9

A Framework for Aggregating CP-nets

In this chapter, we define a new family of voting rules for combinatorial voting that allows voters to use any CP-nets (even cyclic ones) to represent their preferences. The set of all CP-nets, as a language, is compact and is much more expressive than acyclic CP-nets or separable CP-nets, which are used in sequential voting and issue-by-issue voting, respectively.[1] The voting rules we define are parameterized by: (1) the local voting rules that are used on individual attributes—we will use these to define a particular graph on the set of alternatives; and (2) a *choice set function* T that selects the winners based on this induced graph.[2] We show that if T satisfies a very natural assumption, then the voting rules induced by T extend the sequential voting rules (and therefore, also issue-by-issue voting rules) and the *order-independent sequential composition* of local rules from Xia et al. (2007b). We study whether properties of the local rules transfer to the global rule, and vice versa. Then, we focus on a particular choice set function, namely the *Schwartz set* (Schwartz, 1970), which has been argued

[1] Earlier work has also considered social choice for potentially cyclic CP-nets (Purrington and Durfee, 2007). However, that approach does not apply to all possible (cyclic) CP-nets.

[2] In principle, T could select multiple winners from the graph. We can use any tie-breaking mechanism to select the unique winner.

to be the largest reasonable choice set for tournament graphs (Laslier, 1997). For the Schwartz set, we study how to compute the winners under this methodology.

9.1 Acyclic CP-nets Are Restrictive

In this section, we show quantitatively that the set of all acyclic CP-nets lacks general usability as a voting language. We will show that even when each local domain is binary, the number of legal linear orders—the set of all linear orders $\succ$ for which there is some acyclic CP-net that $\succ$ extends—is exponentially smaller than the number of all linear orders. Let $CP(\mathcal{X}) = \{V \in L(\mathcal{X}) :$ There exists a CP-net $\mathcal{N}$ such that $V \sim \mathcal{N}\}$. That is, $CP(\mathcal{X}) = \bigcup_{\mathcal{O}} Legal(\mathcal{O})$.

Theorem 9.1.1. *If* $\mathcal{X} = \{0,1\}^p$*, then* $\frac{|CP(\mathcal{X})|}{|L(\mathcal{X})|} \leqslant \frac{p!}{2^{2^{p-2}}}$.

Proof. We prove the theorem by constructing a set of exponentially many permutations on the set of alternatives, and we prove that for any two different linear orders compatible with the same order over attributes, for any two (not necessarily different) permutations in the set, if we apply the first permutation to the first linear order and the second permutation to the second linear order, the results are different. That is, for any linear order compatible with a given order $\mathcal{O}$, we can find a large set of corresponding linear orders by applying the set of permutations to it; and the sets of linear orders corresponding to different $G_{\mathcal{O}}$-legal linear orders are disjoint.

More precisely, we define a set of permutations on $\mathcal{X}$, denoted by $K(\mathcal{X})$, and show that it satisfies the following two properties:

1. $|K(\mathcal{X})| = 2^{2^{p-2}}$.

2. For any $V_1, V_2 \in Legal(X_1 > \cdots > X_p)$ and any $M_1, M_2 \in K(\mathcal{X})$, if $M_1 \neq M_2$, then $M_1(V_1) \neq M_2(V_2)$.

Now we show how to construct $K(\mathcal{X})$. Given any setting $\overrightarrow{x_{p-2}}$ of $(X_1, \ldots, X_{p-2})$, let $M_{\overrightarrow{x_{p-2}}}$ be the permutation that only exchanges $(\overrightarrow{x_{p-2}}, 0, 0)$ and $(\vec{x}_{p-2}, 0, 1)$. Then, for any $E_{p-2} \subseteq \{0,1\}^{p-2}$, let $M_{E_{p-2}} = \circ_{\overrightarrow{x_{p-2}} \in E_{p-2}} M_{\overrightarrow{x_{p-2}}}$, where $\circ$ is the composition of two permutations. This notion is well-defined because for any $\overrightarrow{x_{p-2}}, \overrightarrow{x_{p-2}}' \in \{0,1\}^{p-2}$, $M_{\overrightarrow{x_{p-2}}}$ and $M_{\overrightarrow{x_{p-2}}'}$ are exchangeable, that is, $M_{\overrightarrow{x_{p-2}}} \circ M_{\overrightarrow{x_{p-2}}'} = M_{\overrightarrow{x_{p-2}}'} \circ M_{\overrightarrow{x_{p-2}}}$.

Let $K(\mathcal{X}) = \{M_{E_{p-2}} : E_{p-2} \subseteq \{0,1\}^{p-2}\}$. Then, $|K(\mathcal{X})| = |\{0,1\}^{\{0,1\}^{p-2}}| = 2^{2^{p-2}}$. For any $V_1, V_2 \in Legal(X_1 > \cdots > X_p)$ any $M_{E^1_{p-2}}, M_{E^2_{p-2}} \in K(\mathcal{X})$ such that $M_{E^1_{p-2}} \neq M_{E^2_{p-2}}$, since $E^1_{p-2} \neq E^2_{p-2}$, w.l.o.g. there exists $\overrightarrow{x_{p-2}}$ such that $\overrightarrow{x_{p-2}} \in E^1_{p-2}$ but $\overrightarrow{x_{p-2}} \notin E^2_{p-2}$. Then, $V_1|_{\{X_{p-1},X_p\}:\overrightarrow{x_{p-2}}}$ extends a CP-net on $\{0,1\}^2$, and the CP-net is compatible with $X_{p-1} > X_p$. Here, $V_1|_{\{X_{p-1},X_p\}:\overrightarrow{x_{p-2}}}$ is the restriction of V_1 to $\{X_{p-1}, X_p\}$, given that $(X_1, \ldots, X_{p-2}) = \overrightarrow{x_{p-2}}$. However, $M_{\overrightarrow{x_{p-2}}}(V_2)|_{\{X_{p-1},X_p\}:\overrightarrow{x_{p-2}}}$ is not compatible with $X_{p-1} > X_p$—it either does not extend a CP-net, or extends a CP-net that is not compatible with $X_{p-1} > X_p$. We note that

$$M_{E^1_{p-2}}(V_1)|_{\{X_{p-1},X_p\}:\overrightarrow{x_{p-2}}} = V_1|_{\{X_{p-1},X_p\}:\overrightarrow{x_{p-2}}},$$

$$M_{E^2_{p-2}}(V_2)|_{\{X_{p-1},X_p\}:\overrightarrow{x_{p-2}}} = M_{\overrightarrow{x_{p-2}}}(V_2)|_{\{X_{p-1},X_p\}:\overrightarrow{x_{p-2}}}$$

Hence, $M_{E^1_{p-2}}(V_1) \neq M_{E^2_{p-2}}(V_2)$, which means that $K(\mathcal{X})$ satisfies the two properties mentioned above.

Therefore, from the two properties of $K(\mathcal{X})$, we know that $|K(\mathcal{X})(Legal(X_1 > \ldots > X_p))| = 2^{2^{p-2}}|Legal(X_1 > \ldots > X_p)|$. Since $K(\mathcal{X})(Legal(X_1 > \ldots > X_p)) \subseteq L(\mathcal{X})$, and $|CP(\mathcal{X})| < p!|L(X_1 > \ldots > X_p)|$ (because there are $p!$ linear orders over $\{x_1, \ldots, x_p\}$, and a CP-net must be compatible with some order), we have $\frac{|CP(\mathcal{X})|}{|L(\mathcal{X})|} \leqslant \frac{p!}{2^{2^{p-2}}}$.

□

We note that $|\mathcal{X}| = 2^p$. Theorem 9.1.1 implies that the expressivity ratio of legal linear orders ($\frac{|CP(\mathcal{X})|}{|L(\mathcal{X})|}$) is $O((2^{0.2})^{-|\mathcal{X}|})$, which is exponentially small even in the number of alternatives.

9.2 H-Composition of Local Voting Rules

In this section, we introduce a new framework for composing local voting rules. We call this *hypercubewise composition (H-composition)* of local voting rules. The reason is that the outcome only depends on preferences between alternatives that differ on only one attribute. We can visualize the set of all alternatives as a hypercube, and alternatives that differ on only one attribute are neighbors on this hypercube, as discussed in Section 8.2. An H-composition of local rules is defined for all profiles in which for each vote, there exists a (possibly cyclic) CP-net that it extends. In fact, for any linear order V on $\mathcal{X}$, there exists a CP-net $\mathcal{N}$ such that V extends $\mathcal{N}$, so we can apply this to *any* linear orders (but also some partial orders). By Theorem 9.1.1, this means that the voting language used by these H-compositions (i.e., possibly cyclic CP-nets) is much more general than the voting language used by sequential voting rules (i.e., $\mathcal{O}$-compatible CP-nets for some ordering $\mathcal{O}$ over $\mathcal{I}$, in the sense we have discussed in Section 8.3.

An H-composition of local rules is defined in two steps. In the first step, an *induced graph* is generated by applying local rules to the input profile. Then, in the second step, a *choice set function* is selected based on the induced graph as the set of winners (the definition and examples of some major choice set functions are deferred to Definition 9.2.4 and the text below it). We first define the induced graph of P w.r.t. local rules (or correspondences) $r_1, \ldots, r_p$.

Definition 9.2.1. *Given a profile $P = (V_1, \ldots, V_n)$ and local rules (or correspondences) $r_1, \ldots, r_p$, the* induced graph *of P w.r.t. $r_1, \ldots, r_p$, denoted by $IG(r_1, \ldots, r_p)(P) = (\mathcal{X}, E)$, is defined by the following edges between alternatives. For any $i \leqslant p$, any setting $\overrightarrow{x_{-i}}$, let $C_i = r_i(P|_{X_i:\overrightarrow{x_{-i}}})$; for any $c_i \in C_i$, any $d_i \in D_i$, let there be an edge*

$(c_i, \overrightarrow{x_{-i}}) \rightarrow (d_i, \overrightarrow{x_{-i}})$.

Example 9.2.2. *Suppose the multi-issue domain consists of two binary attributes:* $\mathbf{S}$ *ranging over* $\{S, \bar{S}\}$ *and* $\mathbf{T}$ *ranging over* $\{T, \bar{T}\}$. *The local rules are both the majority rule. Two votes* V_1, V_2 *and their induced graph* $IG(Maj, Maj)(V_1, V_2)$ *are illustrated in Figure 9.2.2, where* Maj *denotes the majority correspondence. We note that* V_1 *is compatible with* $\mathbf{S} > \mathbf{T}$, V_2 *is compatible with* $\mathbf{T} > \mathbf{S}$.

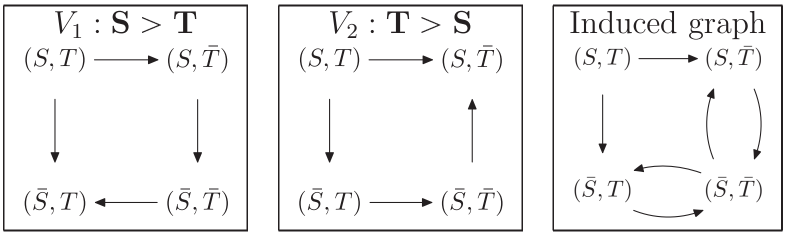

FIGURE 9.1: Two votes and their induced graph.

Next, we define the dominance relation in a directed graph.

Definition 9.2.3. *Given a directed graph* $G = (\mathcal{V}, E)$, *for any* $v_1, v_2 \in \mathcal{V}$, v_1 *is said to* dominate v_2, *denoted by* $v_1 \succ_G v_2$, *if and only if:*

1. there is a directed path from v_1 *to* v_2, *and*

2. there is no directed path from v_2 *to* v_1.

Let $\succeq_G$ be the *transitive closure* of E, that is, $\succeq_G$ is the minimum preorder such that if $(v_1, v_2) \in E$, then $v_1 \succeq_G v_2$. Then, another equivalent way to define the dominance relation is: $\succ_G$ is the strict order induced by $\succeq_G$, that is, $v_1 \succ_G v_2$ if and only if $v_1 \succeq_G v_2$ and $v_2 \not\succeq_G v_1$.

We further define two kinds of special vertices in a directed graph G as follows. The first is a vertex that dominates all the other vertices, and the second is a vertex that dominates all its neighbors. We call the former the *global Condorcet winner* (which must be unique), and the latter a *local Condorcet winner*.

Now, we are ready to define the *choice set function*, which specifies a *choice set* for each graph.

Definition 9.2.4. *A* choice set function T *is a mapping from any graph to a subset of its vertices.*

We now recall the definitions of some major choice sets in a graph $G = (V, E)$.

- The *Schwartz set* is the union of all maximal mutually connected subsets. A maximal mutually connected subset is a subset of vertices such that there is a path between any two vertices in the set, but there is no path from a vertex outside the set to a vertex inside the set.
- The *Smith set* is the smallest set of vertices such that every vertex in the set dominates all the vertices outside the set.
- The *Copeland set*: A vertex c's Copeland score is the number of vertices that are dominated by c minus the number of vertices that dominate c. The vertices with the highest Copeland score are the winners.

Choice sets were originally introduced to make group decisions for tournament graphs. However, the definitions are easily extended to general graphs, as we did above. See Laffond et al. (1995) and Brandt et al. (2007) for more discussion.

We say a choice set function T *always chooses the global Condorcet winner*, if for any graph $G = (V, E)$ in which c is the global Condorcet winner, we have $T(G) = \{c\}$. We say that T *always chooses local Condorcet winners*, if every local Condorcet winner is always in $T(G)$. We emphasize that here, the meaning of a Condorcet winner is different from traditional meaning of a Condorcet winner, which refers to an alternative that wins every pairwise election. We say that T is *monotonic*, if for any graph $(\mathcal{V}, E)$, any $c \in T(\mathcal{V}, E)$, and any $(\mathcal{V}, E')$ that is obtained from $(\mathcal{V}, E)$ by only flipping some of the incoming edges of c, we have $c \in T(\mathcal{V}, E')$.

Theorem 9.2.5 (known/easy). *The Schwartz set, Smith set, and Copeland set are monotonic and always choose the global Condorcet winner; the Smith set and Schwartz set always choose local Condorcet winners.*

We are now ready to define the H-composition of local rules (correspondences).

Definition 9.2.6. *Let T be a choice set function. The Hypercubewise-T (H-T) composition of local rules $r_1, \ldots, r_p$, denoted by $H_T(r_1, \ldots, r_p)$, is defined as follows. For any profile P of linear orders on $\mathcal{X}$,*

$$H_T(r_1, \ldots, r_p)(P) = T(IG(r_1, \ldots, r_p)(P))$$

That is, for any profile P, $H_T(r_1, \ldots, r_p)$ computes the winner in the following two steps. First, the induced graph $IG(r_1, \ldots, r_p)(P)$ is generated by applying local rules $r_1, \ldots, r_p$ to the restrictions of P to all the local domains. Then, in the second step, the set of winners is selected by the choice set function T from the induced graph $IG(r_1, \ldots, r_p)(P)$.

From Theorem 9.1.1, the fact that all linear orders are consistent with some CP-net, and all CP-nets can be used under H-composition, we know that the domain of H-composition of local rules is exponentially larger than the domain of order-independent sequential composition. We note that to build the induced graph, only the preferences between adjacent alternatives are necessary. We note that the H-composition of local rules is a correspondence, and we can use any tie-breaking mechanism to select a unique winner.

One interesting question is how H-compositions are related to (order-independent) composition of local rules. Because the H-compositions are defined by both local rules and the choice set, the relationship should also depend on local rules and the properties of the choice set. The next theorem states that if a choice set function T always chooses the global Condorcet winner, then H-T composition of local rules is an extension of order-independent sequential composition of the same local rules (Xia et al., 2007b). The *order-independent sequential composition* of local rules, denoted by $Seq^{OI}(r_1, \ldots, r_p)$, extends the domain of sequential composition of local rules to the set of all legal profiles P, which means that the order $\mathcal{O}$ is not held fixed in the definition. For any permutation σ on $\{1, \ldots, p\}$, let $\mathcal{O} = X_{\sigma(1)} > \ldots > X_{\sigma(p)}$. Then, for any $\mathcal{O}$-legal profile P, $Seq^{OI}(r_1, \ldots, r_p)(P) = Seq_{\mathcal{O}}(r_{\sigma(1)}, \ldots, r_{\sigma(p)})(P)$. The order-independent sequential composition of local correspondences is defined similarly. This voting rule is well-defined because it has been shown in Lang (2007) that the winner does not depend on which ordering $\mathcal{O}$ that is used in the definition, as long as the profile is $\mathcal{O}$-legal.

Theorem 9.2.7. *Let T be a choice set function that always chooses the global Condorcet winner. Then, for all legal profiles P, $H_T(r_1, \ldots, r_p)(P) = Seq^{OI}(r_1, \ldots, r_p)(P)$.*

The proof is quite straightforward and is thus omitted.

Corollary 9.2.8. *If T is the Schwartz set, Smith set, or Copeland set, then $H_T(r_1, \ldots, r_p)$ is an extension of $Seq^{OI}(r_1, \ldots, r_p)$.*

9.3 Local vs. Global Properties

In this section we examine the "quality" of the H-compositions of local rules in terms of whether they satisfy some common voting axioms described in Section 2.2. We recall that in Section 8.3 we have asked a similar question for sequential voting rules, and whether sequential voting rule satisfies some desired axiomatic properties depends on whether the local voting rules satisfy these axiomatic properties. Lang and Xia (2009) asked the following two questions for any axiomatic property Y, and the answers are summarized in Table 8.2.

1. If the sequential voting rule satisfies Y, is it true that all its local voting rules satisfy Y?

2. If the sequential voting rule satisfies Y, is it true that all its local voting rules satisfy Y?

For H-composition of local rules, we can ask the same question. From Theorem 9.2.7 we know that if T always chooses the global Condorcet winner, then H_T is an extension of Seq^{OI}. We can use this observation to carry over some of the results in Lang (2007); Xia et al. (2007a,b) to H_T. Specifically, if T always chooses the global Condorcet winner, and if a criterion transfers from the order-independent sequential composition of local rules to each local rule, then it also transfers for H-T composition; if a criterion does not transfer from local rules to their order-independent sequential composition, then it also does not transfer for H-T composition. Given the results in Xia et al. (2007b), these observations allow us to resolve everything except how anonymity, homogeneity, monotonicity, and consistency transfer from local rules to their H-T composition. It is easy to see that anonymity and homogeneity always transfer. The next example shows that if T always chooses local Condorcet winners, then consistency does not transfer, even when the votes in the profile extend (possibly different) acyclic CP-nets.

Example 9.3.1. *Let* $\mathcal{X} = \{0_1, 1_1\} \times \{0_2, 1_2\} \times \{0_3, 1_3\}$, *and let all the local rules be the majority rule. Consider the following three CP-nets (the non-specified parts of the CPTs do not matter):*

$\mathcal{N}_1$: *compatible with* $X_1 > X_2 > X_3$, *and* $1_1 \succ 0_1$, $1_1 : 1_2 \succ 0_2$, $1_1 1_2 : 1_3 \succ 0_3$, $0_1 : 0_2 \succ 1_2$, $0_1 0_2 : 0_3 \succ 1_3$.

$\mathcal{N}_2$: *compatible with* $X_2 > X_3 > X_1$, *and* $1_2 \succ 0_2$, $1_2 : 1_3 \succ 0_3$, $1_2 1_3 : 1_1 \succ 0_1$, $0_2 : 0_3 \succ 1_3$, $0_2 0_3 : 0_1 \succ 1_1$.

$\mathcal{N}_1$: *compatible with* $X_3 > X_1 > X_2$, *and* $1_3 \succ 0_3$, $1_3 : 1_1 \succ 0_1$, $1_3 1_1 : 1_2 \succ 0_2$, $0_3 : 0_1 \succ 1_1$, $0_3 0_1 : 0_2 \succ 1_2$.

For any V_1, V_2, V_3 extending $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$, respectively, let $P = (V_1, V_2, V_3)$. Let $H_T(M) = H_T(Maj, Maj, Maj)$. Suppose ties are broken in favor of $0_1 0_2 0_3$. Because $0_1 0_2 0_3$ is a local Condorcet winner, so $H_T(M)(P) = 0_1 0_2 0_3$. However, $H_T(M)(V_1) = H_T(M)(V_2) = H_T(M)(V_3) = 1_1 1_2 1_3$, so $H_T(M)$ does not satisfy consistency (because otherwise, we must have $H_T(M) = 1_1 1_2 1_3$, which we know is not the case).

The next proposition states that for any monotonic choice set function T, the monotonicity is transferred from local rules to their H-T composition. The proof is quite straightforward and is omitted.

Proposition 9.3.2. *Let T be a monotonic choice set function. If all local rules $\{r_1, \ldots, r_p\}$ satisfy monotonicity, then $H_T(r_1, \ldots, r_p)$ also satisfies monotonicity.*

For choice sets T that always choose the global Condorcet winner, whether properties of local rules transfer to their H-T composition and vice versa is summarized in Table 9.1.

Table 9.1: Local vs. global for H-compositions.

Criteria	Global to local	Local to global
Anonymity	Y	Y
Homogeneity	Y	Y
Neutrality	Y	N
Monotonicity	Y	Y for monotonic T
Consistency	Y	N if T always chooses local Condorcet winner
Participation	Y	N
Pareto efficiency	Y	N

9.4 Computing H-Schwartz Winners

Among all choice sets, we are most interested in the Schwartz set, because first, it has been argued that the Schwartz set is the "largest" reasonable choice set for tournaments (Laslier, 1997), and second, it corresponds to the *nondominated set* previously considered in the context of CP-nets (Boutilier et al., 2004). In this section, we investigate the computational complexity of computing H-Schwartz winners. We note that in this section H-Schwartz is a voting correspondence. Recent work on the complexity of computing dominance relations in CP-nets shows that the *dominance* problem in a CP-net is hard (Goldsmith et al., 2008). More precisely, given a CP-net $\mathcal{N}$ and two alternatives a and b, it is PSPACE-complete to compute whether or not $a >_{\mathcal{N}} b$. This can be used to show that checking membership in the Schwartz set is PSPACE-complete (Goldsmith et al., 2008).

Although computing the Schwartz set is hard in general, if the preferences are more structured it can be easy. As an extreme example, if the voters' preferences extend an acyclic CP-net $\mathcal{N}$, then H-Schwartz is equivalent to order-independent sequential composition of local rules, under which computing the winner is easy. In this section, we introduce a technique to exploit more limited independence information in the submitted votes for the purpose of computing the set of H-Schwartz winners.

Definition 9.4.1. *Let* $\{\mathcal{I}_1, \ldots, \mathcal{I}_q\}$ *($q \leqslant p$) be a partition of the set of issues* $\mathcal{I}$. *We say a CP-net* $\mathcal{N}$ *whose graph is* G *is* compatible *with the ordering* $\mathcal{I}_1 > \cdots > \mathcal{I}_q$ *if for any* $l \leqslant q$ *and any* $X \in \mathcal{I}_l$, $Par_G(X) \subseteq \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_l$. *A linear order* V *is compatible with* $\mathcal{I}_1 > \ldots > \mathcal{I}_q$ *if there exists a CP-net* $\mathcal{N}$ *such that* V *extends* $\mathcal{N}$ *and* $\mathcal{N}$ *is compatible with* $\mathcal{I}_1 > \ldots > \mathcal{I}_q$.

Let $\mathcal{O} = X_1 > \ldots > X_p$. One special case is the following: if the input profile is $\mathcal{O}$-legal, then we can use the partition $\mathcal{I}_1 = \{X_1\}, \ldots, \mathcal{I}_p = \{X_p\}$. We can use the following algorithm to find a partition with which the input profile P is compatible. Suppose that we already know the graphs of the CP-nets that the votes in P extend.

Algorithm 1

1. Let G_P be the union of all the graphs of the CP-nets that the votes in P extend.

2. Let $q = 0$; repeat step 3 until $G_P = \emptyset$.

3. Let $q \leftarrow q+1$. Find a maximal mutually connected subset of G_P, and call it $\mathcal{I}_q$. Remove $\mathcal{I}_q$ and all edges connecting it to G_P.

4. Output the partition $\mathcal{I}_1 \cup \ldots \cup \mathcal{I}_q$.

This algorithm runs in time $O(p^3)$. Now we are ready to present the technique for computing the set of H-Schwartz winners more efficiently. Suppose the set of attributes can be partitioned into $\mathcal{I}_1 \cup \mathcal{I}_2$ so that P is compatible with $\mathcal{I}_1 > \mathcal{I}_2$. Let $r_{\mathcal{I}_1}$ denote the sub-vector of $(r_1, \ldots, r_p)$ that contains the local rules r_i if and only if $X_i \in \mathcal{I}_1$.

Process 1

1. Compute the Schwartz set $H_{Schwartz}(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1}) = W_1^1 \cup \ldots \cup W_1^k$, where the W_1^i are the maximal mutually connected subsets in $IG(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$.

2. For each $i \leqslant k$, let $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:W_1^i}) = \bigcup_{\vec{w} \in W_1^i} IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:\vec{w}})$; then, compute the Schwartz set W_2^i for $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:W_1^i})$.

3. Output $W_p = \bigcup_{i=1}^k W_1^i \times W_2^i$.

The next theorem states that we can compute the winners of $H_{Schwartz}(r_1, \ldots, r_p)(P)$ by Process 1.

Theorem 9.4.2. $W_P = H_{Schwartz}(r_1, \ldots, r_p)(P)$.

Proof. Let $\vec{w}_2$ be a setting of $\mathcal{I}_2$ and $\vec{w}_1, \vec{w}_1'$ be settings of $\mathcal{I}_1$ such that $\vec{w}_1$ and $\vec{w}_1'$ differ only on one attribute. Since P is compatible with $\mathcal{I}_1 > \mathcal{I}_2$, we have that there is an edge from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}_1', \vec{w}_2)$ in $IG(r_{\mathcal{I}})(P)$ if and only if there is an edge from $\vec{w}_1$ to $\vec{w}_1'$ in $IG(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$. This implies the following claim.

Claim 9.4.1. *If there is a path from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}_1', \vec{w}_2')$ in $IG(r_{\mathcal{I}})(P)$, then its projection on $\mathcal{I}_1$ is a path from $\vec{w}_1$ to $\vec{w}_1'$ in $IG(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$.*

Proof of Claim 9.4.1: W.l.o.g. we only need to prove the case where there is an edge from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}_1', \vec{w}_2')$ in $IG(r_{\mathcal{I}})(P)$. Because only neighboring alternatives are connected in $IG(r_{\mathcal{I}})(P)$, either $\vec{w}_1 = \vec{w}_1'$ or $\vec{w}_2 = \vec{w}_2'$. If $\vec{w}_1 = \vec{w}_1'$ then the claim is automatically proved, because the projections of the two alternatives in $IG(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$ are the same (that is, $\vec{w}_1$). If $\vec{w}_2 = \vec{w}_2'$, then by definition there is an from $\vec{w}_1$ to $\vec{w}_1'$ in $IG(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$. This proves the claim. □

We note that for any $i \leqslant k$, any $\vec{w}_1, \vec{w}_1' \in W_1^i$ such that there is a path from $\vec{w}_1$ to $\vec{w}_1'$, and any $\vec{w}_2 \in D_{\mathcal{I}_2}$, there is a path from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}_1', \vec{w}_2)$. Therefore, we have the following claim.

Claim 9.4.2. *For any $i \leqslant k$, any $(\vec{w}_1, \vec{w}_2), (\vec{w}_1', \vec{w}_2') \in W_1^i \times D_{\mathcal{I}_2}$, there is a path from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}_1', \vec{w}_2')$ if and only if there is a path from $\vec{w}_2$ to $\vec{w}_2'$ in $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:W_1^i})$.*

Proof of Claim 9.4.2: We first prove the "only if" part. W.l.o.g. we only need to prove the case where there is an edge from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}_1', \vec{w}_2')$ in $IG(r_{\mathcal{I}})(P)$. In this case either $\vec{w}_1 = \vec{w}_1'$ or $\vec{w}_2 = \vec{w}_2'$. If $\vec{w}_1 = \vec{w}_1'$, then there is an edge from $\vec{w}_2$ to $\vec{w}_2'$ in $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:\vec{w}_1})$, which means that there is an edge from $\vec{w}_2$ to $\vec{w}_2'$ in $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:W_1^i})$. If $\vec{w}_2 = \vec{w}_2'$, then the claim is automatically proved.

Now we prove the "if" part. W.l.o.g. we only need to prove the case where there is an edge from $\vec{w}_2$ to $\vec{w}_2'$ in $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:W_1^i})$. By definition of $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:W_1^i})$, there exists $\vec{w}_1^* \in W_1^i$ such that there is an edge from $\vec{w}_2$ to $\vec{w}_2'$ in $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:\vec{w}_1^*})$, which means that there is an edge from $(\vec{w}_1^*, \vec{w}_2)$ to $(\vec{w}_1^*, \vec{w}_2')$ in $IG(r_{\mathcal{I}})(P)$. Because W_1^i is a maximum mutually connected set, there exist a path from $\vec{w}_1$ to $\vec{w}_1^*$ and another path from $\vec{w}_1^*$ to $\vec{w}_1'$ in $IG(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$. Because P is compatible with $\mathcal{I}_1 > \mathcal{I}_2$, there exist two paths in $IG(r_{\mathcal{I}})(P)$, one from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}_1^*, \vec{w}_2)$ and the other from $(\vec{w}_1^*, \vec{w}_2')$ to $(\vec{w}_1', \vec{w}_2')$. These two paths can be connected by the edge from $(\vec{w}_1^*, \vec{w}_2)$ to $(\vec{w}_1^*, \vec{w}_2')$ to form a path from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}_1', \vec{w}_2')$. □

Based on Claim 9.4.1 and Claim 9.4.1 we are now ready to prove that $W_P \subseteq H_{Schwartz}(r_{\mathcal{I}})(P)$ and $H_{Schwartz}(r_{\mathcal{I}})(P) \subseteq W_P$, which mean that $W_P = H_{Schwartz}(r_{\mathcal{I}})(P)$.

We first prove that $W_P \subseteq H_{Schwartz}(r_{\mathcal{I}})(P)$. Equivalently, we need to prove that for any $(\vec{w}_1, \vec{w}_2) \in W_P$, there is no alternative $(\vec{w}'_1, \vec{w}'_2) \in \mathcal{X}$ such that (1) there is a path from $(\vec{w}'_1, \vec{w}'_2)$ to $(\vec{w}_1, \vec{w}_2)$, (2) there is no path from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}'_1, \vec{w}'_2)$. We prove this by contradiction. Suppose there exists $(\vec{w}'_1, \vec{w}'_2)$ that satisfies the above two conditions. Suppose $\vec{w}_1 \in W^i_1$. By Claim 9.4.1, there is a path from $\vec{w}'_1$ to $\vec{w}_1$ in $IG(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$. Because W^i_1 is a maximum mutually connected set, $\vec{w}'_1 \in W^i_1$. By Claim 9.4.2 there exists a path from $\vec{w}'_2$ to $\vec{w}_2$ in $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:W^i_1})$. Because $\vec{w}_2 \in W^i_2$, there must exist a path from $\vec{w}_2$ to $\vec{w}'_2$ in $IG(r_{\mathcal{I}_2})(P|_{\mathcal{I}_2:W^i_1})$. Now, by Claim 9.4.2 there exists a path from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}'_1, \vec{w}'_2)$, which contradicts the condition (2) above.

Next, we prove that $H_{Schwartz}(r_{\mathcal{I}})(P) \subseteq W_P$. Let $(\vec{w}_1, \vec{w}_2) \in H_{Schwartz}(r_{\mathcal{I}})(P)$. We first show that $\vec{w}_1 \in H_{Schwartz}(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$. Suppose for the sake of contradiction $\vec{w}_1 \notin H_{Schwartz}(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$, then there exists $\vec{w}'_1 \in D_{\mathcal{I}_1}$ such that (1) there is a path from $\vec{w}'_1$ to $\vec{w}_1$ in $IG(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$, and (2) there is no path from $\vec{w}_1$ to $\vec{w}'_1$ in $IG(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$. From (1) we know that there is a path from $(\vec{w}'_1, \vec{w}_2)$ to $(\vec{w}_1, \vec{w}_2)$. From (2) we know that there is no path from $(\vec{w}_1, \vec{w}_2)$ to $(\vec{w}'_1, \vec{w}_2)$, because otherwise by Claim 9.4.1 there is a path from $\vec{w}_1$ to $\vec{w}'_1$, which is a contradiction. It follows that $(\vec{w}_1, \vec{w}_2)$ is dominated by $(\vec{w}'_1, \vec{w}_2)$, which contradicts the assumption that $(\vec{w}_1, \vec{w}_2) \in H_{Schwartz}(r_{\mathcal{I}})(P)$. Therefore, $\vec{w}_1 \in H_{Schwartz}(r_{\mathcal{I}_1})(P|_{\mathcal{I}_1})$.

Now, suppose $\vec{w}_1 \in W^i_1$. If $\vec{w}_2 \notin W^i_2$, then there exists $\vec{w}'_2 \in W^i_2$ that dominates $\vec{w}_2$. However, it follows from Claim 9.4.2 that $(\vec{w}_1, \vec{w}'_2)$ dominates $(\vec{w}_1, \vec{w}_2)$ in $IG(r_{\mathcal{I}})(P)$, which contradicts the assumption that $(\vec{w}_1, \vec{w}_2) \in H_{Schwartz}(r_{\mathcal{I}})(P)$. It follows that $\vec{w}_2 \in W^i_2$, which means that $(\vec{w}_1, \vec{w}_2) \in W_P$.

Therefore, $W_P = H_{Schwartz}(r_{\mathcal{I}})(P)$, which completes the proof. □

If the decomposition is $\mathcal{I}_1 > \ldots > \mathcal{I}_q$ with $q > 2$, then Process 1 can be applied recursively to find the Schwartz set, as follows. First, compute the Schwartz set over $\mathcal{I}_1 \times \mathcal{I}_2$ by Process 1, then use this result to compute the Schwartz set over $(\mathcal{I}_1 \times \mathcal{I}_2) \times \mathcal{I}_3$, etc. up to $(\mathcal{I}_1 \times \ldots \times \mathcal{I}_{q-1}) \times \mathcal{I}_q$.

The next example shows how Process 1 works.

Example 9.4.3. *Let $\mathcal{X} = \{0,1\}^3$, and let three votes V_1, V_2, V_3 extend three CP-nets such that V_1 is $(X_1 > X_2 > X_3)$-legal, V_2 is $(X_2 > X_1 > X_3)$-legal, and V_3 is separable. Let the partition be $\mathcal{I}_1 = \{X_1, X_2\}$, $\mathcal{I}_2 = \{X_3\}$. Then, for all $i = 1,2,3$, V_i is compatible with $\mathcal{I}_1 > \mathcal{I}_2$. Suppose that $\{\vec{w}_1, \vec{w}'_1\} = H_{Schwartz}(r_1, r_2)(P|_{\{X_1,X_2\}})$, so that there is no path from $\vec{w}_1$ to $\vec{w}'_1$, and vise versa. Also suppose that $\{\vec{w}_2\} = H_{Schwartz}(r_3)(P|_{X_3:X_{-3}=\vec{w}_1})$ and $\{\vec{w}'_2\} = H_{Schwartz}(r_3)(P|_{X_3:X_{-3}=\vec{w}'_1})$. Then, the winners are $(\vec{w}_1, \vec{w}_2)$ and $(\vec{w}'_1, \vec{w}'_2)$.*

The next theorem states that if P is compatible with $\mathcal{I}_1 > \ldots > \mathcal{I}_q$, then the time required to compute the set of Schwartz winners by applying Process 1 is a polynomial function of the number of winners, the longest time it takes to apply local rules, p, n, and $\max |D_{\mathcal{I}_i}|$.

Theorem 9.4.4. *Suppose an n-vote profile P is compatible with $\mathcal{I}_1 > \ldots > \mathcal{I}_q$. Let $d_{max} = \max_{i \leqslant q} |D_{\mathcal{I}_q}|$. Let $t_{max}(n)$ be the longest time it takes to apply local rules on n inputs. Then, the running time of Process 1 is $O(apd_{max}(np + t_{max}(n)p + d_{max}))$, where a is the number of H-Schwartz winners.*

Usually $t_{max}(n)$ is polynomial. Therefore, the computational complexity of Process 1 mainly comes from the number of H-Schwartz winners, and the size of the largest partition d_{max}.

9.5 Summary

Sequential voting rules require the voters' preferences to extend acyclic CP-nets compatible with a common order on the attributes. We showed that this requirement is very restrictive, by proving that the number of linear orders extending an acyclic CP-net is exponentially smaller than the number of all linear orders. This means that the voting language used in sequential voting rules lacks general usability. To overcome this, in this chapter we introduced a very general methodology that allows us to aggregate preferences when voters express CP-nets that can be cyclic. There does not need to be any common structure among the submitted CP-nets. We studied whether properties of the local rules transfer to the global rule, and vice versa. We also addressed how to compute the winning alternatives.