11

Strategic Sequential Voting

In previous two chapters we have been focusing on designing "good" voting rules for combinatorial voting. In most of the previous work on combinatorial voting, it was assumed that the voters report their true preferences using the voting language we provide to them, when the voting language is expressive enough to do so. Now, if the voters vote on issues sequentially, one issue after another according to some ordering over issues, and are assumed to know the preferences of other voters well enough, then we can expect them to vote strategically at each step, forecasting the outcome at later steps conditional on the outcomes at earlier steps. Let us consider the following motivating example (a similar example was shown in Lacy and Niou (2000)).

Example 11.0.1. Three residents want to vote to decide whether they should build a swimming pool and/or a tennis court. There are two issue **S** and **T**. **S** can take the value of s (meaning "to build the swimming pool") or $\bar{s}$ (meaning "not to build the swimming pool"). Similarly, **T** takes a value in $\{t, \bar{t}\}$. Suppose the preferences of the three voters are, respectively, $st \succ \bar{s}t \succ s\bar{t} \succ \bar{s}\bar{t}$, $s\bar{t} \succ st \succ \bar{s}t \succ \bar{s}\bar{t}$ and

$\bar{s}t \succ \bar{s}\bar{t} \succ s\bar{t} \succ st$. Voter 2 and 3 do not rank st as their first choices, because they thought that the money could be spent on something else. Suppose the voters first vote on issue **S** then on issue **T**. Since both issues are binary, the local rule used at each step is majority (there will be no ties, because the number of voters is odd). Voter 1 is likely to reason in the following way: *if the outcome of the first step is* s*, then voters 2 and 3 will vote for* $\bar{t}$*, since they both prefer* $s\bar{t}$ *to* st*, and the final outcome will be* $s\bar{t}$*; but if the outcome of the first step is* $\bar{s}$*, then voters 2 and 3 will vote for* t*, and the final outcome will be* $\bar{s}t$*; because I prefer* $\bar{s}t$ *to* $s\bar{t}$*, I am better off voting for* $\bar{s}$*, since either it will not make any difference, or it will lead to a final outcome of* $\bar{s}t$ *instead of* $s\bar{t}$*.* If voters 2 and 3 reason in the same way, then 2 will vote for s and 3 for $\bar{s}$; hence, the result of the first step is $\bar{s}$, and then, since two voters out of three prefer $\bar{s}t$ to $\bar{s}\bar{t}$, the final outcome will be $\bar{s}t$. Note that the result is fully determined, provided that (1) it is common knowledge that voters behave strategically according to the principle we have stated informally, (2) the order in which the issues are decided, as well as the local voting rules used in all steps, are also common knowledge, and (3) voters' preferences are common knowledge. Therefore, these three assumptions allow the voters and the modeler (provided he knows as much as the voters) to predict the final outcome.

Let us take a closer look at voter 1 in Example 11.0.1. Her preferences are *separable*: she prefers s to $\bar{s}$ whatever the value of **T** is, and t to $\bar{t}$ whatever the value of **S** is. *And yet she strategically votes for* $\bar{s}$, because the outcome for **S** affects the outcome for **T**. Moreover, while voters 2 and 3 have nonseparable preferences, still, all three voters' preferences enjoy the following property: their preferences over the value of **S** are independent of the value of **T**. That is, the profile is ($\mathbf{S} > \mathbf{T}$)-legal. Hence, we can apply the sequential voting rule w.r.t. the order $\mathbf{S} > \mathbf{T}$, using majority rules for **S** and **T**. For the profile given in Example 11.0.1, the outcome of the first step under the sequential voting rule will be s (since two voters out of three prefer s to $\bar{s}$, unconditionally), and the final outcome will be $s\bar{t}$. This outcome is different from the outcome we obtain if voters behave strategically. The reason for this discrepancy is that in Lang and Xia (2009), voters are not assumed to know the others' preferences and are assumed to vote truthfully.

We have seen that even if the voters' preferences are $\mathcal{O}$-legal, voters may in fact have no incentive to vote truthfully. Consequently, existing results on multiple-election paradoxes are not directly applicable to situations where voters vote strategically.

Overview of this chapter

In this chapter, we analyze the complete-information game-theoretic model of sequential voting that we illustrated in Example 11.0.1. This model applies to any preferences that the voters may have (not just $\mathcal{O}$-legal ones), though they must be strict orders on the set of all alternatives.

We focus on voting in multi-binary-issue domains, that is, for any $i \leqslant p$, X_i must take a value in $\{0_i, 1_i\}$. This has the advantage that for each issue, we can use the majority rule as the local rule for that issue. We use a game-theoretic model to analyze outcomes that result from sequential voting. Specifically, we model the sequential voting process as a p-stage complete-information game as follows. There is an order $\mathcal{O}$ over all issues (without loss of generality, let $\mathcal{O} = X_1 > X_2 > \cdots > X_p$), which indicates the order in which these issues will be voted on. For any $1 \leqslant i \leqslant p$, in stage i, the voters vote on issue X_i simultaneously, and the majority rule is used to choose the winning value for X_i. We make the following game-theoretic assumptions: it is common knowledge that all voters are perfectly rational; the order $\mathcal{O}$ and the fact that in each step, the majority rule is used to determine the winner are common knowledge; all voters' preferences are common knowledge.

We can solve this game by a type of backward induction already illustrated in Example 11.0.1: in the last (pth) stage, only two alternatives remain (corresponding to the two possible settings of the last issue), so at this point it is a weakly dominant strategy for each voter to vote for her more preferred alternative of the two. Then, in the second-to-last ($(p-1)$th) stage, there are two possible local outcomes for the $(p-1)$th issue; for each of them, the voters can predict which alternative will finally be chosen, because they can predict what will happen in the pth stage. Thus, the $(p-1)$th stage is effectively a majority election between two alternatives, and each voter will vote for her more preferred alternative; etc. We call this procedure the *strategic sequential voting procedure (SSP)*.[1]

Given exogenously the order $\mathcal{O}$ over the issues, this game-theoretic analysis maps every profile of strict ordinal preferences to a unique outcome. Since any function from profiles of preferences to alternatives can be interpreted as a voting rule, the voting rule that corresponds to SSP is denoted by $SSP_{\mathcal{O}}$.

Lacy and Niou (2000) showed that whenever there exists a Condorcet winner, it must be the SSP winner. That is, SSP is Condorcet consistent. We will show that, unfortunately, all three major types of multiple-election paradoxes (see Section 8.1) also arise under SSP. To better present our results, we introduce a parameter which we call the *minimax satisfaction index (MSI)*. For an election with m alternatives and n voters, it is defined in the following way. For each profile, consider the highest position that the winner obtains across all input rankings of the alternatives (the ranking where this position is obtained corresponds to the most-satisfied voter); this is the *maximum satisfaction index* for this profile. Then, the minimax satisfaction index is obtained by taking the minimum over all profiles of the maximum satisfaction index. A low minimax satisfaction index means that there exists a profile in which

[1] Lacy and Niou (2000) called such a procedure *sophisticated voting* following the convention of Farquharson (1969).

the winner is ranked in low positions in all votes, thus indicating a multiple-election paradox. Our main theorem is the following.

Theorem 11.3.1 *For any $p \in \mathbb{N}$ and any $n \geqslant 2p^2 + 1$, the minimax satisfaction index of SSP when there are $m = 2^p$ alternatives and n voters is $\lfloor p/2 + 2 \rfloor$. Moreover, in the profile P that we use to prove the upper bound, the winner $SSP_{\mathcal{O}}(P)$ is Pareto-dominated by $2^p - (p+1)p/2$ alternatives.*

We note that an alternative c Pareto-dominates another alternative c' implies that c beats c' in their pairwise election. Therefore, Theorem 11.3.1 implies that the winner for SSP is an almost Condorcet loser. It follows from this theorem that SSP exhibits all three types of multiple-election paradoxes: the winner is ranked almost in the bottom in every vote, the winner is an almost Condorcet loser, and the winner is Pareto-dominated by almost every other alternative. We further show a paradox (Theorem 11.3.6) that states that there exists a profile such that for *any* order $\mathcal{O}$ over the issues, for every voter, the SSP winner w.r.t. $\mathcal{O}$ is ranked almost in the bottom position. We also show that even when the voters' preferences can be represented by CP-nets that are compatible with a common order, multiple-election paradoxes still arise.

Related work and discussion

The setting of SSP has been considered by Lacy and Niou (2000). But at a high level, our motivation, results, and conclusion are quite different from those of Lacy and Niou. We focus on the game-theoretic aspects of SSP, and we aim at examining the equilibrium outcomes in voting games. They viewed SSP as a voting rule (see Section 11.1.3 for more discussion on this point of view), and aimed at proposing solutions to aggregate non-separable profiles in combinatorial voting. They showed that SSP satisfies Condorcet consistency, but did not mention whether the other types of multiple-election paradoxes can be avoided. We, on the other hand, show that the other three types of multiple-election paradoxes still arise in SSP.[2] In terms of their conclusion, Lacy and Niou argued that SSP might not be a good solution as a voting rule, because it requires the voters to have complete information about the other voters' true preferences. The paradoxes that will be shown in this chapter, like the paradoxes we showed for Stackelberg voting games in Chapter 7, are an ordinal version of price-of-anarchy results. Consequently, these paradoxes provide more evidence that strategic behavior of the voters should be prevented, and therefore motivate the study in the next chapter, where the objective is to design strategy-proof voting rules that are computationally tractable for combinatorial domains.

More generally, SSP is closely related to *multi-stage sophisticated voting*, studied by McKelvey and Niemi (1978), Moulin (1979), and Gretlei (1983). They investigated the model where the backward induction outcomes correspond to the truthful outcomes of voting trees. Therefore, SSP is a special case of multi-stage sophisticated voting. However, their work focused on the characterization of the outcomes as the outcomes in sophisticated voting (Farquharson, 1969), and therefore did not shed much light on the quality of the equilibrium outcome. We, on the other hand, are primarily interested in the strategic outcome of the natural procedure of voting sequentially over multiple issues. Also, the relationship between sequential voting and voting trees takes a particularly natural form in the context of domains with multiple binary issues, as we will show. More importantly, we illustrate several multiple-election paradoxes for SSP, indicating that the equilibrium outcome could be extremely undesirable.

Another paper that is closely related to part of this work was written by Dutta and Sen (1993). They showed that social choice rules corresponding to binary voting trees can be implemented via backward induction via a sequential vot-

[2] In fact, those paradoxes were also discovered by Lacy and Niou in the same paper (Lacy and Niou, 2000), but they did not discuss whether they arise in SSP. See Section 8.1.

ing mechanism. This is closely related to the relationship revealed for multi-stage sophisticated voting and will also be mentioned later in this chapter, that is, an equivalence between the outcome of strategic behavior in sequential voting over multiple binary issues, and a particular type of voting tree. It should be pointed out that the sequential mechanism that Dutta and Sen consider is somewhat different from sequential voting as we consider it—in particular, in the Dutta-Sen mechanism, one voter moves at a time, and a move consists not of a vote, but rather of choosing the next player to move (or in some states, choosing the winner).

Nevertheless, the approach by Dutta and Sen and our approach are related at a high level, though they are motivated quite differently: Dutta and Sen are interested in social choice rules corresponding to voting trees, and are trying to create sequential mechanisms that implement them via backward induction. We, on the other hand again, are primarily interested in the strategic outcome of the natural mechanism for voting sequentially over multiple issues, and use voting trees merely as a useful tool for analyzing the outcome of this process.

11.1 Strategic Sequential Voting

11.1.1 Formal Definition

In this chapter, we focus on multi-binary-issue domains. That is, the multi-issue domain is composed of multiple binary issues. Sequential voting on multi-binary-issue domains can be seen as a game where in each step, the voters decide whether to vote for or against the issue under consideration after reasoning about what will happen next. We make the following assumptions.

1. All voters act strategically (in an optimal manner that will be explained later), and this is common knowledge.

2. The order in which the issues will be voted upon, as well as the local voting rules

used at the different steps (namely, majority rules), are common knowledge.

3. All voters' preferences on the set of alternatives are common knowledge.

Assumption 1 is standard in game theory. Assumption 2 merely means that the rule has been announced. Assumption 3 (complete information) is the most significant assumption. It may be interesting to consider more general settings with incomplete information, resulting in a Bayesian game. Nevertheless, because the complete-information setting is a special case of the incomplete-information setting (where the prior distribution is degenerate), in that sense, *all the worst-case negative results obtained for the complete-information setting also apply to the incomplete-information setting.* That is, the restriction to complete information only strengthens negative results. Of course, for incomplete information setting in general, we need a more elaborate model to reason about voters' strategic behavior.

Given these assumptions, the voting process can be modeled as a game that is composed of p stages where in each stage, the voters vote simultaneously on one issue. Let $\mathcal{O}$ be the order over the set of issues, which without loss of generality we assume to be $X_1 > \cdots > X_p$. Let P be the profile of preferences over $\mathcal{X}$. The game is defined as follows: for each $i \leqslant p$, in stage i the voters vote simultaneously on issue i; then, the value of X_i is determined by the majority rule (plus, in the case of an even number of voters, some tie-breaking mechanism), and this local outcome is broadcast to all voters.

We now show how to solve the game. Because of assumptions 1 to 3, at step i the voters vote strategically, by recursively figuring out what the final outcome will be if the local outcome for X_i is 0_i, and what it will be if it is 1_i. More concretely, suppose that steps 1 to $i-1$ resulted in issues $X_1, \ldots, X_{i-1}$ taking the values $d_1, \ldots, d_{i-1}$, and let $\vec{d} = (d_1, \ldots, d_{i-1})$. Suppose also that if X_i takes the value 0_i (respectively, 1_i), then, recursively, the remaining issues will take the tuple of values $\vec{a}$ (respectively,

$\vec{b}$). Then, X_i is determined by a pairwise comparison between $(\vec{d}, 0_i, \vec{a})$ and $(\vec{d}, 1_i, \vec{b})$ in the following way: if the majority of voters prefer $(\vec{d}, 0_i, \vec{a})$ over $(\vec{d}, 1_i, \vec{b})$, then X_i takes the value 0_i; in the opposite case, X_i takes the value 1_i. This process, which corresponds to the strategic behavior in the sequential election, is what we call the *strategic sequential voting (SSP)* procedure, and for any profile P, the winner with respect to the order $\mathcal{O}$ is denoted by $SSP_{\mathcal{O}}(P)$.

As we shall see later, SSP can not only be thought of as the strategic outcome of sequential voting, but also as a voting rule in its own right. The following definition and two propositions merely serve to make the game-theoretic solution concept that we use precise; a reader who is not interested in this may safely skip them.

Definition 11.1.1. *Consider a finite extensive-form game which transitions among* states*. In each nonterminal state s, all players simultaneously take an action; this joint local action profile $(a^s_1, \ldots, a^s_n)$ determines the next state s'.*[3] *Terminal states t are associated with payoffs for the players (alternatively, players have ordinal preferences over the terminal states). The current state is always common knowledge among the players.*[4]

Suppose that in every final nonterminal state s (that is, every state that has only terminal states as successors), every player i has a (weakly) dominant action a^s_i. At each final nonterminal state, its local profile of dominant actions $(a^s_1, \ldots, a^s_n)$ results in a terminal state $t(s)$ and associated payoffs. We then replace each final nonterminal state s with the terminal state $t(s)$ that its dominant-strategy profile leads to. Furthermore suppose that in the resulting smaller tree, again, in every final nonterminal state, every player has a (weakly) dominant strategy. Then, we can repeat this procedure, etc. *If we can repeat this all the way to the root of the tree,*

[3] In the extensive-form representation of the game, each state is associated with multiple nodes, because in the extensive form only one player can move at a node.

[4] Hence, the only imperfect information in the extensive form of the game is due to simultaneous moves within states.

then we say that the game is solvable by within-state dominant-strategy backward induction (WSDSBI).

We note that the backward induction in perfect-information extensive-form games is just the special case of WSDSBI where in each state only one player acts.

Proposition 11.1.2. *If a game is solvable by WSDSBI, then the solution is unique.*

Proposition 11.1.3. *The complete-information sequential voting game with binary issues (with majority as the local rule everywhere) is solvable by WSDSBI when voters have strict preferences over the alternatives.*

Both propositions are straightforward to prove and have been mentioned implicitly in Lacy and Niou (2000). We note that SSP corresponds to a particular balanced voting tree, as illustrated in Figure 11.1 for the case $p = 3$. In this voting tree, in the first round, each alternative is paired up against the alternative that differs only on the pth issue; each alternative that wins the first round is then paired up with the unique other remaining alternative that differs only on the $(p-1)$th and possibly the pth issue; etc. This bottom-up procedure corresponds exactly to the backward induction (WSDSBI) process.

Of course, there are many voting trees that do *not* correspond to an SSP election; this is easily seen by observing that there are only $p!$ different SSP elections (corresponding to the different orders of the issues), but many more voting trees. The voting tree corresponding to the order $\mathcal{O} = X_1 > \cdots > X_p$ is defined by the property that for any node v whose depth is i (where the root has depth 1), the alternative associated with any leaf in the left (respectively, right) subtree of v gives the value 0_i (respectively, 1_i) to X_i.

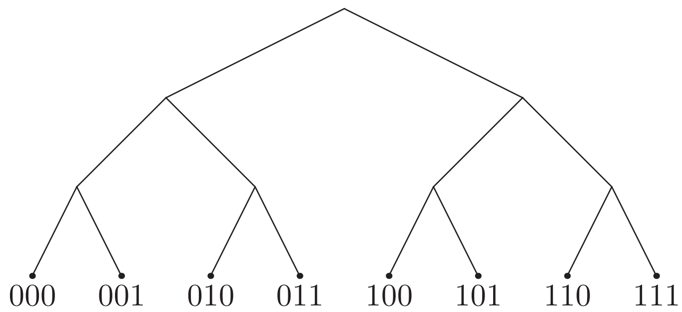

FIGURE 11.1: A voting tree that is equivalent to the strategic sequential voting procedure ($p = 3$). 000 is the abbreviation for $0_1 0_2 0_3$, etc.

11.1.2 Strategic Sequential Voting vs. Truthful Sequential Voting

We have seen on Example 11.0.1 that even when the profile P is $\mathcal{O}$-legal, $SSP_{\mathcal{O}}(P)$ can be different from $Seq_{\mathcal{O}}(Maj, \ldots, Maj)(P)$. This means that even if the profile is $\mathcal{O}$-legal, voters may be better off voting strategically than truthfully. However, $SSP_{\mathcal{O}}(P)$ and $Seq_{\mathcal{O}}(Maj, \ldots, Maj)(P)$ are guaranteed to coincide under the further restriction that P is $\mathcal{O}$-lexicographic.

Proposition 11.1.4. *For any $\mathcal{O}$-lexicographic profile P,*

$$SSP_{\mathcal{O}}(P) = Seq_{\mathcal{O}}(Maj, \ldots, Maj)(P)$$

The intuition for Proposition 11.1.4 is as follows: if P is $\mathcal{O}$-lexicographic, then, as is shown in the proof of the proposition, when voters vote strategically under sequential voting (the *Seq* process), they are best off voting according to their true preferences in each round (their preferences in each round are well-defined because voters have $\mathcal{O}$-legal preferences in this case). When voters with $\mathcal{O}$-legal preferences vote truthfully in each round under sequential voting, the outcome is $Seq_{\mathcal{O}}(Maj, \ldots, Maj)(P)$; when they vote strategically, the outcome is $SSP_{\mathcal{O}}(P)$; and so, these must be the same when preferences are $\mathcal{O}$-lexicographic.

Now, there is another interesting domain restriction under which $SSP_{\mathcal{O}}(P)$ and $Seq(Maj, \ldots, Maj)(P)$ coincide, namely when P is $inv(\mathcal{O})$- legal, where $inv(\mathcal{O}) =$

$(X_p > \ldots > X_1)$.

Proposition 11.1.5. *Let* $inv(\mathcal{O}) = X_p > \ldots > X_1$. *For any* $inv(\mathcal{O})$*-legal profile* P, $SSP_{\mathcal{O}}(P) = Seq_{inv(\mathcal{O})}(Maj, \ldots, Maj)(P)$.

As a consequence, when P is separable, it is *a fortiori* $inv(\mathcal{O})$-legal, and therefore, $SSP_{\mathcal{O}}(P) = Seq_{inv(\mathcal{O})}(Maj, \ldots, Maj)(P)$, which in turn is equal to $Seq_{\mathcal{O}}(Maj, \ldots, Maj)(P)$ and coincides with issue-by-issue voting.

Corollary 11.1.6. *If* P *is separable, then* $SSP_{\mathcal{O}}(P) = Seq_{\mathcal{O}}(Maj, \ldots, Maj)(P)$.

11.1.3 A Second Interpretation of SSP

The first interpretation of SSP (that we follow in this chapter) is the one we have discussed so far, namely, SSP consists in modeling sequential voting as a complete-information game, which allows us to analyze sequential voting on multi-issue domains from a game-theoretic point of view. For this, assumptions 1, 2, and 3 above are crucial. Under this interpretation, $SSP_{\mathcal{O}}(P)$ is a (specific kind of) equilibrium for sequential voting.

However, there is a second interpretation of SSP. It consists in seeing $SSP_{\mathcal{O}}$ as a new voting rule on multi-issue domains (which is implementable in complete-information contexts by using sequential voting).[5] This seems to be the point of view of Lacy and Niou (2000). This defines a family of voting rules (one for each order over issues), which can be applied to any profile. The family of voting rules thus defined is a distinguished subset of the family of voting trees. This interpretation does not say anything about how preferences are to be elicited; unlike in the game-theoretic interpretation, the p-step protocol does not apply here. The communication complexity of finding the outcome of $SSP_{\mathcal{O}}$ (without any complete-information assumption, of

[5] Of course, by Gibbard-Satterthwaite (Gibbard, 1973; Satterthwaite, 1975), SSP is not strategy-proof.

course)[6] is given as follows.

Proposition 11.1.7. *When the voters' preferences over alternatives are unrestricted, the communication complexity of* $SSP_{\mathcal{O}}$ *is* $\Theta(2^p \cdot n)$*.*

Proof of Proposition 11.1.7: This now follows immediately from a result in Conitzer and Sandholm (2005b), where it is established that the communication complexity for balanced voting trees is $\Theta(m \cdot n)$ for m alternatives and n voters. Since we do not place any restrictions on the preferences in the multi-issue domain in the statement of the proposition, the communication complexity is identical, and $m = 2^p$. □

The upper bound in this proposition is obtained simply by eliciting the voters' preferences for every pair of alternatives that face each other in the voting tree.

Now, Propositions 11.1.4 and 11.1.5 immediately give us conditions under which this communication complexity can be reduced. Indeed, these Propositions say that when P is $\mathcal{O}$-lexicographic or $inv(\mathcal{O})$-legal, then the SSP winner coincides with the sequential election winner in the sense of Lang and Xia (2009). Now, the sequential election winner in the sense of Lang and Xia (2009) can be found with $O(pn)$ communication, simply by having each agent vote for a value for the issue at each round. This leads immediately to the following two corollaries (to Propositions 11.1.4 and 11.1.5, respectively).

Corollary 11.1.8. *When the voters' preferences over alternatives are* $\mathcal{O}$*-lexicographic, the communication complexity of* $SSP_{\mathcal{O}}$ *is* $O(pn)$*.*

Corollary 11.1.9. *When the voters' preferences over alternatives are* $inv(\mathcal{O})$*-legal, the communication complexity of* $SSP_{\mathcal{O}}$ *is* $O(pn)$*.*

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[6] The communication complexity of a voting rule is the smallest number of bits that must be transmitted to compute the winner of that rule (i.e., taking the minimum across all correct protocols). See Conitzer and Sandholm (2005b).

11.1.4 The Winner is Sensitive to The Order over The Issues

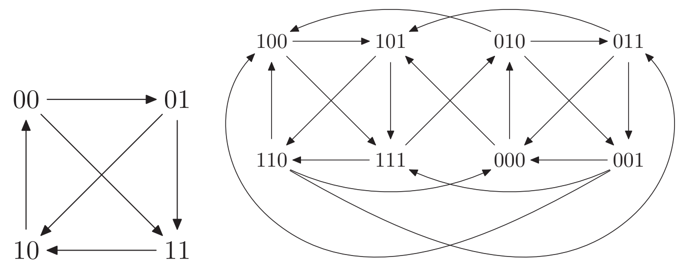
In the definition of SSP, we simply fixed the order $\mathcal{O}$ to be $X_1 > X_2 > \cdots > X_p$. A question worth addressing is, to what extent is the outcome of SSP sensitive to the variation of the order $\mathcal{O}$? More precisely, given a profile P, let $\mathrm{PW}(P) = |\{\vec{d} \in \mathcal{X} \mid \vec{d} = SSP_{\mathcal{O}'}(P)$ for some order $\mathcal{O}'\}|$. $\mathrm{PW}(P)$ is the number of different alternatives that can be made SSP winners by choosing a particular order $\mathcal{O}'$. Then, for a given number of binary issues p, we look for the maximal value of $\mathrm{PW}(P)$, for all profiles P on $\mathcal{X} = D_1 \times \ldots \times D_p$; we denote this number by $\mathrm{MW}(p)$.

A first observation is that there are $p!$ different choices for $\mathcal{O}'$. Therefore, a trivial upper bound on $\mathrm{MW}(p)$ is $p!$. Since there are 2^p alternatives, the $p!$ upper bound is only interesting when $p! < 2^p$, that is, $p \leqslant 3$. Example 11.1.10 shows that when $p = 2$ or $p = 3$, this trivial upper bound is actually tight, i.e. $\mathrm{MW}(2) = 2!$ and $\mathrm{MW}(3) = 3!$: there exists a profile such that by changing the order over the issues, all $p!$ different alternatives can be made winners. Due to McGarvey's Theorem (see Lemma 2.2.3), any complete and asymmetric directed graph G over the alternatives corresponds to the majority graph of some profile (we recall that the majority graph of a profile P is the directed graph whose vertices are the alternatives and containing an edge from c to c' if and only if a majority of voters in P prefer c to c'). Therefore, in the example, we only show the majority graph instead of explicitly constructing the whole profile.

Example 11.1.10. *The majority graphs for $p = 2$ and $p = 3$ are shown in Figure 11.2. Let P (respectively, P') denote an arbitrary profile whose majority graph is the same as Figure 11.2(a) (respectively, Figure 11.2(b)). It is not hard to verify that $SSP_{X_1>X_2}(P) = 00$ and $SSP_{X_2>X_1}(P) = 01$. For P', the value of $SSP_{\mathcal{O}'}(P')$ for the six possible orders is shown on Table 11.1. Note that $2! = 2$ and $3! = 6$. It follows that when $p = 2$ or $p = 3$, there exists a profile for which the SSP winners* *w.r.t. different orders over the issues are all different from each other.*

Table 11.1: The SSP winners for P' w.r.t. different orders over the issues.

The order	$X_1 > X_2 > X_3$	$X_1 > X_3 > X_2$	$X_2 > X_1 > X_3$
SSP winner	010	011	001
The order	$X_2 > X_3 > X_1$	$X_3 > X_1 > X_2$	$X_3 > X_2 > X_1$
SSP winner	100	110	101



(a) (b)

FIGURE 11.2: The majority graphs for $p = 2$ and $p = 3$.

In Figure 11.2, (a) is the majority graph for $p = 2$. (b) is the majority graph for $p = 3$, where four edges are not shown in the graph: $100 \to 000$, $101 \to 001$, $110 \to 010$, *and* $111 \to 011$. *The directions of the other edges are defined arbitrarily.* 000 *is the abbreviation for* $0_1 0_2 0_3$, *etc.*

When $p \geqslant 4$, $p! > 2^p$. However, it is not immediately clear whether $\text{MW}(p) = 2^p$ or not, i.e., whether each of the 2^p alternatives can be made a winner by changing the order over the issues. The next theorem shows that this can actually be done, that is, $\text{MW}(p) = 2^p$.

Theorem 11.1.11. *For any $p \geqslant 4$ and any $n \geqslant 142 + 4p$, there exists an n-profile P such that for every alternative $\vec{d}$, there exists an order $\mathcal{O}'$ over $\mathcal{I}$ such that $SSP_{\mathcal{O}'}(P) = \vec{d}$.*

Proof. We prove the theorem by induction on the number of issues p. Surprisingly, the hardest part in the inductive proof is the base case: when we first show how to construct a desirable majority graph $\mathcal{M}$ for $p = 4$, then we show how to construct a n-profile that corresponds to $\mathcal{M}$.

To define $\mathcal{M}$ when $p = 4$, we first define a majority graph $\mathcal{M}_3$ over $\mathcal{X}_3 = D_2 \times D_3 \times D_4$. Let $\mathcal{M}'$ denote the majority graph defined in Example 11.1.10 when $p = 3$. We note that $\mathcal{M}'$ is defined over $D_1 \times D_2 \times D_3$. The structure of $\mathcal{M}_3$ is exactly the same as $\mathcal{M}'$, except that $\mathcal{M}_3$ is defined over $D_2 \times D_3 \times D_4$. Formally, let $h_1 : D_1 \to D_2$ be a mapping such that $h_1(0_1) = 0_2$ and $h_1(1_1) = 1_2$; let $h_2 : D_2 \to D_3$ be a mapping such that $h_2(0_2) = 0_3$ and $h_2(1_2) = 1_3$; and let $h_3 : D_3 \to D_4$ be a mapping such that $h_3(0_3) = 0_4$ and $h_3(1_3) = 1_4$. Let $h : D_1 \times D_2 \times D_3 \to D_2 \times D_3 \times D_4$ be a mapping such that for any $(a_2, a_3, a_4) \in \{0, 1\}$, $h(a_1, a_2, a_3) = (h_1(a_1), h_2(a_2), h_3(a_3))$. For example, $h(0_1 1_2 0_3) = 0_2 1_3 0_4$. Then, we let $\mathcal{M}_3 = h(\mathcal{M}')$.

For any $\vec{a} = (a_2, a_3, a_4) \in \mathcal{X}_3$, let $f(\vec{a}) = (1_1, \vec{a})$ and let $g(\vec{a}) = (0_1, \overline{a_2}, a_3, a_4)$. That is, f concatenates 1_1 and $\vec{a}$, and g flips the first two components of $f(\vec{a})$. For example, $f(0_2 0_3 0_4) = 1_1 0_2 0_3 0_4$ and $g(0_2 0_3 0_4) = 1_1 1_2 0_3 0_4$. We define $\mathcal{M}$ as follows.

(1) The subgraph of $\mathcal{M}$ over $\{1_1\} \times \mathcal{X}_3$ is $f(\mathcal{M}_3)$. That is, for any $\vec{a}, \vec{b} \in \mathcal{X}_3$, if $\vec{a} \to \vec{b}$ in $\mathcal{M}'$, then $f(\vec{a}) \to f(\vec{b})$ in $\mathcal{M}$.

(2) The subgraph of $\mathcal{M}$ over $\{0_1\} \times \mathcal{X}_3$ is $g(\mathcal{M}_3)$.

(3) For any $\vec{a} \in \mathcal{X}_3$, we have $(1_1, \vec{a}) \to (0_1, \vec{a})$. For any $\vec{a} \in \mathcal{X}_3$ and $\vec{a} \neq 111$, we have $g(\vec{a}) \to f(\vec{a})$.

(4) We then add the following edges to $\mathcal{M}$. $0100 \to 1110$, $1000 \to 0010$, $1101 \to 0111$, $0001 \to 1011$, $1101 \to 0100$, $1000 \to 0001$, $0001 \to 1101$, $0100 \to 1000$, $1111 \to 0110$, $1100 \to 0101$, $0011 \to 1010$, $1001 \to 0000$, $1111 \to 0011$, $0011 \to 1100$, $0011 \to 1001$, $1111 \to 0000$.

(5) Any other edge that is not defined above is defined arbitrarily.

Let P be an arbitrary profile whose majority graph satisfies conditions (1) through (4) above. We make the following observations.

- If X_1 is the first issue in $\mathcal{O}'$, then the first component of $\mathrm{SSP}_{\mathcal{O}'}(P)$ is 1_1. Moreover, every alternative whose first component is 1_1 (except 1111 and 1000) can be made to win by changing the order of X_2, X_3, X_4.

- If X_1 is the last issue in $\mathcal{O}'$, then the first component of $\mathrm{SSP}_{\mathcal{O}'}(P)$ is 0_1. Moreover, every alternative whose first component is 0_1 (except 0011 and 0100) can be made to win by changing the order of X_2, X_3, X_4.

- Let $\mathcal{O}' = X_3 > X_1 > X_2 > X_4$, we have $\mathrm{SSP}_{\mathcal{O}'}(P) = 0100$; let $\mathcal{O}' = X_3 > X_1 > X_4 > X_2$, we have $\mathrm{SSP}_{\mathcal{O}'}(P) = 1000$; let $\mathcal{O}' = X_4 > X_1 > X_3 > X_2$, we have $\mathrm{SSP}_{\mathcal{O}'}(P) = 0011$; let $\mathcal{O}' = X_2 > X_4 > X_1 > X_3$, we have $\mathrm{SSP}_{\mathcal{O}'}(P) = 1111$.

In summary, every alternative is a winner of SSP w.r.t. at least one order over the issues. The reader can also check out the java program online at `http://www.cs.duke.edu/~lxia/Files/SSP.zip`, to verify the correctness of such a construction. We notice that conditions (1) through (4) impose 79 constraints on pairwise comparisons. Therefore, using McGarvey's trick (Lemma 2.2.3), for any $n \geqslant 2 \times 79 = 158$, we can construct an n-profile whose majority graph satisfies conditions (1) through (4). This means that the theorem holds for $p = 4$.

Now, suppose that the theorem holds for $p = p'$. Let $P = (V_1, \ldots, V_n)$ be an n-profile over $\mathcal{X}' = D_2 \times \ldots \times D_{p'+1}$ such that $n \geqslant 142 + 4p'$ and each alternative in $\mathcal{X}'$ can be made to win in SSP by changing the order over $X_2, \ldots, X_{p'+1}$. Let $\mathcal{X} = D_1 \times \cdots \times D_{p'+1}$. Let $f : \mathcal{X}' \to \mathcal{X}$ be the mapping defined as follows. For any $\vec{a} \in \mathcal{X}'$, $f(\vec{a}) = (1_1, \vec{a})$. That is, for any $\vec{a} \in \mathcal{X}'$, f concatenates 1_1 and $\vec{a}$. Let $g : \mathcal{X}' \to \mathcal{X}$ be the mapping defined as follows. For any $\vec{a} = (a_2, \ldots, a_{p'+1}) \in \mathcal{X}'$,

$g(\vec{a}) = (0_1, \overline{a_2}, a_3, \ldots, a_{p'+1})$. That is, for any $\vec{a} \in \mathcal{X}'$, g flips the first two components of $f(\vec{a})$. Next, we define an $(n+4)$-profile $P' = (V'_1, \ldots, V'_{n+4})$ as follows.

For any $i \leqslant 2\lfloor (n-1)/2 \rfloor$, we let $V'_i = \begin{cases} f(V_i) \succ g(V_i) & \text{if } i \text{ is odd} \\ g(V_i) \succ f(V_i) & \text{if } i \text{ is even} \end{cases}$. For any $2\lfloor (n-1)/2 \rfloor + 1 \leqslant i \leqslant n$, we let $V'_i = [f(V_i) \succ g(V_i)]$. For any $j \leqslant 4$, we let

$$V'_{n+j} = \begin{cases} \begin{array}{c} g(0_2 \ldots 0_{p+1}) \succ f(0_2 \ldots 0_{p+1}) \succ g(0_2 \ldots 0_p 1_{p+1}) \\ \succ f(0_2 \ldots 0_p 1_{p+1}) \succ g(1_2 \ldots 1_{p+1}) \succ f(1_2 \ldots 1_{p+1}) \end{array} & \text{if } j \text{ is odd} \\ \begin{array}{c} g(1_2 \ldots 1_{p+1}) \succ f(1_2 \ldots 1_{p+1}) \succ g(1_2 \ldots 1_p 0_{p+1}) \\ \succ f(1_2 \ldots 1_p 0_{p+1}) \succ g(0_2 \ldots 0_{p+1}) \succ f(0_2 \ldots 0_{p+1}) \end{array} & \text{if } j \text{ is even} \end{cases}$$

For any pair of alternatives c, c', and any profile P^*, we let $D_{P*}(c, c')$ denote the number of times that c is preferred to c', minus the number of times c' is preferred to c, both in the profile P^*. That is, $D_{P*}(c, c') > 0$ if and only if c beats c' in their pairwise election. We make the following observations on P'.

- For any $\vec{a} \in \mathcal{X}'$, $D_{P'}(f(\vec{a}), g(\vec{a})) > 0$ and $D_{P'}((1_1, \vec{a}), (0_1, \vec{a})) > 0$.
- For any $\vec{a}, \vec{b} \in \mathcal{X}'$ (with $\vec{a} \neq \vec{b}$), $D_{P'}(f(\vec{a}), f(\vec{b})) > 0$ if and only if $D_P(\vec{a}, \vec{b}) > 0$; $D_{P'}(g(\vec{a}), g(\vec{b})) > 0$ if and only if $D_P(\vec{a}, \vec{b}) > 0$.

It follows that for any order $\mathcal{O}'$ over $\{X_2, \ldots, X_{p+1}\}$, we have $\text{SSP}_{[X_1 \succ \mathcal{O}']}(P') = f(SSP_{\mathcal{O}'}(P))$ (because after voting on issue X_1, all alternatives whose first component is 0_1 are eliminated, then it reduces to SSP over $\mathcal{X}'$); we also have that $\text{SSP}_{[\mathcal{O}' \succ X_1]}(P') = g(SSP_{\mathcal{O}'}(P))$ (because in the last round, the two competing alternatives are considering are $f(SSP_{\mathcal{O}'}(P))$ and $g(SSP_{\mathcal{O}'}(P))$, and the majority of voters prefer the latter). We recall that each alternative in $\mathcal{X}'$ can be made to win w.r.t. an order O' over $\{X_2, \ldots, X_{p'+1}\}$. It follows that each alternative in $\mathcal{X}$ can also be made to win w.r.t. an order over $\{X_1, \ldots, X_{p'+1}\}$, which means that the theorem holds for $p = p' + 1$. Therefore, the theorem holds for any $p \geqslant 4$. □

11.2 Minimax Satisfaction Index

In the rest of this chapter, we will show that strategic sequential voting on multi-issue domains is prone to paradoxes that are almost as severe as previously studied multiple-election paradoxes under models that are not game-theoretic (Brams et al., 1998; Lacy and Niou, 2000).[7] To facilitate the presentation of these results, we define an index that is intended to measure one aspect of the quality of a voting rule, called the *minimax satisfaction index*.

Definition 11.2.1. *For any voting rule r, the* minimax satisfaction index (MSI) *of r is defined as*

$$MSI_r(m,n) = \min_{P \in L(\mathcal{X})^n} \max_{V \in P} \big(m + 1 - rank_V(r(P))\big)$$

where m is the number of alternatives, n is the number of voters, and $rank_V(r(P))$ is the position of $r(P)$ in vote V.

We note that in this chapter $m = 2^p$, where p is the number of issues. The MSI of a voting rule is not the final word on it. For example, the MSI for dictatorships is m, the maximum possible value, which is not to say that dictatorships are desirable. However, if the MSI of a voting rule is low, then this implies the existence of a paradox for it, namely, a profile that results in a winner that makes all voters unhappy.

The third type of multiple-election paradoxes (see Section 8.1) implicitly refer to such an index. We recall that the third type of multiple-election paradoxes state that if voters vote on issues separately and optimistically, then there exists a profile such that in each vote, the winner is ranked near the bottom; therefore this rule has a very low MSI.

[7] Even though Lacy and Niou (2000) have studied SSP, they actually did not examine whether there are any multiple-election paradoxes in SSP.

11.3 Multiple-Election Paradoxes for Strategic Sequential Voting

In this section, we show that over multi-binary-issue domains, for any natural number n that is sufficiently large (we will specify the number in our theorems), there exists an n-profile P such that $SSP_{\mathcal{O}}(P)$ is ranked almost in the bottom position in each vote in P. That is, the minimax satisfaction index is extremely low for the strategic sequential voting procedure.

We first calculate the MSI for $SSP_{\mathcal{O}}$ when the winner does not depend on the tie-breaking mechanism. That is, either n is odd, or n is even and there is never a tie in any stage of running the election sequentially. This is our main multiple-election paradox result.

Theorem 11.3.1. *For any $p \in \mathbb{N}$ ($p \geqslant 2$) and any $n \geqslant 2p^2 + 1$, $MSI_{SSP_{\mathcal{O}}}(m, n) = \lfloor p/2 + 2 \rfloor$.*[8] *Moreover, in the profile P that we use to prove the upper bound, the winner $SSP_{\mathcal{O}}(P)$ is Pareto-dominated by $2^p - (p+1)p/2$ alternatives.*

Proof of Theorem 11.3.1: The upper bound on $MSI_{SSP_{\mathcal{O}}}(m, n)$ is constructive, that is, we explicitly construct a paradox.

For any n-profile $P = (V_1, \ldots, V_n)$, we define the mapping $f_P : \mathcal{X} \to \mathbb{N}^n$ as follows: for any $c \in \mathcal{X}$, $f_P(c) = (h_1, \ldots, h_n)$ such that for any $i \leqslant n$, h_i is the number of alternatives that are ranked below c in V_i. For any $l \leqslant p$, we denote $\mathcal{X}_l = D_l \times \cdots \times D_p$ and $\mathcal{O}_l = X_l > X_{l+1} > \cdots > X_p$. For any vector $\vec{h} = (h_1, \ldots, h_n)$ and any $l \leqslant p$, we say that $\vec{h}$ is *realizable* over $\mathcal{X}_l$ (through a balanced binary tree) if there exists a profile $P_l = (V_1, \ldots, V_n)$ over $\mathcal{X}_l$ such that $f_{P_l}(SSP_{\mathcal{O}_l}(P_l)) = \vec{h}$. We first prove the following lemma.

[8] If n is even, then to prove $MSI_{SSP_{\mathcal{O}}}(m, n) \geqslant \lfloor p/2 + 2 \rfloor$, we restrict attention to profiles without ties.

Lemma 11.3.2. *For any l such that $1 \leqslant l < p$,*

$$\vec{h}_* = (\underbrace{0, \ldots, 0}_{\lfloor n/2 \rfloor - p + l}, \underbrace{1, \ldots, 1}_{p-l+1}, \underbrace{2^{p-l+1} - 1, \ldots, 2^{p-l+1} - 1}_{\lceil n/2 \rceil - 1})$$

is realizable over $\mathcal{X}_l$.

Proof of Lemma 11.3.2: We prove that there exists an n-profile P_l over $\mathcal{X}_l$ such that $SSP_{\mathcal{O}_l}(P_l) = 1_l \cdots 1_p$ and $\vec{h}_*$ is realized by P_l. For any $1 \leqslant i \leqslant p - l + 1$, we let $\vec{b}_i = 1_l \cdots 1_{p-i} 0_{p+1-i} 1_{p+2-i} \cdots 1_p$. That is, $\vec{b}_i$ is obtained from $1_l \cdots 1_p$ by flipping the value of X_{p+1-i}. We obtain $P_l = (V_1, \ldots, V_n)$ in the following steps.

1. Let $W_1, \ldots, W_n$ be null partial orders over $\mathcal{X}_l$. That is, for any $i \leqslant n$, the preference relation W_i is empty.

2. For any $j \leqslant \lfloor n/2 \rfloor - p + l$, we put $1_l \cdots 1_p$ in the bottom position in W_j; we put $\{\vec{b}_1, \ldots, \vec{b}_{p-l+1}\}$ in the top positions in W_j.

3. For any j with $\lfloor n/2 \rfloor + 2 \leqslant j \leqslant n$, we put $1_l \cdots 1_p$ in the top position of W_j, and we put $\{\vec{b}_1, \ldots, \vec{b}_{p-l+1}\}$ in the positions directly below the top.

4. For j with $\lfloor n/2 \rfloor - p + l + 1 \leqslant j \leqslant \lfloor n/2 \rfloor + 1$, we define preferences as follows. For any $i \leqslant p - l + 1$, in $W_{\lfloor n/2 \rfloor - p + l + i}$, we put $\vec{b}_i$ in the bottom position, $1_l \cdots 1_p$ in the second position from the bottom, and all the remaining b_j (with $j \neq i$) at the very top.

5. Finally, we complete the profile arbitrarily: for any $j \leqslant n$, we let V_j be an arbitrary extension of W_j.

Let $P_l = (V_1, \ldots, V_n)$. We note that for any $i \leqslant p - l + 1$, $\vec{b}_i$ beats any alternative in $\mathcal{X}_l \setminus \{1_l \cdots 1_p, \vec{b}_1, \ldots, \vec{b}_{p-l+1}\}$ in pairwise elections. Therefore, for any $i \leqslant p - l + 1$, the ith alternative that meets $1_l \cdots 1_p$ is $\vec{b}_i$, which loses to $1_l \cdots 1_p$ (just barely). It follows that $1_l \cdots 1_p$ is the winner, and it is easy to check that $f_{P_l}(1_l \cdots 1_p) = \vec{h}_*$. This completes the proof of the lemma. $\square$

Because the majority rule is anonymous, for any permutation π over $1, \ldots, n$ and any $l < p$, if $(h_1, \ldots, h_n)$ is realizable over $\mathcal{X}_l$, then $(h_{\pi(1)}, \ldots, h_{\pi(n)})$ is also realizable over $\mathcal{X}_l$. For any $k \in \mathbb{N}$, we define $H_k = \{\vec{h} \in \{0,1\}^n : \sum_{j \leqslant n} h_j \geqslant k\}$. That is, H_k is composed of all n-dimensional binary vectors in each of which at least k components are 1. We next show a lemma to derive a realizable vector over $\mathcal{X}_{l-1}$ from two realizable vectors over $\mathcal{X}_l$.

Lemma 11.3.3. *Let $l < p$, and let $\vec{h}_1$, $\vec{h}_2$ be vectors that are realizable over $\mathcal{X}_l$. For any $\vec{h} \in H_{\lfloor n/2 \rfloor + 1}$, $\vec{h}_1 + (\vec{h}_2 + \vec{1}) \cdot \vec{h}$ is realizable over $\mathcal{X}_{l-1}$, where $\vec{1} = (1, \ldots, 1)$, and for any $\vec{a} = (a_1, \ldots, a_n)$ and any $\vec{b} = (b_1, \ldots, b_n)$, we have $\vec{a} \cdot \vec{b} = (a_1 b_1, \ldots, a_n b_n)$.*

Proof of Lemma 11.3.3: Without loss of generality, we prove the lemma for $\vec{h} = (\underbrace{0, \ldots, 0}_{\lceil n/2 \rceil - 1}, \underbrace{1, \ldots, 1}_{\lfloor n/2 \rfloor + 1})$. Let P_1, P_2 be two profiles over $\mathcal{X}_l$, each of which is composed of n votes, such that $f(P_1) = \vec{h}_1$ and $f(P_2) = \vec{h}_2$. Let $P_1 = (V_1^1, \ldots, V_n^1)$, $P_2 = (V_1^2, \ldots, V_n^2)$, $\vec{a} = SSP_{\mathcal{O}_l}(P_1)$, $\vec{b} = SSP_{\mathcal{O}_l}(P_2)$. We define a profile $P = (V_1, \ldots, V_n)$ over $\mathcal{X}_{l-1}$ as follows.

1. Let $W_1, \ldots, W_n$ be n null partial orders over $\mathcal{X}_{l-1}$.

2. For any $j \leqslant n$ and any $\vec{e}_1, \vec{e}_2 \in \mathcal{X}_l$, we let $(1_{l-1}, \vec{e}_1) \succ_{W_j} (1_{l-1}, \vec{e}_2)$ if $\vec{e}_1 \succ_{V_j^1} \vec{e}_2$; and we let $(0_{l-1}, \vec{e}_1) \succ_{W_j} (0_{l-1}, \vec{e}_2)$ if $\vec{e}_1 \succ_{V_j^2} \vec{e}_2$.

3. For any $\lceil n/2 \rceil \leqslant j \leqslant n$, we let $(1_{l-1}, \vec{a}) \succ_{W_j} (0_{l-1}, \vec{b})$.

4. Finally, we complete the profile arbitrarily: for any $j \leqslant n$, we let V_j be an (arbitrary) extension of W_j such that $(1_{l-1}, \vec{a})$ is ranked as low as possible.

We note that $(1_{l-1}, \vec{a})$ is the winner of the subtree in which $X_{l-1} = 1_{l-1}$, $(0_{l-1}, \vec{b})$ is the winner of the subtree in which $X_{l-1} = 0_{l_1}$, and $(1_{l-1}, \vec{a})$ beats $(0_{l-1}, \vec{b})$ in their pairwise election (because the votes from $\lceil n/2 \rceil$ to n rank $(1_{l-1}, \vec{a})$ above $(0_{l-1}, \vec{b})$). Therefore, $SSP_{\mathcal{O}_{l-1}}(P) = (1_{l-1}, \vec{a})$.

Finally, we have that $f_P((1_{l-1}, \vec{a})) = \vec{h}_1 + (\vec{h}_2 + \vec{1}) \cdot \vec{h}$. This is because $(1_{l-1}, \vec{a})$ is ranked just as low as in the profile P_1 for voters 1 through $\lceil n/2 \rceil - 1$; for any voter j with $\lceil n/2 \rceil \leqslant j \leqslant n$, additionally, $(0_{l-1}, \vec{b})$ needs to be placed below $(1_{l-1}, \vec{a})$, which implies that also, all the alternatives $(0_{l-1}, \vec{b}')$ for which j ranked $\vec{b}'$ below $\vec{b}$ in P_2 must be below $(1_{l-1}, \vec{a})$ in j's new vote in P. This completes the proof of the lemma.
□

Now we are ready to prove the main part of the theorem. It suffices to prove that for any $n \geqslant 2p^2 + 1$, there exists a vector $\vec{h}_p \in \mathbb{N}^n$ such that each component of $\vec{h}_p$ is no more than $\lfloor p/2 + 1 \rfloor$, and $\vec{h}_p$ is realizable over $\mathcal{X}$. We first prove the theorem for the case in which n is odd. We show the construction by induction in the proof of the following lemma.

Lemma 11.3.4. *Let n be odd. For any $l' < p$ (such that l' is odd),*

$$\vec{h}_{l'} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{\lceil n/2 \rceil - (l'^2+1)/2}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{\lfloor n/2 \rfloor + (l'^2+1)/2})$$

is realizable over $\mathcal{X}_{p-l'+1}$, and if $l' < p$, then

$$\vec{h}_{l'+1} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{l'+1}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{n-(l'+5)(l'+1)/2}, \underbrace{\lceil l'/2 \rceil + 1, \ldots, \lceil l'/2 \rceil + 1}_{(l'+3)(l'+1)/2})$$

is realizable over $\mathcal{X}_{p-l'}$.

Proof of Lemma 11.3.4: The base case in which $l' = 1$ corresponds to a single-issue majority election over two alternatives, where $\lceil n/2 \rceil - 1$ voters vote for one alternative, and $\lfloor n/2 \rfloor + 1$ vote for the other, so that only the latter get their preferred alternative.

Now, suppose the claim holds for some $l' \leqslant p - 2$; we next show that the claim also holds for $l' + 2$. To this end, we apply Lemma 11.3.3 twice. Let $l = p - l' + 1$.

First, let $\vec{h}_* = (\underbrace{1, \ldots, 1}_{l'}, \underbrace{2^{l'} - 1, \ldots, 2^{l'} - 1}_{l'+1}, \underbrace{0, \ldots, 0}_{\lfloor n/2 \rfloor - l' + 1}, \underbrace{2^{l'} - 1, \ldots, 2^{l'} - 1}_{\lceil n/2 \rceil - l' - 2})$

By Lemma 11.3.2, $\vec{h}_*$ is realizable over $\mathcal{X}_l$ (via a permutation of the voters). Let $\vec{h} = (\underbrace{1,\ldots,1}_{l'}, \underbrace{0,\ldots,0}_{l'+1}, \underbrace{1,\ldots,1}_{\lfloor n/2\rfloor - l'+1}, \underbrace{0,\ldots,0}_{\lceil n/2\rceil - l'-2})$.

Then, by Lemma 11.3.3, $\vec{h}_{l'} + (\vec{h}_* + \vec{1}) \cdot \vec{h}$ is realizable over $\mathcal{X}_{l-1}$. We have the following calculation.

$$
\begin{aligned}
&\vec{h}_{l'} + (\vec{h}_* + \vec{1}) \cdot \vec{h} \\
=&(\underbrace{\lfloor l'/2\rfloor + 1, \ldots, \lceil l'/2\rceil + 1}_{l'}, \\
&\underbrace{\lfloor l'/2\rfloor, \ldots, \lfloor l'/2\rfloor}_{l'+1}, \underbrace{\lceil l'/2\rceil, \ldots, \lceil l'/2\rceil}_{\lceil n/2\rceil - (l'+3)(l'+1)/2}, \\
&\underbrace{\lceil l'/2\rceil + 1, \ldots, \lceil l'/2\rceil + 1}_{(l'+1)^2/2+1}, \underbrace{\lceil l'/2\rceil, \ldots, \lceil l'/2\rceil}_{\lfloor n/2\rfloor - l'-1})
\end{aligned}
$$

The partition of the set of voters into these five groups uses the fact that $n \geqslant 2p^2 + 1$ implies $\lceil n/2\rceil - (l'+3)(l'+1)/2 \geqslant 0$. After permuting the voters in this vector, we obtain the following vector which is realizable over $\mathcal{X}_{l-1}$:

$$\vec{h}_{l'+1} = (\underbrace{\lfloor l'/2\rfloor, \ldots, \lfloor l'/2\rfloor}_{l'+1}, \underbrace{\lceil l'/2\rceil, \ldots, \lceil l'/2\rceil}_{n-(l'+5)(l'+1)/2}, \underbrace{\lceil l'/2\rceil + 1, \ldots, \lceil l'/2\rceil + 1}_{(l'+3)(l'+1)/2})$$

We next let $\vec{h}' = (\underbrace{1,\ldots,1}_{\lfloor n/2\rfloor+1}, \underbrace{0,\ldots,0}_{\lceil n/2\rceil-1})$ and

$$\vec{h}'_* = (\underbrace{1,\ldots,1}_{l'+1}, \underbrace{0,\ldots,0}_{\lfloor n/2\rfloor - l'}, \underbrace{2^{l'+1}-1, \ldots, 2^{l'+1}-1}_{\lceil n/2\rceil - 1})$$

By Lemma 11.3.2, the latter is realizable over $\mathcal{X}_{l-1}$. Thus, by Lemma 11.3.3, $\vec{h}_{l'+1} + (\vec{h}'_* + \vec{1}) \cdot \vec{h}'$ is realizable over $\mathcal{X}_{l-2}$. Through a permutation over the voters, we obtain the desired vector:

$$\vec{h}_{l'+2} = (\underbrace{\lfloor l'/2\rfloor + 1, \ldots, \lfloor l'/2\rfloor + 1}_{\lceil n/2\rceil - (l'+2)(l'+1)/2 - 1}, \underbrace{\lceil l'/2\rceil + 1, \ldots, \lceil l'/2\rceil + 1}_{\lfloor n/2\rfloor + (l'+2)(l'+1)/2+1})$$

which is realizable over $\mathcal{X}_{l-2}$. Therefore, the claim holds for $l'+2$. This completes the proof of the lemma. □

If p is odd, from Lemma 11.3.4 we know that the theorem is true, by setting $l' = p$. If p is even, then we first set $l' = p-1$; then, the maximum component of $\vec{h}_{l'+1}$ is $\lceil l'/2 \rceil + 1 = \lceil (p-1)/2 \rceil + 1 = p/2+1$. Thus we have proved the upper bound in the theorem when n is odd.

When n is even, we have the following lemma (the proof is similar to the proof of Lemma 11.3.4, so we omitted its proof).

Lemma 11.3.5. *Let n be even. For any $l' < p$ (such that l' is odd),*

$$\vec{h}_{l'} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{n/2-(l'^2-l'+1)/2}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{n/2+(l'^2-l'+1)/2})$$

is realizable over $\mathcal{X}_{p-l'+1}$, and if $l'+1 \leqslant p$, then

$$\vec{h}_{l'+1} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{l'+1}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{n-1-(l'+4)(l'+1)/2}, \underbrace{\lceil l'/2 \rceil + 1, \ldots, \lceil l'/2 \rceil + 1}_{(l'+2)(l'+1)/2+1})$$

is realizable over $\mathcal{X}_{p-l'}$.

The upper bound in the theorem when n is even follows from Lemma 11.3.5. Moreover, we note that in the step from l' to $l'+1$ (respectively, from $l'+1$ to $l'+2$), no more than l' new alternatives are ranked lower than the winner in the profile that realizes $\vec{h}_{l'+1}$ (respectively, $\vec{h}_{l'+2}$). It follows that in the profile that realizes $\vec{h}_{l'+1}$ (respectively, $\vec{h}_{l'+2}$) in Lemma 11.3.4 or Lemma 11.3.5, the number of alternatives that are ranked lower than the winner by at least one voter is no more than $(l'+1)l'/2+l'+1 = (l'+1)(l'+2)/2$ (respectively, $(l'+2)(l'+3)/2$), which equals $(p+1)p/2$ if $l'+1 = p$ (respectively, $(p+1)p/2$ if $l'+2 = p$). Therefore, in the profile that we use to obtain the upper bound, the winner under $SSP_{\mathcal{O}}$ is Pareto-dominated by $2^p - (p+1)p/2$ alternatives.

Finally, we show that $\lfloor p/2 + 2 \rfloor$ is a lower bound on $MSI_{SSP_{\mathcal{O}}}(m, n)$. Let P be an n-profile; let $SSP_{\mathcal{O}}(P) = \vec{a}$, and let $\vec{b}_1, \ldots, \vec{b}_p$ be the alternatives that $\vec{a}$ defeats in pairwise elections in rounds $1, \ldots, p$. It follows that in round j, more than half of the voters prefer $\vec{a}$ to $\vec{b}_j$, because we assume that there are no ties in the election. Therefore, summing over all votes, there are at least $p \times (\lfloor n/2 \rfloor + 1)$ occasions where $\vec{a}$ is preferred to one of $\vec{b}_1, \ldots, \vec{b}_p$. It follows that there exists some $V \in P$ in which $\vec{a}$ is ranked higher than at least $\lceil p \times (\lfloor n/2 \rfloor + 1)/n \rceil \geqslant \lfloor p/2 + 1 \rfloor$ of the alternatives $\vec{b}_1, \ldots, \vec{b}_p$. Thus $MSI_{SSP_{\mathcal{O}}}(m, n) \geqslant \lfloor p/2 + 2 \rfloor$.

(End of proof for Theorem 11.3.1.) □

We note that the number of alternatives is $m = 2^p$. Therefore, $\lfloor p/2 + 2 \rfloor$ is exponentially smaller than the number of alternatives, which means that there exists a profile for which every voter ranks the winner very close to the bottom. Moreover, $(p + 1)p/2$ is still exponentially smaller than 2^p, which means that the winner is Pareto-dominated by almost every other alternative.

Naturally, we wish to avoid such paradoxes. One may wonder whether the paradox occurs only if the ordering of the issues is particularly unfortunate with respect to the preferences of the voters. If not, then, for example, perhaps a good approach is to randomly choose the ordering of the issues.[9] Unfortunately, our next result shows that we can construct a single profile that results in a paradox for *all* orderings of the issues. While it works for all orders, the result is otherwise somewhat weaker than Theorem 11.3.1: it does not show a Pareto-dominance result, it requires a number of voters that is at least twice the number of alternatives, the upper bound shown on the MSI is slightly higher than in Theorem 11.3.1, and unlike Theorem 11.3.1, no matching lower bound is shown.

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[9] Of course, for any ordering of the issues, there exists a profile that results in the paradoxes in Theorem 11.3.1; but this does not directly imply that there exists a single profile that works for all orderings over the issues.

Theorem 11.3.6. *For any $p, n \in \mathbb{N}$ (with $p \geqslant 2$ and $n \geqslant 2^{p+1}$), there exists an n-profile P such that for any order $\mathcal{O}$ over $\{X_1, \ldots, X_p\}$, $SSP_{\mathcal{O}}(P) = 1_1 \cdots 1_p$, and any $V \in P$ ranks $1_1 \cdots 1_p$ somewhere in the bottom $p + 2$ positions.*

Proof of Theorem 11.3.6: We first prove a lemma.

Lemma 11.3.7. *For any $c \in \mathcal{X}$, $\mathcal{X}' \subset \mathcal{X}$ such that $c \notin \mathcal{X}'$, and any $n \in \mathbb{N}$ ($n \geqslant 2m = 2^{p+1}$), there exists an n-profile that satisfies the following conditions. Let $F = \mathcal{X} \backslash (\mathcal{X}' \cup \{c\})$.*

- *For any $c' \in \mathcal{X}'$, c defeats c' in their pairwise election.*
- *For any $c' \in \mathcal{X}'$ and $d \in F$, c' defeats d in their pairwise election.*
- *For any $V \in P$, c is ranked somewhere in the bottom $|\mathcal{X}'| + 2$ positions.*

Proof of Lemma 11.3.7: We let $P = (V_1, \ldots, V_n)$ be the profile defined as follows. Let $F_1, \ldots, F_{\lfloor n/2 \rfloor + 1}$ be a partition of F such that for any $j \leqslant \lfloor n/2 \rfloor + 1$, $|F_j| \leqslant \lceil 2m/n \rceil = 1$. For any $j \leqslant \lfloor n/2 \rfloor + 1$, we let $V_j = [(F \backslash F_j) \succ c \succ \mathcal{X}' \succ F_j]$. For any $\lfloor n/2 \rfloor + 2 \leqslant j \leqslant n$, we let $V_j = [\mathcal{X}' \succ F \succ c]$. It is easy to check that P satisfies all conditions in the lemma. □

Now, let $c = 1_1 \cdots 1_p$ and $\mathcal{X}' = \{0_1 1_2 \cdots 1_p, 1_1 0_2 1_3 \cdots 1_p, \ldots, 1_1 \cdots 1_{p-1} 0_p\}$. By Lemma 11.3.7, there exists a profile P such that c beats any alternative in $\mathcal{X}'$ in pairwise elections, any alternative in $\mathcal{X}'$ beats any alternative in $\mathcal{X} \backslash (\mathcal{X}' \cup \{c\})$ in pairwise elections, and c is ranked somewhere in the bottom $p + 2$ positions. This is the profile that we will use to prove the paradox.

Without loss of generality , we assume that $\mathcal{O} = X_1 > X_2 > \cdots > X_p$. (This is without loss of generality because all issues have been treated symmetrically so far.) c beats $1_1 \cdots 1_{p-1} 0_p$ in the first round; c will meet $1_1 \cdots 1_{p-2} 0_{p-1} 1_p$ in the next pairwise election, because $1_1 \cdots 1_{p-2} 0_{p-1} 1_p$ beats every other alternative in that branch (they are all in F), and c will win; and so on. It follows that $c = SSP_{\mathcal{O}}(P)$. Moreover, all voters rank c in the bottom $p+2$ positions.

(End of proof for Theorem 11.3.6.) □

11.4 Multiple-Election Paradoxes for SSP with Restrictions on Preferences

The paradoxes exhibited so far placed no restriction on the voters' preferences. While SSP is perfectly well defined for any preferences that the voters may have over the alternatives, we may yet wonder what happens if the voters' preferences over alternatives are restricted in a way that is natural with respect to the multi-issue structure of the setting. In particular, we may wonder if paradoxes are avoided by such restrictions. It is well known that natural restrictions on preferences sometimes lead to much more positive results in social choice and mechanism design—for example, single-peaked preferences allow for good strategy-proof mechanisms (Black, 1948; Moulin, 1980). In the next chapter we will see that we can characterize all strategy-proof voting rules when the voters' preferences are lexicographic, and their local preferences over issues are restricted.

In this section, we study the MSI for $SSP_{\mathcal{O}}$ for the following three cases: (1) voters' preferences are separable; (2) voters' preferences are $\mathcal{O}$-lexicographic; and (3) voters' preferences are $\mathcal{O}$-legal. For case (1), we show a mild paradox (and that this is effectively the strongest paradox that can be obtained); for case (2), we show a positive result; for case (3), we show a paradox that is nearly as bad as the unrestricted case.

Theorem 11.4.1. *For any $n \geqslant 2p$, when the profile is separable, the MSI for $SSP_{\mathcal{O}}$ is between $2^{[p/2]}$ and $2^{[p/2]+1}$.*

That is, the MSI of $SSP_{\mathcal{O}}$ when votes are separable is $\Theta(\sqrt{m})$. We still have that $\lim_{m\to\infty} \Theta(\sqrt{m})/m = 0$, so in that sense this is still a paradox. However, its convergence rate to 0 is much slower than for $\Theta(\log m)/m$, which corresponds to the convergence rate for the earlier paradoxes.

Proof of Theorem 11.4.1: Let $P = (V_1, \ldots, V_n)$. For any $i \leqslant p$, we let $d_i = maj(P|_{X_i})$. That is, d_i is the majority winner for the projection of the profile to the ith issue. Because any separable profile is compatible with any order over the issues, P is an $\mathcal{O}^{-1}$-legal profile. It follows from Corollary 11.1.6 that $SSP_{\mathcal{O}}(P) = (d_1, \ldots, d_p)$. Without loss of generality $(d_1, \ldots, d_p) = (1_1, \ldots, 1_p)$.

First, we prove the lower bound. Because for any $i \leqslant p$, at least half of the voters prefer 1_i to 0_i, the total number of times that a voter prefers 1 to 0 for an issue, counted across all voters and issues, is at least $p \cdot (n/2)$. Therefore, there exists $j \leqslant n$ such that voter j prefers 1 to 0 on at least $p/2$ issues, otherwise the total number of times that a voter prefers 1 to 0 for an issue, counted across all voters and issues, is no more than $n \cdot (p/2) - 1 < p \cdot (n/2)$, which is a contradiction. Formally put, there exists $j \leqslant n$ such that $|\{i \leqslant p : 1_i \succ_{V_j} 0_i\}| \geqslant p/2$. Without loss of generality for every $i \leqslant \lceil p/2 \rceil$, $1_i \succ_{V_j} 0_i$. It follows that for any $\vec{a} \in D_1 \times \cdots D_{\lceil p/2 \rceil}$, we have that $(1_1, \ldots, 1_p) \succ_{V_j} (\vec{a}, 1_{\lceil p/2 \rceil+1}, \ldots, 1_p)$. Therefore, the minimax satisfaction index is at least $2^{\lceil p/2 \rceil}$.

Next, we prove the upper bound. We first show that there exists a set of n CP-nets $\mathcal{N}_1, \ldots, \mathcal{N}_n$ that satisfies the following two conditions.

1. For each $j \leqslant n$, the number of issues on which $\mathcal{N}_j$ prefers 1 to 0 is exactly $\lfloor p/2 \rfloor + 1$.

2. For each $i \leqslant p$, $maj(\mathcal{N}_1|_{X_i}, \ldots, \mathcal{N}_n|_{X_i}) = 1_i$.

The proof is by explicitly constructing the profile through the following n-step process. Informally, we will allocate $p(\lfloor n/2 \rfloor + 1)$ CPT entries "1 is preferred to 0", $\lfloor n/2 \rfloor + 1$ entries per issue, to n CP-nets as even as possible. Let $k_1 = \cdots = k_p = \lfloor n/2 \rfloor + 1$. In the jth step, we let $I_j = \{i_1, \ldots, i_{\lfloor p/2 \rfloor+1}\}$ be the set of indices of the highest k's. Then, for any $i \in I_j$, we let $\mathcal{N}_j|_{X_i} = [1_i \succ 0_i]$ and $k_i \leftarrow k_i - 1$; for any $i \notin I_j$, we let $\mathcal{N}_j|_{X_i} = [0_i \succ 1_i]$. Because of the assumption that $n \geqslant 2p$, we have that $n(\lfloor p/2 \rfloor + 1) \geqslant p(\lfloor n/2 \rfloor + 1)$, which means that after n steps, for all $i \leqslant p$, $k_i \leqslant 0$.

It left to show that there exist extensions of $\mathcal{N}_1, \ldots, \mathcal{N}_n$ such that in each of these extensions, $1_1 \cdots 1_p$ is ranked within bottom $2^{\lfloor p/2 \rfloor + 1}$ positions. To show this, we use the following lemma.

Lemma 11.4.2. *For any partial order W and any alternative c, we let $|Down_W(c)| = \{c' : c \succeq_W c'\}$, that is, $|Down_W(c)|$ is the set of all alternatives (including c) that are less preferred to c in W. There exists a linear order V such that V extends W and c is ranked in the $|Down_W(c)|$th position from the bottom.*

The proof of Lemma 11.4.2 is quite straightforward: for every alternative d such that $d \notin \text{Down}_W(c)$, we put $d \succ c$ in the partial order. This does not violate transitivity, which means that the ordering relation obtained in this way is a partial order, denoted by W'. Then, let V be an arbitrary linear order that extends W'. It follows that c is ranked at the $\text{Down}_W(c)$th position from the bottom in V.

We note that for any $j \leqslant n$, the number of entries in $\mathcal{N}_j$ where $1 \succ 0$ is no more than $\lfloor p/2 + 1 \rfloor$. Therefore, for any $j \leqslant n$, $|\text{Down}_{\succ_{\mathcal{N}_j}}(1_1 \cdots 1_p)| \leqslant 2^{\lfloor p/2+1 \rfloor}$ (we recall that $\succ_{\mathcal{N}_j}$ is the partial order that $\mathcal{N}_j$ encodes). Let $V_1, \ldots, V_n$ be extensions of $\mathcal{N}_1, \ldots, \mathcal{N}_n$, respectively, where for all $j \leqslant n$, $1_1 \cdots 1_p$ is ranked as low as possible in any V_j. It follows from Lemma 11.4.2 that for any $j \leqslant n$, $1_1 \cdots 1_p$ is ranked in the $2^{\lfloor p/2+1 \rfloor}$th position from the bottom in V_j. This proves the upper bound.

(End of proof for Theorem 11.4.1.) □

Theorem 11.4.3. *For any $p \in \mathbb{N}$ ($p \geqslant 2$) and any $n \geqslant 5$, when the profile is $\mathcal{O}$-lexicographic, $MSI(SSP_{\mathcal{O}}) = 3 \cdot 2^{p-2} + 1$. Moreover, $SSP_{\mathcal{O}}(P)$ is ranked somewhere in the top 2^{p-1} positions in at least $n/2$ votes.*

Naturally $\lim_{m\to\infty}(3m/4+1)/m = 3/4$, so in that sense there is no paradox when votes are $\mathcal{O}$-lexicographic.

Proof of Theorem 11.4.3: The proof is for profiles without ties. The other cases can be proved similarly. Without loss of generality $SSP_{\mathcal{O}}(P) = 1_1 \cdots 1_p$ and for every $j \leqslant \lfloor n/2 \rfloor + 1$, $1_1 \succ_{V_j|X_1} 0_1$. It follows that in $V_1, \ldots, V_{\lfloor n/2 \rfloor+1}$, $1_1 \cdots 1_p$ is ranked within top $2^{p-1} = m/2$ positions. Because in at least $\lfloor n/2 \rfloor + 1$ votes $1_1 : 1_2 \succ 0_2$, there exists a vote $V \in P$ such that $1_1 \succ_{V|X_1} 0_1$ and $1_1 : 1_2 \succ_{V|X_2:1_1} 0_2$. It follows that $1_1 \cdots 1_p$ is ranked in the $(3 \cdot 2^{p-2} + 1)$th position from the bottom. This proves that when the profile is $\mathcal{O}$-lexicographic, $MSI(SSP_{\mathcal{O}}) \geqslant 3 \cdot 2^{p-2} + 1$.

We next prove that $3 \cdot 2^{p-2} + 1$ is also an upper bound. Consider the profile $P = (V_1, \ldots, V_n)$ defined as follows. For any $j \leqslant \lfloor n/2 \rfloor + 1$, $1_1 \succ_{V_j|X_1} 0_1$; for any $\lfloor n/2 \rfloor + 2 \leqslant j \leqslant n$, $1_2 \succ_{V_j|X_2:1_1} 0_2$; for $j = 1, 2$, $1_2 \succ_{V_j|X_2:1_1} 0_2$; for any $3 \leqslant j \leqslant n$ and any $3 \leqslant i \leqslant p$, $1_i \succ_{V_j|X_i:1_1 \cdots 1_{i-1}} 0_j$; for any local preferences of any voter that is not defined above, let 0 be preferred to 1.

We note that for any $i \leqslant p$, more than $n/2$ votes in $P|_{X_i:1_1 \cdots 1_{i-1}}$ prefer 1_i to 0_i, which means that $SSP_{\mathcal{O}}(P) = 1_1 \cdots 1_p$. It is easy to check that in any vote, $1_1 \cdots 1_p$ is ranked somewhere within the bottom $3 \cdot 2^{p-2} + 1$ positions.

(End of proof for Theorem 11.4.3.) □

Under the previous two restrictions (separability and $\mathcal{O}$-lexicographicity), $SSP_{\mathcal{O}}$ coincides with $Seq(Maj, \ldots, Maj)$ (by Corollary 11.1.6 and Proposition 11.1.4, respectively). Therefore, Theorems 11.4.1 and 11.4.3 also apply to sequential voting rules as well as issue-by-issue voting rules.

Finally, we study the MSI for $SSP_{\mathcal{O}}$ when the profile is $\mathcal{O}$-legal. Theorem 11.4.6 shows that it is nearly as bad as the unrestricted case (Theorem 11.3.1). The proof of Theorem 11.4.6 is the most involved proof in this chapter. The idea of the proof is similar to that of the proof for Theorem 11.3.1, but now we cannot apply Lemma 11.3.3, because $\mathcal{O}$-legality must be preserved. We start with a simpler result that shows the idea of the construction.

Theorem 11.4.4. *There exists a way to break ties in $SSP_{\mathcal{O}}$ such that the following is true. Let $SSP'_{\mathcal{O}}$ be the rule corresponding to $SSP_{\mathcal{O}}$ plus the tiebreaking mechanism. For any $p \in \mathbb{N}$, there exists an $\mathcal{O}$-legal profile that consists of two votes, such that in one of the two votes, no more than $\lceil p/2 \rceil$ alternatives are ranked lower than the winner $SSP'_{\mathcal{O}}(P)$; and in the other vote, no more than $\lfloor p/2 \rfloor$ alternatives are ranked lower than $SSP'_{\mathcal{O}}(P)$.*

Proof of Theorem 11.4.4: The proof is by induction on p. When $p = 2$, let the CPT of $\mathcal{N}_1$ be $0_1 \succ 1_1, 0_1 : 1_2 \succ 0_2, 1_1 : 1_2 \succ 0_2$; let the CPT of $\mathcal{N}_2$ be $1_1 \succ 0_1, 0_1 : 0_2 \succ 1_2, 1_1 : 0_2 \succ 1_2$; $V_1 = [0_1 1_2 \succ 0_1 0_2 \succ 1_1 1_2 \succ 1_1 0_2]$; $V_2 = [1_1 0_2 \succ 0_1 0_2 \succ 1_1 1_2 \succ 0_1 1_2]$. In the first step, ties are broken in favor of $1_1 1_2$. Given 1_1, ties are broken in favor of 1_2; given 0_1, ties are broken in favor of 1_2.

Suppose the claim is true for $p = l$. Next we construct $\mathcal{N}_1$ and $\mathcal{N}_2$ for $p = l + 1$. Let $\mathcal{N}'_1$, $\mathcal{N}'_2$, V'_1, V'_2 be the CP-nets and the votes for the case of $p = l$, where the multi-issue domain is $D_2 \times \cdots \times D_{l+1}$. Without loss of generality $|\text{Down}_{V_1}(1_2 \cdots 1_{l+1})| \leqslant \lceil l/2 \rceil$ and $|\text{Down}_{V_2}(1_2 \cdots 1_{l+1})| \leqslant \lfloor l/2 \rfloor$. We recall that for any vote V and any alternative c, $\text{Down}_V(c)$ (defined in Lemma 11.4.2) is the set of all alternatives that are ranked below c in V, including c. Let $\vec{e} \in D_2 \times \cdots \times D_{l+1}$ be an arbitrary alternative such that $1_2 \cdots 1_{l+1} \succ_{V_2} \vec{e}$. Such an $\vec{e}$ always exists, because if on the contrary $1_2 \cdots 1_{l+1}$ is in the bottom of V_2, it must be ranked higher than at least l other alternatives in V_1 to win the election, which contradicts the assumption that $|\text{Down}_{V_1}(1_2 \cdots 1_{l+1})| \leqslant \lceil l/2 \rceil$. We will explain later why we choose $\vec{e}$ in such a way.

Let $\mathcal{N}_1^*$ (respectively, $\mathcal{N}_2^*$) be the separable CP-net (we recall that a CP-net is separable if its graph has no edges) $D_2 \times \cdots \times D_{l+1}$ in which $\vec{e}$ is in the top (respectively, bottom) position. For $i = 1, 2$, we let $\mathcal{N}_i$ be a CP-net over $D_1 \times \cdots \times$

D_{l+1}, defined as follows:

- $0_1 \succ_{\mathcal{N}_i} 1_1$.
- The sub-CP-net of $\mathcal{N}_i$ restricted on $X_1 = 1_1$ is $\mathcal{N}_i'$;
- The sub-CP-net of $\mathcal{N}_i$ restricted on $X_1 = 0_1$ is $\mathcal{N}_i^*$;

Let V_1, V_2 be the extension of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, that satisfy the following conditions:

- For any $\vec{b}, \vec{d} \in D_2 \times \cdots \times D_{l+1}$ such that $\vec{b} \neq \vec{e}$, and any $i = 1, 2$, we have $(0_1, \vec{b}) \succ_{V_i} (1_1, \vec{d})$. This condition can be satisfied, because we have $0_1 \succ_{\mathcal{N}_i} 1_1$.
- For any $\vec{b}, \vec{d} \in D_2 \times \cdots \times D_{l+1}$, and any $i = 1, 2$, we have that $(1_1, \vec{b}) \succ_{V_i} (1_1, \vec{d})$ if and only if $\vec{b} \succ_{V_i'} \vec{d}$. This condition says that if we focus on the order of the alternatives whose X_1 component is 1_1 in V_i, then it is the same as in V_i'.
- For any $\vec{d} \in D_2 \times \cdots \times D_{l+1}$, we have that $(0_1, \vec{e}) \succ_{V_1} (1_1, \vec{d})$.
- $(1_1, \ldots, 1_{l+1}) \succ_{V_2} (0_1, \vec{e}) \succ_{V_2} (1_1, \vec{e})$.

We let the tie-breaking mechanism be defined as follows: in the first step, ties are broken in favor of 1_1; in the subgame in which $X_1 = 1_1$, ties are broken in the same way as for the profile (V_1', V_2') (such that $1_2 \cdots 1_{l_1}$ is the winner for the profile); in the subgame in which $X_1 = 0_1$, ties are broken in such a way that $\vec{e}$ is the winner (because $\vec{e}$ is ranked in the top position in one vote, and in the bottom position in the other, there exists a tie-breaking mechanism under which $\vec{e}$ is the winner).

We note that $1_1 \cdots 1_p \succ_{V_1} \vec{d}$ if and only if $\vec{d} = (1_1, \vec{d}')$ for some $\vec{d}' \in D_2 \times \cdots \times D_{l+1}$ such that $1_2 \cdots 1_p \succ_{V_1'} \vec{d}'$. It follows that $|\text{Down}_{V_1}(1_1 \cdots 1_{l+1})| = |\text{Down}_{V_1'}(1_2 \cdots 1_{l+1})|$. We also note that $1_1 \cdots 1_{l+1} \succ_{V_2} \vec{b}$ if and only if $\vec{b} = (0_1, \vec{e})$ or $\vec{b} = (1_1, \vec{b}')$ for some

$\vec{b}' \in D_2 \times \cdots \times D_{l+1}$ such that $1_2 \cdots 1_p \succ_{V_2'} \vec{b}'$. It follows that $|\mathrm{Down}_{V_2}(1_1 \cdots 1_{l+1})| = |\mathrm{Down}_{V_2'}(1_2 \cdots 1_{l+1})| + 1$. Therefore, we have the following inequalities.

$$|\mathrm{Down}_{V_1}(1_1 \cdots 1_{l+1})| \leqslant \lfloor (l+1)/2 \rfloor$$

$$|\mathrm{Down}_{V_2}(1_1 \cdots 1_p)| \leqslant \lfloor l/2 \rfloor + 1 \leqslant \lceil (l+1)/2 \rceil$$

Here the trick to choose $\vec{e}$ such that $1_2 \cdots 1_{l+1} \succ_{V_2'} \vec{e}$ is crucial, because we force $0_1 \succ_{\mathcal{N}_2} 1_1$ and $1_1 \cdots 1_{l+1} \succ_{V_2} (0_1, \vec{e})$, which implies that $1_1 \cdots 1_{l+1} \succ_{V_2} (0_1, \vec{e}) \succ_{V_2} (1_1, \vec{e})$ (since V_2 extends $\mathcal{N}_2$). If we chose $\vec{e}$ such that $\vec{e} \succ_{V_2'} 1_2 \cdots 1_{l+1}$, then we would have that $|\mathrm{Down}_{V_2}(1_1 \cdots 1_{l+1})| = |\mathrm{Down}_{V_2'}(1_2 \cdots 1_{l+1})| + 2$, which does not prove the claim for $p = l + 1$.

Next, we verify that $SSP_{\mathcal{O}}(V_1, V_2) = 1_1 \cdots 1_{l+1}$. We note that $(0_1, \vec{e}) \succ_{V_1} 1_1 \cdots 1_{l+1}$. Therefore, in the first step voter 1 will vote for 0_1. Meanwhile, $1_1 \cdots 1_{l+1} \succ_{V_2} (0_1, \vec{e})$, which means that in the first step voter 2 will vote for 1_1. Because ties are broken in favor of 1_1 in the first step, we will fix $X_1 = 1_1$. Then, in the following steps (step $2, \ldots, l+1$), $1_2, \ldots, 1_{l+1}$ will be the winners by induction hypothesis, which means that $SSP_{\mathcal{O}}(V_1, V_2) = 1_1 \cdots 1_{l+1}$.

Therefore, the claim is true for $p = l + 1$. This means that the claim is true for any $p \in \mathbb{N}$.

Example 11.4.5. *Let us show an example of the above construction from $p = 2$ to $p = 3$. In $\mathcal{N}_1$, we have $0_1 \succ 1_1$, $1_1 : \mathcal{N}_1^*$, and $0_1 : \mathcal{N}_1'$, where $\mathcal{N}_1'$ is $0_2 \succ 1_2, 0_2 : 1_3 \succ 0_3, 1_2 : 1_3 \succ 0_3$. (We note that $\mathcal{N}_1'$ is defined over $D_2 \times D_3$.) V_1 restricted to 1_1 is $V_1' = [0_2 1_3 \succ 0_2 0_3 \succ 1_2 1_3 \succ 1_2 0_3]$ (which is, again, over $D_2 \times D_3$). Let $\vec{e} = 0_2 1_3$. Therefore, we have the following construction:*

$V_1 = 0_1 0_2 1_3 \succ 0_1 1_2 1_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 0_3 \succ 1_1 0_2 1_3 \succ 1_1 0_2 0_3 \succ 1_1 1_2 1_3 \succ 1_1 1_2 0_3$

$V_2 = 0_1 1_2 0_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 1_3 \succ 1_1 1_2 0_3 \succ 1_1 0_2 0_3 \succ 1_1 1_2 1_3 \succ 0_1 0_2 1_3 \succ 1_1 0_2 1_3$

Ties are broken in a way such that if we are in the branch in which $X_1 = 1_1$, then $1_2 1_3$ is the winner; and if we are in the branch in which $X_1 = 0_1$, then $\vec{e} = 0_2 1_3$

is the winner. In the first step, ties are broken in favor of 1_1*. Then, the sub-game winners are* $1_1 1_2 1_3$ *and* $0_1 0_2 1_3$*. Since exactly one vote (*V_1*) prefers* $0_1 0_2 1_3$ *to* $1_1 1_2 1_3$*, and the other vote* V_2 *prefers* $1_1 1_2 1_3$ *to* $0_1 0_2 1_3$*, the winner is* $1_1 1_2 1_3$*.*

(End of proof for Theorem 11.4.4.) □

We emphasize that, unlike any of our other results, Theorem 11.4.4 is based on a specific tie-breaking mechanism. The next theorem concerns the more general and complicated case in which n can be either odd or even, and the winner does not depend on the tie-breaking mechanism. That is, there are no ties in the election. The situation is almost the same as in Theorem 11.3.1.

Theorem 11.4.6. *For any* $p, n \in \mathbb{N}$ *with* $n \geqslant 2p^2 + 2p + 1$*, there exists an* $\mathcal{O}$*-legal profile such that in each vote, no more than* $\lceil p/2 \rceil + 4$ *alternatives are ranked lower than* $SSP_{\mathcal{O}}(P)$*. Moreover,* $SSP_{\mathcal{O}}(P)$ *is Pareto-dominated by at least* $2^p - 4p^2$ *alternatives.*

Of course, the lower bound on the MSI from Theorem 11.3.1 still applies when the profile is $\mathcal{O}$-legal, so together with Theorem 11.4.6 this proves that the MSI for $SSP_{\mathcal{O}}$ when the profile is $\mathcal{O}$-legal is $\Theta(\log m)$, just as in the unrestricted case.

Proof of Theorem 11.4.6: For simplicity, we prove the theorem for the case in which $n = 2p^2 + 2p + 1$. The proof for the case in which $n > 2p^2 + 2p + 1$ is similar. For any $l \leqslant p$, we let $\mathcal{X}_l = \{0_l, 1_l\} \times \{0_{l+1}, 1_{l+1}\} \times \cdots \times \{0_p, 1_p\}$; let $\mathcal{O}_l = X_l > X_{l+1} > \cdots > X_p$. We first prove the following claim by induction.

Claim 11.4.1. *For any* $l \leqslant p$*, there exists a* $\mathcal{O}_l$*-legal profile* $P_l = A_l \cup B_l \cup \hat{A}_l \cup \hat{B}_l \cup \{c^l\}$ *over* $\mathcal{X}_l$*, where* $A_l = \{a^l_1, \ldots, a^l_{p^2}\}$*,* $B_l = \{b^l_1, \ldots, b^l_{p^2}\}$*,* $\hat{A}_l = \{\hat{a}^l_1, \ldots, \hat{a}^l_p\}$*,* $\hat{B}_l = \{\hat{b}^l_1, \ldots, \hat{b}^l_p\}$*, that satisfies the following conditions.*

- $SSP_{\mathcal{O}_l}(P_l) = 1_l \cdots 1_p$*.*

- *For any* $V \in P_l$*,* $|Down_V(1_l \cdots 1_p)| \leqslant \lceil (p - l + 1)/2 \rceil + 5$*.*

- *For any* $(p-l)p \leqslant j \leqslant p^2$, $|Down_{a^l_j}(1_l \cdots 1_p)| \leqslant \lceil (p-l+1)/2 \rceil + 3$, $|Down_{b^l_j}(1_l \cdots 1_p)| \leqslant \lceil (p-l+1)/2 \rceil + 3$.

- *For any* $p-l \leqslant j \leqslant p$, $|Down_{\hat{a}^l_j}(1_l \cdots 1_p)| \leqslant \lceil (p-l+1)/2 \rceil + 3$, $|Down_{\hat{b}^l_j}(1_l \cdots 1_p)| \leqslant \lceil (p-l+1)/2 \rceil + 3$.

- *If* $p-l+1$ *is odd, then*

  - *for any* $V_B \in B$, $|Down_{V_B}(1_l \cdots 1_p)| \leqslant \lceil (p-l+1)/2 \rceil + 4$;
  - *for any* $(p-l)p \leqslant j \leqslant p^2$, $|Down_{b^l_j}(1_l \cdots 1_p)| \leqslant \lceil (p-l+1)/2 \rceil + 2$;
  - *and for any* $p-l \leqslant j \leqslant p$, $|Down_{\hat{b}^l_j}(1_l \cdots 1_p)| \leqslant \lceil (p-l+1)/2 \rceil + 2$.

- $1_l \cdots 1_p$ *is ranked higher than* $1_l \cdots 1_{p-2}0_{p-1}0_p$ *in all votes in* P_l.

Proof of Claim 11.4.1: We prove the claim by induction on l. When $l = p-1$, we let all votes in P_{p-1} be $1_{p-1}1_p \succ 1_{p-1}0_p \succ 0_{p-1}1_p \succ 0_{p-1}0_p$. It is easy to check that P_{p-1} satisfies all the conditions in the claim. Suppose the claim is true for $l \leqslant p$, we next prove that the claim is also true for $l-1$. We show the existence of P_{l-1} by construction for the following two cases.

Case 1: $p-l+1$ is even.

We let $\mathcal{N}^l_1, \ldots, \mathcal{N}^l_{p-l+1}, \mathcal{N}^l_A, \mathcal{N}^l_B$ be separable CP-nets over $\mathcal{X}_l$, defined as follows.

- Let $1_l \cdots 1_{p-2}0_{p-1}0_p$ be in the bottom position of $\mathcal{N}^l_A$; let $1_l \cdots 1_{p-2}0_{p-1}0_p$ be in the top position of $\mathcal{N}^l_B$.

- For any $1 \leqslant i \leqslant p-l-1$, let $1_l \cdots 1_{l+i-2}0_{l+i-1}1_{l+i} \cdots 1_{p-2}0_{p-1}0_p$ be in the top position of $\mathcal{N}^l_i$; let $1_l \cdots 1_{p-2}1_{p-1}0_p$ be in the top position of $\mathcal{N}^l_{p-l}$; let $1_l \cdots 1_{p-2}0_{p-1}1_p$ be in the top position of $\mathcal{N}^l_{p-l+1}$.

For any linear order V over $\mathcal{X}_l$, we let the *composition* of V and $\mathcal{N}$ (where $\mathcal{N} \in \{\mathcal{N}^l_1, \ldots, \mathcal{N}^l_{p-l+1}, \mathcal{N}^l_A, \mathcal{N}^l_B\}$) be a partial order O^{l-1} over $\mathcal{X}_{l-1}$, defined as follows.

- The restriction of O^{l-1} on $X_{l-1} = 1_{l-1}$ is V. That is, for any $\vec{d}_1, \vec{d}_2 \in \mathcal{X}_l$ such that $\vec{d}_1 \succ_V \vec{d}_2$, we let $(1_{l-1}, \vec{d}_1) \succ_{O^{l-1}} (1_{l-1}, \vec{d}_2)$.

- The restriction of O^{l-1} on $X_{l-1} = 0_{l-1}$ is the partial order encoded by $\mathcal{N}$. That is, for any $\vec{d}_1, \vec{d}_2 \in \mathcal{X}_l$ such that $\vec{d}_1 \succ_{\mathcal{N}} \vec{d}_2$, we let $(0_{l-1}, \vec{d}_1) \succ_{O^{l-1}} (0_{l-1}, \vec{d}_2)$.

- For any $\vec{d} \in \mathcal{X}_l$, we let $(0_{l-1}, \vec{d}) \succ_{O^{l-1}} (1_{l-1}, \vec{d})$.

- If $\mathcal{N} \in \{\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l, \mathcal{N}_A^l\}$, we let $1_{l-1}1_l \cdots 1_p \succ_{O^{l-1}} 0_{l-1}1_l \cdots 1_{p-2}0_{p-1}0_p$.

We are now ready to define P_{l-1}. Any $V \in P_{l-1}$ has a counterpart in P_l. For example, the counterpart of $\hat{a}_1^{l-1}$ is $\hat{a}_1^l$. For any $V \in P_{l-1}$, V is defined to be the extension of the composition of V's counterpart in P_l and some $\mathcal{N}$ (where $\mathcal{N} \in \{\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l, \mathcal{N}_A^l, \mathcal{N}_B^l\}$), in which $1_{l-1} \cdots 1_p$ is ranked as low as possible. Next we specify which $\mathcal{N}$ that each $V \in P_{l-1}$ corresponds to in the following table.

Table 11.2: From P_l to P_{l-1}.

for all	**votes in P_{l-1}**	**is composed of**	
$1 \leqslant j \leqslant p$	$\hat{a}_j^{l-1}$	$\hat{a}_j^l$	$\mathcal{N}_A^l$
$j \leqslant (p-l)p$	a_j^{l-1}	a_j^l	$\mathcal{N}_A^l$
$(p-l)p+1 \leqslant j \leqslant (p-l+1)p$	a_j^{l-1}	a_j^l	$\mathcal{N}_{j-(p-l)p}^l$
$(p-l+1)p+1 \leqslant j \leqslant p^2$	a_j^{l-1}	a_j^l	$\mathcal{N}_A^l$
	$\hat{b}_{p-l+2}^{l-1}$	$\hat{b}_{p-l+2}^l$	$\mathcal{N}_A^l$
$j \neq p-l+2$	$\hat{b}_j^{l-1}$	$\hat{b}_j^l$	$\mathcal{N}_B^l$
$j \leqslant p^2$	b_j^{l-1}	b_j^l	$\mathcal{N}_B^l$
	c^{l-1}	c^l	$\mathcal{N}_B^l$

It follows that P_{l-1} is $\mathcal{O}_{l-1}$-legal. By Lemma 11.4.2, we have the following calculation.

- For any $1 \leqslant j \leqslant p$, $|\text{Down}_{\hat{a}_j^{l-1}}(1_{l-1} \cdots 1_p)| = |\text{Down}_{\hat{a}_j^l}(1_l \cdots 1_p)| + 1$. This is because for any $\vec{d} \in \mathcal{X}_l$ such that $\vec{d} \in \text{Down}_{\hat{a}_j^l}(1_l \cdots 1_p)$, $1_{l-1} \cdots 1_p$ is ranked higher than $(1_{l-1}, \vec{d})$ in $\hat{a}_j^{l-1}$; and moreover, $1_{l-1} \cdots 1_p$ is ranked higher than $0_{l-1}1_l \cdots 1_{p-2}0_{p-1}0_p$ in $\hat{a}_j^{l-1}$.

- For any $1 \leqslant j \leqslant p$, $|\text{Down}_{a^{l-1}_{(p-l)p+j}}(1_{l-1} \cdots 1_p)| = |\text{Down}_{a^{l}_{(p-l)p+j}}(1_l \cdots 1_p)| + 3$. This is because for any $\vec{d} \in \mathcal{X}_l$ such that $\vec{d} \in \text{Down}_{a_{(p-l)p+j}}(1_l \cdots 1_p)$, $1_{l-1} \cdots 1_p$ is ranked higher than $(1_{l-1}, \vec{d})$ in $a_{(p-l)p+j}$; and moreover, $1_{l-1} \cdots 1_p$ is ranked higher than $0_{l-1}1_l \cdots 1_{p-2}0_{p-1}0_p$ in $a_{(p-l)p+j}$.

- For any $j \leqslant (p-l)p$ or $(p-l+1)p+1 \leqslant j \leqslant p^2$, $|\text{Down}_{a_j^{l-1}}(1_{l-1} \cdots 1_p)| = |\text{Down}_{a_j^l}(1_l \cdots 1_p)| + 1$.

- $|\text{Down}_{\hat{b}^{l-1}_{p-l+2}}(1_{l-1} \cdots 1_p)| = |\text{Down}_{\hat{b}^{l}_{p-l+2}}(1_l \cdots 1_p)| + 1$.

- For any $V_B \in (B_{l-1} \cup \hat{B}_{l-1} \cup \{c\}) \backslash \{\hat{b}_{p-l+2}\}$,

$$|\text{Down}_{V_B}(1_{l-1} \cdots 1_p)| = |\text{Down}_{V_B^l}(1_l \cdots 1_p)|,$$

  where V_B^l is the counterpart of V_B in P_l.

We next prove that $SSP_{\mathcal{O}_{l-1}}(P_{l-1}) = 1_{l-1} \cdots l_p$. We note that $P_{l-1}|_{X_{l-1}=1_{l-1}} = P_l$. Therefore, if in the first step 1_{l-1} is chosen, then the winner is $1_{l-1} \cdots 1_p$. We also note that $P_{l-1}|_{X_{l-1}=0_{l-1}}$ is separable (and the CP-nets are $\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l$, p^2+p copies of $\mathcal{N}_A^l$ and p^2+p copies of $\mathcal{N}_B^l$). Therefore, if in the first step 0_{l-1} is chosen, then the winner is $0_{l-1}1_l \cdots 1_{p-2}1_{p-1}1_p$. Because exactly p^2+p-1 votes in P_{l-1} prefer $0_{l-1}1_l \cdots 1_{p-2}1_{p-1}1_p$ to $1_{l-1} \cdots 1_p$ (those votes corresponds to $\mathcal{N}_B^l$ in the construction), we have that 1_{l-1} is the winner in the first step. Therefore, $SSP_{\mathcal{O}_{l-1}}(P_{l-1}) = 1_{l-1} \cdots l_p$. It is also easy to verify that P_{l-1} satisfies all conditions in the claim.

Case 2: $p-l+1$ is odd. The construction is similar as in the even case. The only difference is that we switch the role of A_l and B_l (also $\hat{A}_l$ and $\hat{B}_l$). □

The theorem follows from Claim 11.4.1 by letting $l = 1$, and it is easy to check that in P_1 in Claim 11.4.1 ($l = 1$), no more than $4p^2$ alternatives have been ranked lower than $SSP_{\mathcal{O}}(P_1)$ in any vote, which means that $SSP_{\mathcal{O}}(P_1)$ is Pareto-dominated by at least $2^p - 4p^2$ alternatives.

(End of proof for Theorem 11.4.6.) □

11.5 Summary

In this chapter, we considered a complete-information game-theoretic analysis of sequential voting on binary issues, which we called strategic sequential voting. Specifically, given that voters have complete information about each other's preferences and their preferences are strict, the game can be solved by a natural backward induction process (WSDSBI), which leads to a unique solution. We showed that under some conditions on the preferences, this process leads to the same outcome as the truthful sequential voting, but in general it can result in very different outcomes. We analyzed the effect of changing the order over the issues that voters vote on and showed that, in some elections, every alternative can be made a winner by voting according to an appropriate order over the issues.

Most significantly, we showed that strategic sequential voting is prone to multiple-election paradoxes; to do so, we introduced a concept called minimax satisfaction index, which measures the degree to which at least one voter is made happy by the outcome of the election. We showed that the minimax satisfaction index for strategic sequential voting is exponentially small, which means that there exists a profile for which the winner is ranked almost in the bottom positions in all votes; even worse, the winner is Pareto-dominated by almost every other alternative. We showed that changing the order of the issues in sequential voting cannot completely avoid the paradoxes. These negative results indicate that the solution of the sequential game can be extremely undesirable for every voter. We also showed that multiple-election paradoxes can be avoided to some extent by restricting voters' preferences to be separable or lexicographic, but the paradoxes still exist when the voters' preferences are $\mathcal{O}$-legal.