Fixed-Parameter Tractability of Integer Generalized Scoring Rules

(Extended Abstract)

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ABSTRACT

We prove that for any integer generalized scoring rules (GSRs), winner determination and computing a wide range of strategic behavior are fixed-parameter tractable (FPT) w.r.t. the number of alternatives.

Categories and Subject Descriptors

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General Terms

Algorithms, Economics, Theory

Keywords

generalized scoring rules; computational complexity; winner determination; strategic behavior

1. INTRODUCTION

An important topic in computational social choice is to characterize computational complexity of winner determination and strategic behavior including manipulation, bribery, and various procedure controls. Recently researchers have started to study the *parameterized complexity* of these problems for specific voting rules. Perhaps the most interesting type of results are showing that some NP-hard problems in social choice are *fixed-parameter tractable (FPT)*. That is, suppose that each input instance is associated with a parameter p, for any FPT problem there exists an algorithm that runs in time $h(p)|I|^{O(1)}$, where h is a function of p and $|I|$ is the input size. One natural interpretation of an FPT problem is that when the parameter is small, the problem can be computed in polynomial time (despite the constant $h(p)$ might be very large). For example, Betzler et al. [1] showed that winner determination for the Kemeny rule is FPT w.r.t. some parameters including the number of alternatives. Hemaspaandra et al. [5] showed that computing many types of strategic behavior for Schulze and ranked pairs is FPT. Previously all these problems were shown to be NP-hard.

In this paper, we prove that winner determination and computing many types of strategic behavior are FPT w.r.t. the number of alternatives for a large class of voting rules called *integer generalized scoring rules*, which include but are not limited to Kemeny, Schulze and ranked pairs. For winner determination, FPT results are usually positive. Our results suggest that winner determination for many natural voting rules can be done in polynomial time when the number of alternatives is bounded, especially for Slater. On the other hand, FPT results for strategic behavior are usually negative, suggesting that computational complexity may not be a very strong barrier against these types of strategic behavior.

Our theorems applied to specific voting rules may not be particularly exciting but we feel that the biggest selling point of our theorems is their generality. We also note that for anonymous voting rules, the total number of different manipulations is $O(n^{m!})$ (Proposition 1 in [2]), which is polynomial in the number of voters (manipulators) n when the number of alternatives m is constant. Still we think that our results provides some useful and new information (as Hemaspaandra et al. did for Schulze and ranked pairs [5]), since an algorithm with running time $O(n^{m!})$ is in the class called XP, which is a proper superset of FPT [3].

One limitation of our theorem is that it only deals with voting rules that select a single winner. Extending our results to multi-winner rules is an interesting direction for future research.

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2. PRELIMINARIES

Suppose there are n agents whose preferences are represented by *linear orders* (antisymmetric, transitive, and total binary relations) over a set of m *alternatives* $\mathcal{C} = \{c_1, \dots, c_m\}$. The set of all linear orders over $\mathcal{C}$ is denoted by $\mathcal{L}(\mathcal{C})$ and the collection of all agents' preferences is called a *preference-profile*. In this paper, a *voting rule* r is a mapping that chooses a single *winner* for any preference profile.

Generalized Scoring Rules

For any $K \in \mathbb{N}$, let $\mathcal{O}_K = \{o_1, \dots, o_K\}$, which represents the K components in $\mathbb{R}^K$. Let $\text{Pre}(\mathcal{O}_K)$ denote the set of all total preorders[1] over $\mathcal{O}_K$. For any $\vec{p} = (p_1, \ldots, p_K) \in \mathbb{R}^K$, let Order($\vec{p}$) denote the preorder $\trianglerighteq$ over $\mathcal{O}_K$ where $o_{k_1} \trianglerighteq o_{k_2}$ if and only if $p_{k_1} \geq p_{k_2}$. That is, the k_1-th component of $\vec{p}$ is at least as large as the k_2-th component of $\vec{p}$. For any preorder $\trianglerighteq$, if $o \trianglerighteq o'$ and $o' \trianglerighteq o$, then we write $o =_{\trianglerighteq} o'$. Each preorder $\trianglerighteq$ naturally induces a (partial) strict order $\triangleright$, where $o \triangleright o'$ if and only if $o \trianglerighteq o'$ and $o' \not\trianglerighteq o$.

We now recall the definition of *generalized scoring rules* [7].

Definition 1 *Given m and $K \in \mathbb{N}$, a* generalized scoring rule (GSR) *$GS(f, g)$ is defined by two functions: $f : L(\mathcal{C}) \to \mathbb{R}^K$ and $g : Pre(\mathcal{O}_K) \to \mathcal{C}$. For any preference profile P over m alternatives, we let $f(P) = \sum_{V \in P} f(V)$, and let $GS(f, g)(P) = g(Order(f(P)))$. Each $f(V)$ is called a* generalized scoring vector. *If $f(V) \in \mathbb{Z}^K$ holds for all $V \in L(\mathcal{C})$, then we call $GS(f, g)$ an* integer GSR.

Intuitively, a GSR first uses f to map the input preference-profile P to a vector $f(P)$ in $\mathbb{R}^K$, then use g to select the winner based on the *preorder* over the components of $f(P)$. Xia and Conitzer [8] showed by construction that many commonly studied voting rules using lexicographic tie-breaking are generalized scoring rules.

Proposition 1 *All positional scoring rules, Baldwin's rule, Bucklin, Copeland, Kemeny, maximin, Nanson's rule, Plurality with runoff, ranked paris, STV, Schulze, Slater[2] are integer GSRs.*

Strategic Behavior and Vote Operations

Given a GSR, in many cases a strategic individual's behavior can be modeled by its influence on $f(P)$ as follows.

Definition 2 *Given an integer GSR $GS(f, g)$ of order K, let $\Delta = [\vec{\delta}_1 \cdots \vec{\delta}_T]$ denote the set of* vote operations*, where for each $i \leq T$, $\vec{\delta}_i \in \mathbb{Z}^K$ is a column vector that represents the changes made to the generalized scoring vector by applying the i-th vote operation. For each $l \leq K$, let Δ_l denote the l-th row of Δ.*

It was shown that (constructive or destructive) manipulation, bribery, and control by adding (or deleting) voters are vote operations [7].

3. MAIN RESULTS

Theorem 1 *Then winner determination for any integer GSR[3] is FPT w.r.t. the number of alternatives.*

Proof sketch: The theorem is proved by observing that 1) given the preference profile, computing $f(P)$ takes $h(m) \cdot n$ steps for some function h, 2) computing the ordering over $\mathcal{O}_K$ takes no more than $K^2 h'(m) \cdot n$ steps, and 3) computing g takes time that only depends on K. □

Special cases of Theorem 1 have been proved for specific rules e.g. Kemeny [1] but its application to Slater is new to the best of our knowledge.

Theorem 2 *For any integer GSR that satisfies all premises in Theorem 1 and any vote operation where T and each δ_l can be computed from $l \leq T$ and m, computing whether the strategic individual can achieve her objective (constructive or destructive) is in FPT w.r.t. the number of alternatives.*

Proof sketch: We only present the idea behind the constructive case. In GSR, c_1 can be made to win by using no more than k vote operations if and only if the following ILP (which is similar to the ILP in [7]) has a feasible solution for some $\trianglerighteq$ with $g(\trianglerighteq) = c_1$ (c.f. the *winner-set certification frameworks* in [4]).

$$
\begin{array}{rl}
 & \text{Does there exist } \vec{v} \\
\text{s.t.} & \sum_{i=1}^{T} v_i \leq k \\
\forall o_{j_1} \trianglerighteq o_{j_2}: & (\Delta_{j_1} - \Delta_{j_2}) \cdot \vec{v} \geq [f(P)]_{j_2} - [f(P)]_{j_1} \\
\forall o_{j_1} \triangleright o_{j_2}: & (\Delta_{j_1} - \Delta_{j_2}) \cdot \vec{v} \geq [f(P)]_{j_2} - [f(P)]_{j_1} + 1 \\
\forall i \leq T: & v_i \geq 0 \text{ and are integers}
\end{array}
$$

Since K is a function that only depends on m and there are no more than 3^{K^2} total preorders over $\mathcal{O}_K$ to make a given alternative win, there are no more than 3^{K^2} ILPs to check. For each ILP, the number of variables is T and the size of the ILP is $h'(K)n$ for some function h'. By Lenstra's theorem [6], checking whether the ILP has a feasible solution takes time in $h(K)n$. Since K can be computed from m, the problem is FPT w.r.t. m. □

The following proposition follows after proofs in [7].

Proposition 2 *Coalitional manipulation, bribery, control by adding voters, and control by deleting voters are vote operations that satisfy the premises in Theorem 2.*

Theorem 3 *For any integer GSR that satisfies all premises in Theorem 1, constructive or destructive control by adding (or deleting) alternatives and control by partitioning alternatives are FPT w.r.t. the number of alternatives.*

The proof uses similar techniques by Hemaspaandra et al. [5, 4]. We note that the number of ways to add/delete/partition alternatives only depends on m, and by Theorem 1 winner determination is FPT w.r.t. m.

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4. REFERENCES

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[1]Reflexive, transitive, and total binary relation.

[2]See [8, 1, 4] for definitions. A lexicographic tie-breaking mechanism is used to select a single winner.

[3]For technical reasons, we assume that K can be computed from m and $f(V)$ can be computed from m and V.