4 Ranked pairs

In this section, we prove that the UCMU and UCMC problems under ranked pairs are NP-complete (even for a single manipulator) by giving a reduction from 3SAT.

**Definition 5** The 3SAT problem is: Given a set of variables $X = \{x_1, \ldots, x_q\}$ and a formula $Q = Q_1 \land \ldots \land Q_t$ such that

1. for all $1 \leq i \leq t$, $Q_i = l_{i,1} \lor l_{i,2} \lor l_{i,3}$, and
2. for all $1 \leq i \leq t$ and $1 \leq j \leq 3$, $l_{i,j}$ is either a variable $x_k$, or the negation of a variable $\neg x_k$.

we are asked whether the variables can be set to true or false so that $Q$ is true.

**Theorem 2** The UCMU and UCMC problems under ranked pairs are NP-complete, even when there is only one manipulator.

**Proof of Theorem 2:** It is easy to verify that the UCMU and UCMC problems under ranked pairs are in NP. We first prove that UCMU is NP-complete. Given an instance of 3SAT, we construct a UCMU instance as follows. Without loss of generality, we assume that for any variable $x$, $x$ and $\neg x$ appears in at least one clause, and none of the clauses contain both $x$ and $\neg x$.

**Set of alternatives:** $\mathcal{C} = \{c, Q_1, \ldots, Q_t, Q_{1}', \ldots, Q_{t}'\} \cup \{x_1, \ldots, x_q, \neg x_1, \ldots, \neg x_q\}$

$\bigcup \{Q_{l_{1,1}}, Q_{l_{1,2}}, Q_{l_{1,3}}, \ldots, Q_{l_{t,1}}, Q_{l_{t,2}}, Q_{l_{t,3}}\} \bigcup \{Q_{l_{1,1}}, Q_{l_{1,2}}, Q_{l_{1,3}}, \ldots, Q_{l_{t,1}}, Q_{l_{t,2}}, Q_{l_{t,3}}\}$

**Alternative preferred by the manipulator:** $c$.

**Number of unweighted manipulators:** $|M| = 1$.

**Non-manipulators’ profile:** $P_{NM}$ satisfying the following conditions.

1. For any $i \leq t$, $D_{PNM}(c, Q_i) = 30, D_{PNM}(Q_i', c) = 20$; for any $x \in C \setminus \{Q_i, Q_i' : 1 \leq i \leq t\}$, $D_{PNM}(c, x) = 10$.
2. For any $j \leq q$, $D_{PNM}(x_j, \neg x_j) = 20$.
3. For any $i \leq t, j \leq 3$, if $l_{i,j} = x_k$, then $D_{PNM}(Q_i, Q_{i,x_k}) = 30$, $D_{PNM}(Q_{i,x_k}, Q_{i,x_k}) = 30$, $D_{PNM}(\neg x_k, Q_{i,x_k}) = 30$, $D_{PNM}(Q_{i,x_k}, Q_{i,x_k}) = 30$; if $l_{i,j} = \neg x_k$, then $D_{PNM}(Q_i, Q_{i-x_k}) = 30$, $D_{PNM}(Q_{i-x_k}, Q_{i-x_k}) = 30$, $D_{PNM}(Q_{i-x_k}, Q_{i-x_k}) = 30$, $D_{PNM}(Q_{i-x_k}, Q_{i-x_k}) = 30$.
4. For any $x, y \in C$, if $D_{PNM}(x, y)$ is not defined in the above steps, then $D_{PNM}(x, y) = 0$.

For example, when $Q_1 = x_1 \lor \neg x_2 \lor x_3$, $D_{PNM}$ is illustrated in Figure 1.

The existence of such a $P_{NM}$ is guaranteed by Lemma 1, and the size of $P_{NM}$ is in polynomial in $t$ and $q$.

First, we prove that if there exists an assignment $v$ of truth values to $X$ so that $Q$ is satisfied, then there exists a vote $R_M$ for the manipulator such that $RP(P_{NM} \cup \{R_M\}) = \{c\}$. We construct $R_M$ as follows.

- Let $c$ be on the top of $R_M$.
- For any $k \leq q$, if $v(x_k) = \top$ (that is, $x_k$ is true), then $x_k \succ_{R_M} \neg x_k$, and for any $i \leq t, j \leq 3$ such that $l_{i,j} = \neg x_k$, let $Q_{i,x_k} \succ_{R_M} Q_{i-x_k}$.
- For any $k \leq q$, if $v(x_k) = \bot$ (that is, $x_k$ is false), then $\neg x_k \succ_{R_M} x_k$, and for any $i \leq t, j \leq 3$ such that $l_{i,j} = \neg x_k$, let $Q_{i-x_k} \succ_{R_M} Q_{i-x_k}$.
- The remaining pairs of alternatives are ranked arbitrarily.
If \( x_k = \top \), then \( \text{DPNM} \cup \{ R_M \} \left( x_k, \neg x_k \right) = 21 \), and for any \( i \leq t, j \leq 3 \) such that \( l_{i,j} = \neg x_k \), \( \text{DPNM} \cup \{ R_M \} \left( Q^{\neg x_k}, Q^i_{x_k} \right) = 19 \). It follows that in the final ranking, this is because there is no directed path from \( \neg x_k \) to \( x_k \).

Similarly if \( x_k = \bot \), then \( \text{DPNM} \cup \{ R_M \} \left( x_k, \neg x_k \right) = 19 \), and for any \( i \leq t, j \leq 3 \) such that \( l_{i,j} = \neg x_k \), \( \text{DPNM} \cup \{ R_M \} \left( Q^{\neg x_k}, Q^i_{x_k} \right) = 21 \). It follows that if \( x_k = \bot \), then \( \neg x_k > x_k \), and for any \( i \leq t, j \leq 3 \) such that \( l_{i,j} = \neg x_k \), \( Q^{\neg x_k} > Q^i_{x_k} \) in the final ranking, which means that for any alternatives \( x \) in \( c \setminus \{ c, Q_1, \ldots, Q_3 \} \), \( x \) in the final ranking because \( \text{DPNM} \cup \{ R_M \} \left( c, x \right) > 0 \). Hence, \( c \) is the unique winner of \( \text{DPNM} \cup \{ R_M \} \) under ranked pairs.

Next, we prove that if there exists a vote \( R_M \) for the manipulator such that \( \text{RP} \left( \text{DPNM} \cup \{ R_M \} \right) = \{ c \} \), then there exists an assignment \( v \) of truth values to \( x \) such that \( Q \) is satisfied. We construct the assignment \( v \) so that \( v(x_k) = \top \) if and only if \( x_k > R_M \neg x_k \), and \( v(x) = \bot \) if and only if \( \neg x_k > R_M x_k \). We claim that \( v(Q) = \top \). If, on the contrary, \( v(Q) = \bot \), then there exists \( c \), without loss of generality, such that \( v(Q_1) = \bot \). We now construct a way to fix the pairwise rankings such that \( c \) is not the winner ranked pairs, as follows. For any \( j \leq 3 \), if there exists \( k \leq q \) such that \( l_{i,j} = \neg x_k \), then \( x_k > R_M \neg x_k \), which is a contradiction.

For any \( j \leq 3 \), if there exists \( k \leq q \) such that \( l_{i,j} = x_k \), then \( \neg x_k > R_M x_k \) because \( v(x_k) = \bot \). Therefore, \( \text{DPNM} \cup \{ R_M \}(x_k, \neg x_k) = 21 \). Then, after trying to add all pairs \( x \succ x' \) such that \( \text{DPNM} \cup \{ R_M \}(x, x') > 21 \) (that is, all solid directed edges in Figure 1), it follows that \( x_k \succ \neg x_k \) can be added to the final ranking.

We choose to add \( x_k \succ \neg x_k \) first, which means that \( Q^1_{x_k} > Q^{\neg x_k} \) in the final ranking (otherwise, we have \( Q^1_{x_k} > Q^{\neg x_k} \succ x_k \succ \neg x_k \succ Q^{\neg x_k} \), which is a contradiction).

For any \( j \leq 3 \), if there exists \( k \leq q \) such that \( l_{i,j} = \neg x_k \), then \( \neg x_k > R_M x_k \) because \( v(x_k) = \bot \). Therefore, \( \text{DPNM} \cup \{ R_M \}(x_k, \neg x_k) = 19 \). We note that after trying to add all pairs \( x \succ x' \) such that \( D_{\text{PNM}}(x, x') > 19 \), \( Q^1_{x_k} \neq Q^{\neg x_k} \). We recall that for any \( j \leq 3 \), if there exists \( k \leq q \) such that \( l_{i,j} = \neg x_k \), then \( Q^{\neg x_k} \neq Q^i_{x_k} \). Hence, it follows that \( Q^i_{x_k} \succ c \) is consistent with all pairwise rankings.
added so far. Then, since \( D_{P^NM∪R_M}(Q'_1, c) ≥ 19 \), if \( Q'_1 ∗ c \) has not been added, we choose to add it first of all pairwise rankings of alternatives \( x ∗ x' \) such that \( D_{P^NM∪R_M}(x, x') = 19 \), which means that \( Q'_1 ∗ c \) in the final ranking—in other words, \( c \) is not at the top in the final ranking. Therefore, \( c \) is not the unique winner, which contradicts the assumption that \( RP(P^{NM}∪\{R_M\}) = \{c\} \).

For UCMC, we modify the reduction as follows: we let \( P^{NM} \) be such that for any \( i ≤ t \), \( D_{P^NM}(Q'_i, c) = 22 \), and for any \( j ≤ q \), \( D_{P^NM}(x_j, ¬x_j) = 22 \).

Similarly, we can prove that when \(|M|\) is a constant greater than one, UCMU and UCMC under ranked pairs remain NP-complete.

**Theorem 3** The UCMU and UCMC problems under ranked pairs are NP-complete, even when the number of manipulators is fixed to some constant \(|M| > 1\).

**Proof of Theorem 3:** We prove UCMU is NP-complete. The proof is similar to that of Theorem 2.

We let \( P^{NM} \) satisfy the following conditions.

1. For any \( i ≤ t \), \( D_{P^NM}(c, Q_i) = 30|M|, D_{P^NM}(Q'_i, c) = 22|M| − 2 \); for any \( x ∈ C \setminus \{Q_i, Q'_i : 1 ≤ i ≤ t\}, D_{P^NM}(c, x) = 10|M| \).
2. For any \( j ≤ q \), \( D_{P^NM}(x_j, ¬x_j) = 22|M| − 2 \).
3. For any \( i ≤ t, j ≤ 3 \), if \( l_{i,j} = x_k \), then \( D_{P^NM}(Q_i, Q^{x_k}_i) = 30|M|, D_{P^NM}(Q^{x_k}_i, x_k) = 30|M|, D_{P^NM}(¬x_k, Q^{x_k}_i) = 30|M| \); if \( l_{i,j} = ¬x_k \), then \( D_{P^NM}(Q_i, Q^{¬x_k}_i) = 30|M|, D_{P^NM}(Q^{¬x_k}_i, x_k) = 30|M|, D_{P^NM}(¬x_k, Q^{¬x_k}_i) = 30|M| \), \( D_{P^NM}(Q^{x_k}_i, Q^{¬x_k}_i) = 30|M|, D_{P^NM}(Q^{¬x_k}_i, Q^{x_k}_i) = 20|M| \).
4. For any \( x, y ∈ C \), if \( D_{P^NM}(x, y) \) is not defined in the above steps, then \( D_{P^NM}(x, y) = 0 \).

First, if there exists an assignment \( v \) of truth values to \( X \) so that \( Q \) is satisfied, then we let \( R_M \) be defined as in the proof for Theorem 2. It follows that \( RP(P^{NM}∪\{M|R_M\}) = \{c\} \) (all the manipulators can vote \( R_M \)).

Next, if there exists a profile \( P^M \) for the manipulators such that \( RP(P^{NM}∪P^M) = \{c\} \), then we construct the assignment \( v \) so that \( v(x_k) = V \) if \( x_k ∗ V \ x_k \) for all \( V ∈ P^M \), and \( v(¬x_k) = ⊥ \) if \( ¬x_k ∗ V \ x_k \) for all \( V ∈ P^M \); the values of all the other variables are assigned arbitrarily. Then by similar reasoning as in the proof for Theorem 2, we know that \( Q \) is satisfied under \( v \).

For UCMC, the proof is similar (by slightly modifying the \( D_{P^NM} \) as we did in the proof of Theorem 2).

**5 Bucklin**

In this section, we present a polynomial-time algorithm for the UCM problem under Bucklin (a polynomial-time algorithm for the UCMC problem under Bucklin can be obtained similarly). For any alternative \( x \), any natural number \( d \), and any profile \( P \), let \( B(x, d, P) \) denote the number of times that \( x \) is ranked among the top \( d \) alternatives in \( P \). The idea behind the algorithm is as follows. Let \( d_{min} \) be the minimal depth so that \( c \) is ranked among the top \( d_{min} \) alternatives in more than half of the votes (when all of the manipulators rank \( c \) first). Then, we check if there is a way to assign the manipulators’ votes so that none of the other alternatives is ranked among the top \( d_{min} \) alternatives in more than half of the votes.

**Algorithm 1**

**Input:** A UCM instance \((\text{Bucklin}, P^{NM}, c, M), C = \{c, c_1, \ldots, c_{m−1}\}\).

1. Calculate the minimal depth \( d_{min} \) such that \( B(c, d_{min}, P^{NM}) + |M| > \frac{1}{2}(|NM| + |M|) \).