4 Ranked pairs

In this section, we prove that the UCMU and UCMC problems under ranked pairs are NP-complete (even for a single manipulator) by giving a reduction from 3SAT.

Definition 5 *The* 3SAT *problem is: Given a set of variables $X = \{x_1, \dots, x_q\}$ and a formula $Q = Q_1 \wedge \dots \wedge Q_t$ such that*

1. *for all $1 \leq i \leq t$, $Q_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$, and*
2. *for all $1 \leq i \leq t$ and $1 \leq j \leq 3$, $l_{i,j}$ is either a variable x_k, or the negation of a variable $\neg x_k$,*

we are asked whether the variables can be set to true or false so that Q is true.

Theorem 2 *The UCMU and UCMC problems under ranked pairs are NP-complete, even when there is only one manipulator.*

Proof of Theorem 2: It is easy to verify that the UCMU and UCMC problems under ranked pairs are in NP. We first prove that UCMU is NP-complete. Given an instance of 3SAT, we construct a UCMU instance as follows. Without loss of generality, we assume that for any variable x, x and $\neg x$ appears in at least one clause, and none of the clauses contain both x and $\neg x$.
Set of alternatives: $\mathcal{C} = \{c, Q_1, \dots, Q_t, Q_1', \dots, Q_t'\} \bigcup \{x_1, \dots, x_q, \neg x_1, \dots, \neg x_q\}$ $\bigcup \{Q_{l_{1,1}}, Q_{l_{1,2}}, Q_{l_{1,3}}, \dots, Q_{l_{t,1}}, Q_{l_{t,2}}, Q_{l_{t,3}}\} \bigcup \{Q_{\neg l_{1,1}}, Q_{\neg l_{1,2}}, Q_{\neg l_{1,3}}, \dots, Q_{\neg l_{t,1}}, Q_{\neg l_{t,2}}, Q_{\neg l_{t,3}}\}$.
Alternative preferred by the manipulator: c.
Number of unweighted manipulators: $|M| = 1$.
Non-manipulators' profile: P^{NM} satisfying the following conditions.

1. For any $i \leq t$, $D_{P^{NM}}(c, Q_i) = 30$,$D_{P^{NM}}(Q_i', c) = 20$; for any $x \in \mathcal{C} \setminus \{Q_i, Q_i' : 1 \leq i \leq t\}$, $D_{P^{NM}}(c, x) = 10$.
2. For any $j \leq q$, $D_{P^{NM}}(x_j, \neg x_j) = 20$.
3. For any $i \leq t, j \leq 3$, if $l_{i,j} = x_k$, then $D_{P^{NM}}(Q_i, Q^i_{x_k}) = 30$, $D_{P^{NM}}(Q^i_{x_k}, x_k) = 30$, $D_{P^{NM}}(\neg x_k, Q^i_{\neg x_k}) = 30$, $D_{P^{NM}}(Q^i_{\neg x_k}, Q_i') = 30$; if $l_{i,j} = \neg x_k$, then $D_{P^{NM}}(Q_i, Q^i_{\neg x_k}) = 30$, $D_{P^{NM}}(Q^i_{x_k}, x_k) = 30$, $D_{P^{NM}}(\neg x_k, Q^i_{\neg x_k}) = 30$, $D_{P^{NM}}(Q^i_{x_k}, Q_i') = 30$, $D_{P^{NM}}(Q^i_{\neg x_k}, Q^i_{x_k}) = 20$.
4. For any $x, y \in \mathcal{C}$, if $D_{P^{NM}}(x, y)$ is not defined in the above steps, then $D_{P^{NM}}(x, y) = 0$.

For example, when $Q_1 = x_1 \vee \neg x_2 \vee x_3$, $D_{P^{NM}}$ is illustrated in Figure 1.
The existence of such a P^{NM} is guaranteed by Lemma 1, and the size of P^{NM} is in polynomial in t and q.
First, we prove that if there exists an assignment v of truth values to X so that Q is satisfied, then there exists a vote R_M for the manipulator such that $RP(P^{NM} \cup \{R_M\}) = \{c\}$. We construct R_M as follows.

- Let c be on the top of R_M.
- For any $k \leq q$, if $v(x_k) = \top$ (that is, x_k is true), then $x_k \succ_{R_M} \neg x_k$, and for any $i \leq t, j \leq 3$ such that $l_{i,j} = \neg x_k$, let $Q^i_{x_k} \succ_{R_M} Q^i_{\neg x_k}$.
- For any $k \leq q$, if $v(x_k) = \bot$ (that is, x_k is false), then $\neg x_k \succ_{R_M} x_k$, and for any $i \leq t, j \leq 3$ such that $l_{i,j} = \neg x_k$, let $Q^i_{\neg x_k} \succ_{R_M} Q^i_{x_k}$.
- The remaining pairs of alternatives are ranked arbitrarily.

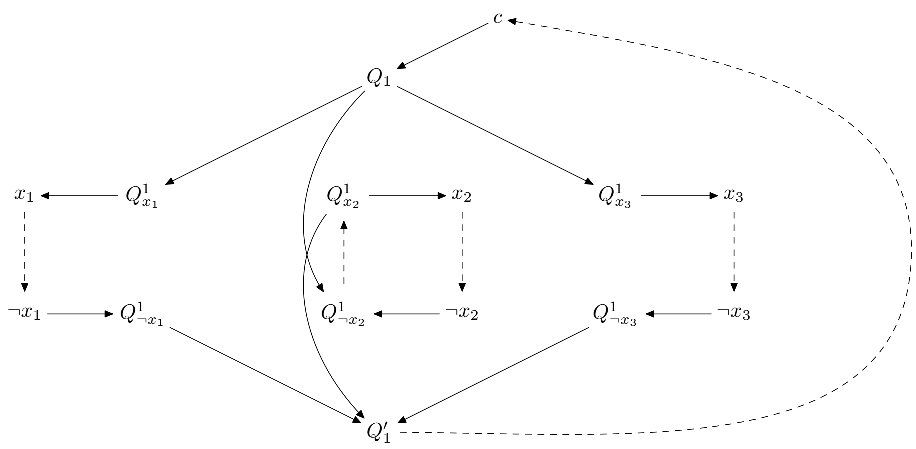

Figure 1: For any vertices v_1, v_2, if there is a solid edge from v_1 to v_2, then $D_{P^{NM}}(v_1, v_2) = 30$; if there is a dashed edge from v_1 to v_2, then $D_{P^{NM}}(v_1, v_2) = 20$; if there is no edge between v_1 and v_2 and $v_1 \neq c$, $v_2 \neq c$, then $D_{P^{NM}}(v_1, v_2) = 0$; for any x such that there is no edge between c and x, $D_{P^{NM}}(c, x) = 10$.

If $x_k = \top$, then $D_{P^{NM}\cup\{R_M\}}(x_k, \neg x_k) = 21$, and for any $i \leq t, j \leq 3$ such that $l_{i,j} = \neg x_k$, $D_{P^{NM}\cup\{R_M\}}(Q^i_{\neg x_k}, Q^i_{x_k}) = 19$. It follows that no matter how ties are broken when applying ranked pairs to $P^{NM} \cup \{R_M\}$, if $x_k = \top$, then $x_k \succ \neg x_k$ in the final ranking. This is because for any $l_{i,j} = \neg x_k$, $D_{P^{NM}\cup\{R_M\}}(Q^i_{\neg x_k}, Q^i_{x_k}) = 19 < 21 = D_{P^{NM}\cup\{R_M\}}(x_k, \neg x_k)$, which means that before trying to fix $x_k \succ \neg x_k$, there is no directed path from $\neg x_k$ to x_k.

Similarly if $x_k = \bot$, then $D_{P^{NM}\cup\{R_M\}}(x_k, \neg x_k) = 19$, and for any $i \leq t, j \leq 3$ such that $l_{i,j} = \neg x_k$, $D_{P^{NM}\cup\{R_M\}}(Q^i_{\neg x_k}, Q^i_{x_k}) = 21$. It follows that if $x_k = \bot$, then $\neg x_k \succ x_k$, and for any $i \leq t, j \leq 3$ such that $l_{i,j} = \neg x_k$, $Q^i_{\neg x_k} \succ Q^i_{x_k}$ in the final ranking. This is because $Q^i_{\neg x_k} \succ Q^i_{x_k}$ will be fixed before $x_k \succ \neg x_k$.

Because Q is satisfied under v, for each clause Q_i, at least one of its three literals is true under v. Without loss of generality, we assume $v(l_{i,1}) = \top$. If $l_{i,1} = x_k$, then before trying to add $Q'_i \succ c$, the directed path $c \to Q_i \to Q_{x_k} \to x_k \to \neg x_k \to Q_{\neg x_k} \to Q'_i$ has already been fixed. Therefore, $c \succ Q'_i$ in the final ranking, which means that for any alternatives x in $\mathcal{C} \setminus \{c, Q_1, \ldots, Q_t, Q'_1, \ldots, Q'_t\}$, $c \succ x$ in the final ranking because $D_{P^{NM}\cup\{R_M\}}(c, x) > 0$. Hence, c is the unique winner of $P^{NM} \cup \{R_M\}$ under ranked pairs.

Next, we prove that if there exists a vote R_M for the manipulator such that $RP(P^{NM}\cup\{R_M\}) = \{c\}$, then there exists an assignment v of truth values to X such that Q is satisfied. We construct the assignment v so that $v(x_k) = \top$ if and only if $x_k \succ_{R_M} \neg x_k$, and $v(x_k) = \bot$ if and only if $\neg x_k \succ_{R_M} x_k$. We claim that $v(Q) = \top$. If, on the contrary, $v(Q) = \bot$, then there exists a clause (Q_1, without loss of generality) such that $v(Q_1) = \bot$. We now construct a way to fix the pairwise rankings such that c is not the winner under ranked pairs, as follows. For any $j \leq 3$, if there exists $k \leq q$ such that $l_{i,j} = \neg x_k$, then $x_k \succ_{R_M} \neg x_k$ because $v(\neg x_k) = \bot$. Therefore, $D_{P^{NM}\cup R_M}(x_k, \neg x_k) = 21$. Then, after trying to add all pairs $x \succ x'$ such that $D_{P^{NM}\cup R_M}(x, x') > 21$ (that is, all solid directed edges in Figure 1), it follows that $x_k \succ \neg x_k$ can be added to the final ranking. We choose to add $x_k \succ \neg x_k$ first, which means that $Q^1_{x_k} \succ Q^1_{\neg x_k}$ in the final ranking (otherwise, we have $Q^1_{\neg x_k} \succ Q^1_{x_k} \succ x_k \succ \neg x_k \succ Q^1_{\neg x_k}$, which is a contradiction).

For any $j \leq 3$, if there exists $k \leq q$ such that $l_{i,j} = x_k$, then $\neg x_k \succ_{R_M} x_k$ because $v(x_k) = \bot$. Therefore, $D_{P^{NM}\cup R_M}(x_k, \neg x_k) = 19$. We note that after trying to add all pairs $x \succ x'$ such that $D_{P^{NM}\cup R_M}(x, x') > 19$, $Q^1_{x_k} \not\succ Q^1_{\neg x_k}$. We recall that for any $j \leq 3$, if there exists $k \leq q$ such that $l_{i,j} = \neg x_k$, then $Q^1_{\neg x_k} \not\succ Q^1_{x_k}$. Hence, it follows that $Q'_1 \succ c$ is consistent with all pairwise rankings added so far. Then, since $D_{P^{NM} \cup R_M}(Q'_1, c) \geq 19$, if $Q'_1 \succ c$ has not been added, we choose to add it first of all pairwise rankings of alternatives $x \succ x'$ such that $D_{P^{NM} \cup R_M}(x, x') = 19$, which means that $Q'_1 \succ c$ in the final ranking—in other words, c is not at the top in the final ranking. Therefore, c is not the unique winner, which contradicts the assumption that $RP(P^{NM} \cup \{R_M\}) = \{c\}$.

For UCMC, we modify the reduction as follows: we let P^{NM} be such that for any $i \leq t$, $D_{P^{NM}}(Q'_i, c) = 22$, and for any $j \leq q$, $D_{P^{NM}}(x_j, \neg x_j) = 22$. □

Similarly, we can prove that when $|M|$ is a constant greater than one, UCMU and UCMC under ranked pairs remain NP-complete.

Theorem 3 *The UCMU and UCMC problems under ranked pairs are NP-complete, even when the number of manipulators is fixed to some constant* $|M| > 1$.

Proof of Theorem 3: We prove UCMU is NP-complete. The proof is similar to that of Theorem 2. We let P^{NM} satisfy the following conditions.

1. For any $i \leq t$, $D_{P^{NM}}(c, Q_i) = 30|M|$,$D_{P^{NM}}(Q'_i, c) = 22|M| - 2$; for any $x \in \mathcal{C} \setminus \{Q_i, Q'_i : 1 \leq i \leq t\}$, $D_{P^{NM}}(c, x) = 10|M|$.
2. For any $j \leq q$, $D_{P^{NM}}(x_j, \neg x_j) = 22|M| - 2$.
3. For any $i \leq t, j \leq 3$, if $l_{i,j} = x_k$, then $D_{P^{NM}}(Q_i, Q^i_{x_k}) = 30|M|$, $D_{P^{NM}}(Q^i_{x_k}, x_k) = 30|M|$, $D_{P^{NM}}(\neg x_k, Q^i_{\neg x_k}) = 30|M|$, $D_{P^{NM}}(Q^i_{\neg x_k}, Q'_i) = 30|M|$; if $l_{i,j} = \neg x_k$, then $D_{P^{NM}}(Q_i, Q^i_{\neg x_k}) = 30|M|$, $D_{P^{NM}}(Q^i_{x_k}, x_k) = 30|M|$, $D_{P^{NM}}(\neg x_k, Q^i_{\neg x_k}) = 30|M|$, $D_{P^{NM}}(Q^i_{x_k}, Q'_i) = 30|M|$, $D_{P^{NM}}(Q^i_{\neg x_k}, Q^i_{x_k}) = 20|M|$.
4. For any $x, y \in \mathcal{C}$, if $D_{P^{NM}}(x, y)$ is not defined in the above steps, then $D_{P^{NM}}(x, y) = 0$.

First, if there exists an assignment v of truth values to X so that Q is satisfied, then we let R_M be defined as in the proof for Theorem 2. It follows that $RP(P^{NM} \cup \{|M|R_M\}) = \{c\}$ (all the manipulators can vote R_M).

Next, if there exists a profile P^M for the manipulators such that $RP(P^{NM} \cup P^M) = \{c\}$, then we construct the assignment v so that $v(x_k) = \top$ if $x_k \succ_V \neg x_k$ for all $V \in P^M$, and $v(x_k) = \bot$ if $\neg x_k \succ_V x_k$ for all $V \in P^M$; the values of all the other variables are assigned arbitrarily. Then by similar reasoning as in the proof for Theorem 2, we know that Q is satisfied under v.

For UCMC, the proof is similar (by slightly modifying the $D_{P^{NM}}$ as we did in the proof of Theorem 2). □

5 Bucklin

In this section, we present a polynomial-time algorithm for the UCMU problem under Bucklin (a polynomial-time algorithm for the UCMC problem under Bucklin can be obtained similarly). For any alternative x, any natural number d, and any profile P, let $B(x, d, P)$ denote the number of times that x is ranked among the top d alternatives in P. The idea behind the algorithm is as follows. Let d_{min} be the minimal depth so that c is ranked among the top d_{min} alternatives in more than half of the votes (when all of the manipulators rank c first). Then, we check if there is a way to assign the manipulators' votes so that none of the other alternatives is ranked among the top d_{min} alternatives in more than half of the votes.

Algorithm 1
Input: *A UCM instance* $(Bucklin, P^{NM}, c, M)$, $C = \{c, c_1, \ldots, c_{m-1}\}$.

1. *Calculate the minimal depth* d_{min} *such that* $B(c, d_{min}, P^{NM}) + |M| > \frac{1}{2}(|NM| + |M|)$.