CSCI-4110/6110: Computational Social Processes, 2016 Fall
Homework 2

You can either upload your solution to LMS or hand in an hard copy (please do not do both). If you have questions about the statements please post on Piazza.

Problem 1. ILP for Kemeny (2pts) Prove that it suffices to check conditions for all cycles of length 3 in the ILP of Kemeny. I.e. if x_{ab}'s do not correspond to a linear order then one of the constraints must be violated.

Problem 2. Manipulation (8pts) Suppose agent 1 is a manipulator, whose preferences are $a \succ b \succ c \succ d$. Let P_{-1} denote the votes of other agents.

$$P_{-1} = 1@[b \succ c \succ d \succ a] + 2@[c \succ d \succ a \succ b] + 1@[d \succ a \succ b \succ c]$$

For each of the following rule, either show a beneficial manipulation of agent 1, or say that agent 1 has no incentive to manipulate (no proof is needed for the latter). Notice that **a beneficial manipulation means that agent 1 must prefers the new winner to the old one w.r.t. to her true preferences** $a \succ b \succ c \succ d$**.** Ties are broken alphabetically in favor of alternatives with higher priority (for example, when eliminating tied alternatives, those with lower priority will be eliminated).

1. Plurality.

2. Borda.

3. Veto.

4. Plurality with runoff.

5. STV.

6. Bucklin.

7. Copeland.

8. Ranked Pairs.

Problem 3. Single-Peaked Preferences (3pts)

1. Is $b \succ c \succ a \succ d$ compatible with the social axis $a \rhd b \rhd c \rhd d$? (The best way to verify this is to draw the plot as we did in the class).

2. Is $a \succ c \succ d \succ b$ compatible with the social axis $a \rhd b \rhd c \rhd d$?

3. Prove that if a linear order V is compatible with a social axis S if and only if it is compatible with the reserves ranking $rev(S)$. For example, if $S = a \rhd b \rhd c$ then $rev(S) = c \rhd b \rhd a$. **Your proof should work for all S, not just this example. Notice that it must be a formal proof by verifying the definition of single-peakedness. Drawing the plot is usually not a valid formal proof.**

Problem 4. Mallows' model (3pts) Prove that the MLE of Mallows' model outputs a ranking with minimum KentallTau distance to the profile. This holds for arbitrary $0 < \varphi < 1$.

Problem 5. Mallows' model (4pts) Let the data be generated from Mallows' model. Let

$$P = 1@[b \succ a \succ c] + 1@[c \succ a \succ b] + 1@[a \succ b \succ c]$$

1. Calculate the probability distribution for all linear orders given ground truth parameter $a \succ b \succ c$. Each probability should be a function of φ and the normalization factor.

2. Calculate the likelihood of all parameters when the data is P. What is the outcome of MLE?

3. Suppose the prior is $\pi(b \succ a \succ c) = \pi(b \succ c \succ a) = q$, and the other linear orders have the same prior probability. Write down the posterior distribution (as a function of φ, q, and any normalization factor you want to introduce).

4. Continuing: Suppose $\varphi = 0.6$. What is the smallest q such that the MAP chooses a linear order where b is ranked at the top?

Problem 6. Plackett-Luce model (4pts) Let the parameter space be $\Theta = \{(1, 4, 5), (4, 3, 3), (2, 7, 1)\}$, where $(1, 4, 5)$ means that the λ values for a, b, c are $1, 4, 5$, respectively. Let P denote the same profile as in the previous question, i.e.

$$P = 1@[b \succ a \succ c] + 1@[c \succ a \succ b] + 1@[a \succ b \succ c]$$

1. Calculate the probability distribution over all linear orders given ground truth parameter $(4, 3, 3)$.

2. Calculate the likelihood of all three parameters in Θ when the data is P. What is the outcome of MLE?

3. Suppose the prior is $\pi(1, 4, 5) = q$, and the other two parameters have the same prior probability $(1 - q)/2$. Write down the posterior distribution over Θ as a function of q and any normalization factor.

4. Continuing: what is the smallest q such that the MAP chooses $(1, 4, 5)$?

Problem 7. ILP bonus question (5pts) Given $m \in \mathbb{N}$ and m positional scoring rules with scoring vectors $(\vec{s^1}, \vec{s^2}, \ldots, \vec{s^m})$, where $\vec{s^i} = (s^i_1, \ldots, s^i_m)$.

Design a mixed integer programming to find a profile P with the smallest number of votes so that all these m positional scoring rules output different winners. **Your ILP should use only polynomially many (in m and n) variables and constraints.**

Remarks: The ILP should be able to identify "failures", that is, situations where such a profile does not exist. You don't need to write down all constraints explicitly, but make sure that every parameter you use in the ILP is well defined.

Hint: Check out "Doubly stochastic matrix" and "Birkhoffvon Neumann theorem" at `http://en.wikipedia.org/wiki/Doubly_stochastic_matrix`