\text{Down}_{V_1}(1_1 \cdots 1_{l+1})| = |\text{Down}_{V_1'}(1_2 \cdots 1_{l+1})|$. We also note that $1_1 \cdots 1_{l+1} \succ_{V_2} \vec{b}$ if and only if $\vec{b} = (0_1, \vec{e})$ or $\vec{b} = (1_1, \vec{b'})$ for some $\vec{b'} \in D_2 \times \cdots \times D_{l+1}$ such that $1_2 \cdots 1_p \succ_{V_2'} \vec{b'}$. It follows that $|\text{Down}_{V_2}(1_1 \cdots 1_{l+1})| = |\text{Down}_{V_2'}(1_2 \cdots 1_{l+1})| + 1$. Therefore, $|\text{Down}_{V_1}(1_1 \cdots 1_{l+1})| \leq \lfloor (l+1)/2 \rfloor$ and $|\text{Down}_{V_2}(1_1 \cdots 1_p)| \leq \lfloor l/2 \rfloor + 1 \leq \lceil (l+1)/2 \rceil$.

Here the trick to choose $\vec{e}$ such that $1_2 \cdots 1_{l+1} \succ_{V_2'} \vec{e}$ is crucial, because we force $0_1 \succ_{\mathcal{N}_2} 1_1$ and $1_1 \cdots 1_{l+1} \succ_{V_2} (0_1, \vec{e})$, which implies that $1_1 \cdots 1_{l+1} \succ_{V_2} (0_1, \vec{e}) \succ_{V_2} (1_1, \vec{e})$ (since V_2 extends $\mathcal{N}_2$). If we chose $\vec{e}$ such that $\vec{e} \succ_{V_2'} 1_2 \cdots 1_{l+1}$, then we would have that $|\text{Down}_{V_2}(1_1 \cdots 1_{l+1})| = |\text{Down}_{V_2'}(1_2 \cdots 1_{l+1})| + 2$, which does not prove the claim for $p = l+1$.

Next, we verify that $SSP_{\mathcal{O}}(V_1, V_2) = 1_1 \cdots 1_{l+1}$. We note that $(0_1, \vec{e}) \succ_{V_1} 1_1 \cdots 1_{l+1}$. Therefore, in the first step voter 1 will vote for 0_1. Meanwhile, $1_1 \cdots 1_{l+1} \succ_{V_2} (0_1, \vec{e})$, which means that in the first step voter 2 will vote for 1_1. Because ties are broken in favor of 1_1 in the first step, we will fix $\mathbf{x}_1 = 1_1$. Then, in the following steps (step $2, \ldots, l+1$), $1_2, \ldots, 1_{l+1}$ will be the winners by induction hypothesis, which means that $SSP_{\mathcal{O}}(V_1, V_2) = 1_1 \cdots 1_{l+1}$.

Therefore, the claim is true for $p = l+1$. This means that the claim is true for any $p \in \mathbb{N}$.

Example 5 *Let us show an example of the above construction from $p = 2$ to $p = 3$. In $\mathcal{N}_1$, we have $0_1 \succ 1_1$, $1_1 : \mathcal{N}_1^*$, and $0_1 : \mathcal{N}_1'$, where $\mathcal{N}_1'$ is $0_2 \succ 1_2, 0_2 : 1_3 \succ 0_3, 1_2 : 1_3 \succ 0_3$. (We note that $\mathcal{N}_1'$ is defined over $D_2 \times D_3$.) V_1 restricted to 1_1 is $V_1' = [0_2 1_3 \succ 0_2 0_3 \succ 1_2 1_3 \succ 1_2 0_3]$ (which is, again, over $D_2 \times D_3$). Let $\vec{e} = 0_2 1_3$. Therefore, we have the following construction:*

$V_1 = 0_1 0_2 1_3 \succ 0_1 1_2 1_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 0_3 \succ 1_1 0_2 1_3 \succ 1_1 0_2 0_3 \succ 1_1 1_2 1_3 \succ 1_1 1_2 0_3$

$V_2 = 0_1 1_2 0_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 1_3 \succ 1_1 1_2 0_3 \succ 1_1 0_2 0_3 \succ 1_1 1_2 1_3 \succ 0_1 0_2 1_3 \succ 1_1 0_2 1_3$

Ties are broken in a way such that if we are in the branch in which $\mathbf{x}_1 = 1_1$, then $1_2 1_3$ is the winner; and if we are in the branch in which $\mathbf{x}_1 = 0_1$, then $\vec{e} = 0_2 1_3$ is the winner. In the first step, ties are broken in favor of 1_1. Then, the sub-game winners are $1_1 1_2 1_3$ and $0_1 0_2 1_3$. Since exactly one vote (V_1) prefers $0_1 0_2 1_3$ to $1_1 1_2 1_3$, and the other vote V_2 prefers $1_1 1_2 1_3$ to $0_1 0_2 1_3$, the winner is $1_1 1_2 1_3$.

(End of proof of Theorem 7.) □

Proof of Theorem 8 (the profile is $\mathcal{O}$-legal, for arbitrary tie-breaking mechanism): For simplicity, we prove the theorem for the case in which $n = 2p^2 + 2p + 1$. The proof for the case in which $n > 2p^2 + 2p + 1$ is similar. For any $l \leq p$, we let $\mathcal{X}_l = \{0_l, 1_l\} \times \{0_{l+1}, 1_{l+1}\} \times \cdots \times \{0_p, 1_p\}$; let $\mathcal{O}_l = \mathbf{x}_l > \mathbf{x}_{l+1} > \cdots > \mathbf{x}_p$. We first prove the following claim by induction.

Claim 1 *For any $l \leq p$, there exists a $\mathcal{O}_l$-legal profile $P_l = A_l \cup B_l \cup \hat{A}_l \cup \hat{B}_l \cup \{c^l\}$ over $\mathcal{X}_l$, where $A_l = \{a^l_1, \ldots, a^l_{p^2}\}$, $B_l = \{b^l_1, \ldots, b^l_{p^2}\}$, $\hat{A}_l = \{\hat{a}^l_1, \ldots, \hat{a}^l_p\}$, $\hat{B}_l = \{\hat{b}^l_1, \ldots, \hat{b}^l_p\}$, that satisfies the following conditions.*

- $SSP_{\mathcal{O}_l}(P_l) = 1_l \cdots 1_p$.
- *For any $V \in P_l$, $|Down_V(1_l \cdots 1_p)| \leq \lceil (p-l+1)/2 \rceil + 5$.*
- *For any $(p-l)p \leq j \leq p^2$, $|Down_{a^l_j}(1_l \cdots 1_p)| \leq \lceil (p-l+1)/2 \rceil + 3$, $|Down_{b^l_j}(1_l \cdots 1_p)| \leq \lceil (p-l+1)/2 \rceil + 3$.*
- *For any $p-l \leq j \leq p$, $|Down_{\hat{a}^l_j}(1_l \cdots 1_p)| \leq \lceil (p-l+1)/2 \rceil + 3$, $|Down_{\hat{b}^l_j}(1_l \cdots 1_p)| \leq \lceil (p-l+1)/2 \rceil + 3$.*
- *If $p-l+1$ is odd, then*
  - *for any $V_B \in B$, $|Down_{V_B}(1_l \cdots 1_p)| \leq \lceil (p-l+1)/2 \rceil + 4$;*
  - *for any $(p-l)p \leq j \leq p^2$, $|Down_{b^l_j}(1_l \cdots 1_p)| \leq \lceil (p-l+1)/2 \rceil + 2$;*
  - *and for any $p-l \leq j \leq p$, $|Down_{\hat{b}^l_j}(1_l \cdots 1_p)| \leq \lceil (p-l+1)/2 \rceil + 2$.*
- *$1_l \cdots 1_p$ is ranked higher than $1_l \cdots 1_{p-2} 0_{p-1} 0_p$ in all votes in P_l.*

Proof of Claim 1: We prove the claim by induction on l. When $l = p-1$, we let all votes in P_{p-1} be $1_{p-1}1_p \succ 1_{p-1}0_p \succ 0_{p-1}1_p \succ 0_{p-1}0_p$. It is easy to check that P_{p-1} satisfies all the conditions in the claim. Suppose the claim is true for $l \leq p$, we next prove that the claim is also true for $l-1$. We show the existence of P_{l-1} by construction for the following two cases.

Case 1: $p-l+1$ is even.

We let $\mathcal{N}^l_1, \ldots, \mathcal{N}^l_{p-l+1}, \mathcal{N}^l_A, \mathcal{N}^l_B$ be separable CP-nets over $\mathcal{X}_l$, defined as follows.

- Let $1_l \cdots 1_{p-2} 0_{p-1} 0_p$ be in the bottom position of $\mathcal{N}^l_A$; let $1_l \cdots 1_{p-2} 0_{p-1} 0_p$ be in the top position of $\mathcal{N}^l_B$.

- For any $1 \leq i \leq p-l-1$, let $1_l \cdots 1_{l+i-2} 0_{l+i-1} 1_{l+i} \cdots 1_{p-2} 0_{p-1} 0_p$ be in the top position of $\mathcal{N}_i^l$; let $1_l \cdots 1_{p-2} 1_{p-1} 0_p$ be in the top position of $\mathcal{N}_{p-l}^l$; let $1_l \cdots 1_{p-2} 0_{p-1} 1_p$ be in the top position of $\mathcal{N}_{p-l+1}^l$.

For any linear order V over $\mathcal{X}_l$, we let the *composition* of V and $\mathcal{N}$ (where $\mathcal{N} \in \{\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l, \mathcal{N}_A^l, \mathcal{N}_B^l\}$) be a partial order O^{l-1} over $\mathcal{X}_{l-1}$, defined as follows.

- The restriction of O^{l-1} on $\mathbf{x}_{l-1} = 1_{l-1}$ is V. That is, for any $\vec{d}_1, \vec{d}_2 \in \mathcal{X}_l$ such that $\vec{d}_1 \succ_V \vec{d}_2$, we let $(1_{l-1}, \vec{d}_1) \succ_{O^{l-1}} (1_{l-1}, \vec{d}_2)$.
- The restriction of O^{l-1} on $\mathbf{x}_{l-1} = 0_{l-1}$ is the partial order encoded by $\mathcal{N}$. That is, for any $\vec{d}_1, \vec{d}_2 \in \mathcal{X}_l$ such that $\vec{d}_1 \succ_{\mathcal{N}} \vec{d}_2$, we let $(0_{l-1}, \vec{d}_1) \succ_{O^{l-1}} (0_{l-1}, \vec{d}_2)$.
- For any $\vec{d} \in \mathcal{X}_l$, we let $(0_{l-1}, \vec{d}) \succ_{O^{l-1}} (1_{l-1}, \vec{d})$.
- If $\mathcal{N} \in \{\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l, \mathcal{N}_A^l\}$, we let $1_{l-1} 1_l \cdots 1_p \succ_{O^{l-1}} 0_{l-1} 1_l \cdots 1_{p-2} 0_{p-1} 0_p$.

We are now ready to define P_{l-1}. Any $V \in P_{l-1}$ has a counterpart in P_l. For example, the counterpart of $\hat{a}_1^{l-1}$ is $\hat{a}_1^l$. For any $V \in P_{l-1}$, V is defined to be the extension of the composition of V's counterpart in P_l and some $\mathcal{N}$ (where $\mathcal{N} \in \{\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l, \mathcal{N}_A^l, \mathcal{N}_B^l\}$), in which $1_{l-1} \cdots 1_p$ is ranked as low as possible. Next we specify which $\mathcal{N}$ that each $V \in P_{l-1}$ corresponds to in the following table.

for all	**votes in P_{l-1}**	**is composed of**	
$1 \leq j \leq p$	$\hat{a}_j^{l-1}$	$\hat{a}_j^l$	$\mathcal{N}_A^l$
$j \leq (p-l)p$	a_j^{l-1}	a_j^l	$\mathcal{N}_A^l$
$(p-l)p+1 \leq j \leq (p-l+1)p$	a_j^{l-1}	a_j^l	$\mathcal{N}_{j-(p-l)p}^l$
$(p-l+1)p+1 \leq j \leq p^2$	a_j^{l-1}	a_j^l	$\mathcal{N}_A^l$
	$\hat{b}_{p-l+2}^{l-1}$	$\hat{b}_{p-l+2}^l$	$\mathcal{N}_A^l$
$j \neq p-l+2$	$\hat{b}_j^{l-1}$	$\hat{b}_j^l$	$\mathcal{N}_B^l$
$j \leq p^2$	b_j^{l-1}	b_j^l	$\mathcal{N}_B^l$
	c^{l-1}	c^l	$\mathcal{N}_B^l$

Table A.3: From P_l to P_{l-1}.

It follows that P_{l-1} is $\mathcal{O}_{l-1}$-legal. By Lemma 8, we have the following calculation.

- For any $1 \leq j \leq p$, $|\text{Down}_{\hat{a}_j^{l-1}}(1_{l-1}\cdots 1_p)| = |\text{Down}_{\hat{a}_j^l}(1_l\cdots 1_p)| + 1$. This is because for any $\vec{d} \in \mathcal{X}_l$ such that $\vec{d} \in \text{Down}_{\hat{a}_j^l}(1_l\cdots 1_p)$, $1_{l-1}\cdots 1_p$ is ranked higher than $(1_{l-1}, \vec{d})$ in $\hat{a}_j^{l-1}$; and moreover, $1_{l-1}\cdots 1_p$ is ranked higher than $0_{l-1}1_l\cdots 1_{p-2}0_{p-1}0_p$ in $\hat{a}_j^{l-1}$.

- For any $1 \leq j \leq p$, $|\text{Down}_{a^{l-1}_{(p-l)p+j}}(1_{l-1}\cdots 1_p)| = |\text{Down}_{a^{l}_{(p-l)p+j}}(1_l\cdots 1_p)|+3$. This is because for any $\vec{d} \in \mathcal{X}_l$ such that $\vec{d} \in \text{Down}_{a_{(p-l)p+j}}(1_l\cdots 1_p)$, $1_{l-1}\cdots 1_p$ is ranked higher than $(1_{l-1}, \vec{d})$ in $a_{(p-l)p+j}$; and moreover, $1_{l-1}\cdots 1_p$ is ranked higher than $0_{l-1}1_l\cdots 1_{p-2}0_{p-1}0_p$ in $a_{(p-l)p+j}$.

- For any $j \leq (p-l)p$ or $(p-l+1)p+1 \leq j \leq p^2$, $|\text{Down}_{a_j^{l-1}}(1_{l-1}\cdots 1_p)| = |\text{Down}_{a_j^l}(1_l\cdots 1_p)| + 1$.

- $|\text{Down}_{\hat{b}^{l-1}_{p-l+2}}(1_{l-1}\cdots 1_p)| = |\text{Down}_{\hat{b}^{l}_{p-l+2}}(1_l\cdots 1_p)| + 1$.

- For any $V_B \in (B_{l-1} \cup \hat{B}_{l-1} \cup \{c\}) \setminus \{\hat{b}_{p-l+2}\}$, $|\text{Down}_{V_B}(1_{l-1}\cdots 1_p)| = |\text{Down}_{V_B^l}(1_l\cdots 1_p)|$, where V_B^l is the counterpart of V_B in P_l.

We next prove that $SSP_{\mathcal{O}_{l-1}}(P_{l-1}) = 1_{l-1}\cdots l_p$. We note that $P_{l-1}|_{\mathbf{x}_{l-1}=1_{l-1}} = P_l$. Therefore, if in the first step 1_{l-1} is chosen, then the winner is $1_{l-1}\cdots 1_p$. We also note that $P_{l-1}|_{\mathbf{x}_{l-1}=0_{l-1}}$ is separable (and the CP-nets are $\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l$, p^2+p copies of $\mathcal{N}_A^l$ and p^2+p copies of $\mathcal{N}_B^l$). Therefore, if in the first step 0_{l-1} is chosen, then the winner is $0_{l-1}1_l\cdots 1_{p-2}1_{p-1}1_p$. Because exactly p^2+p-1 votes in P_{l-1} prefer $0_{l-1}1_l\cdots 1_{p-2}1_{p-1}1_p$ to $1_{l-1}\cdots 1_p$ (those votes corresponds to $\mathcal{N}_B^l$ in the construction), we have that 1_{l-1} is the winner in the first step. Therefore, $SSP_{\mathcal{O}_{l-1}}(P_{l-1}) = 1_{l-1}\cdots l_p$. It is also easy to verify that P_{l-1} satisfies all conditions in the claim.

Case 2: $p-l+1$ is odd. The construction is similar as in the even case. The only difference is that we switch the role of A_l and B_l (also $\hat{A}_l$ and $\hat{B}_l$). □

The theorem follows from Claim 1 by letting $l = 1$, and it is easy to check that in P_1 in Claim 1 ($l = 1$), no more than $4p^2$ alternatives has been ranked lower than $SSP_{\mathcal{O}}(P_1)$ in any vote, which means that $SSP_{\mathcal{O}}(P_1)$ is Pareto-dominated by at least $2^p - 4p^2$ alternatives.

(End of proof of Theorem 8.) □