Matching

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Nobel prize in Economics 2013

Alvin E. Roth

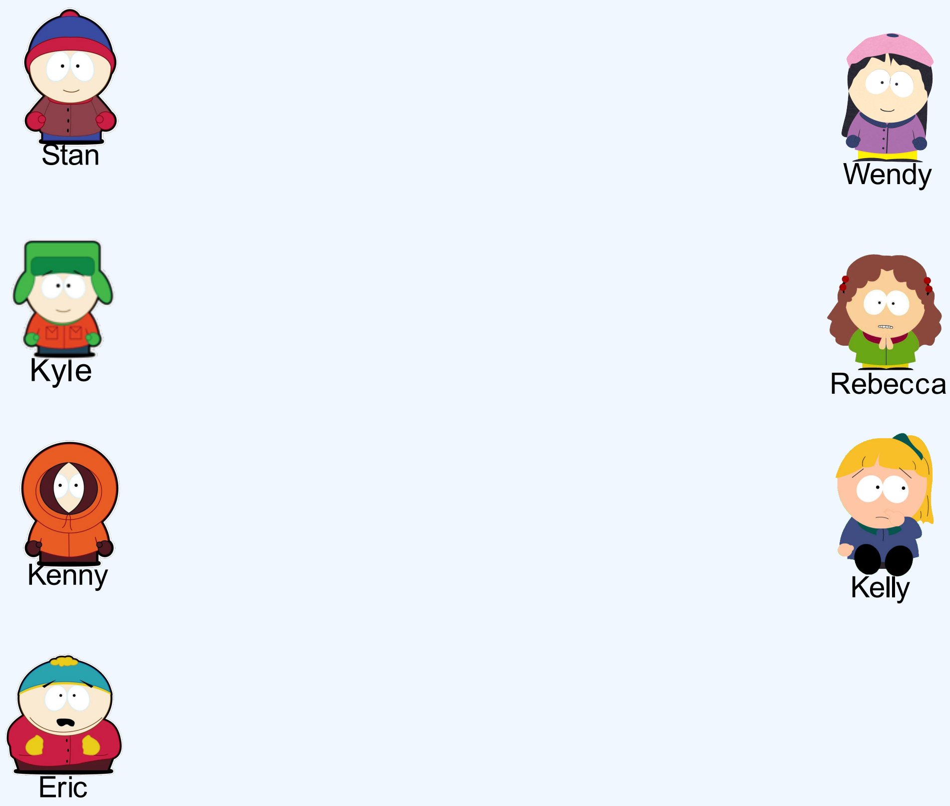
Lloyd Shapley

- "for the theory of stable allocations and the practice of market design."

Two-sided one-one matching

Boys

Girls

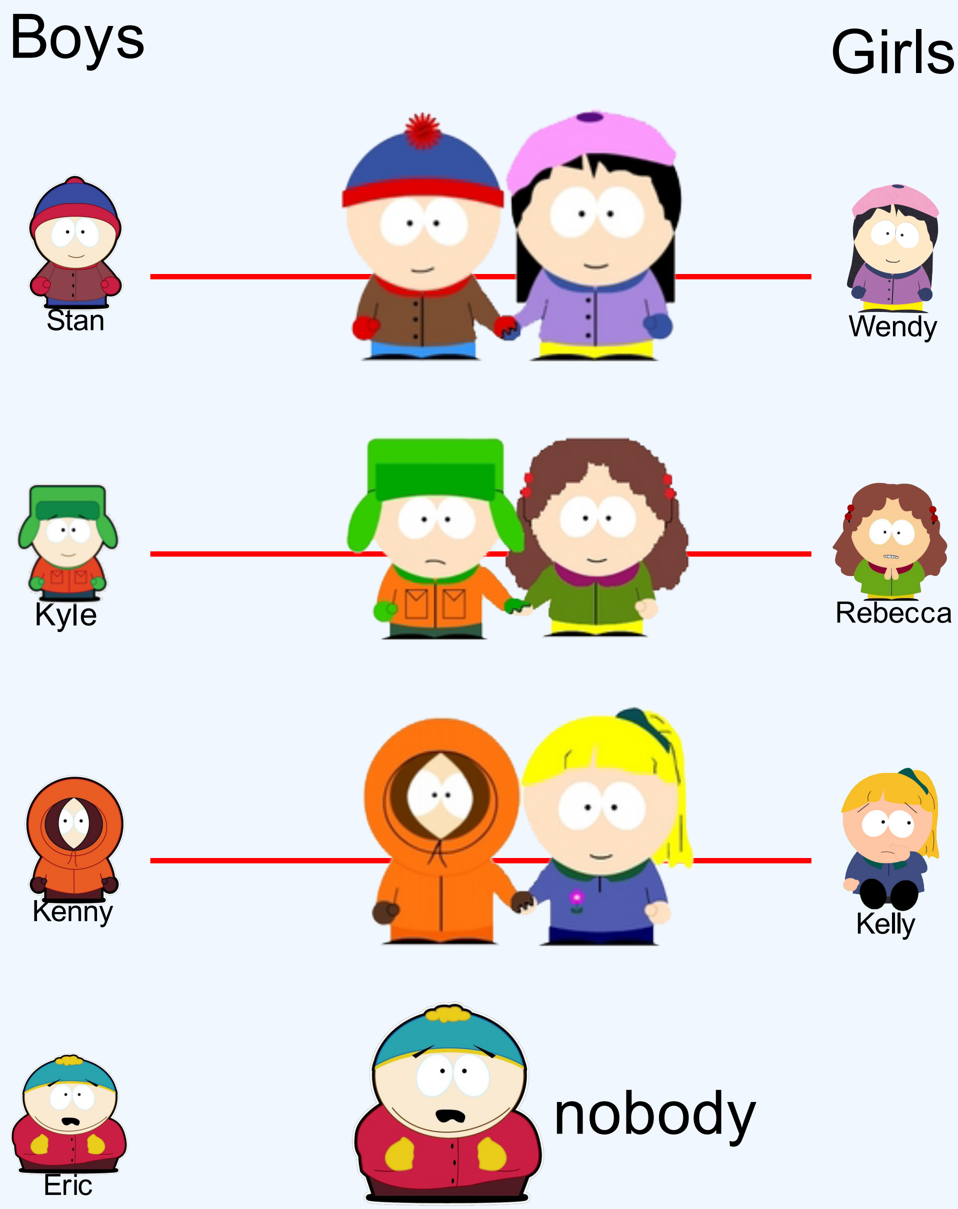


Applications: student/hospital, National Resident Matching Program

Formal setting

- Two groups: B and G
- Preferences:
  - members in B: full ranking over $G \cup$ {nobody}
  - members in G: full ranking over $B \cup$ {nobody}
- Outcomes: a matching M: $B \cup G \rightarrow B \cup G \cup$ {nobody}
  - M(B) $\subseteq G \cup$ {nobody}
  - M(G) $\subseteq B \cup$ {nobody}
  - [M(a)=M(b)≠nobody] $\Rightarrow$ [a=b]
  - [M(a)=b] $\Rightarrow$ [M(b)=a]

Example of a matching



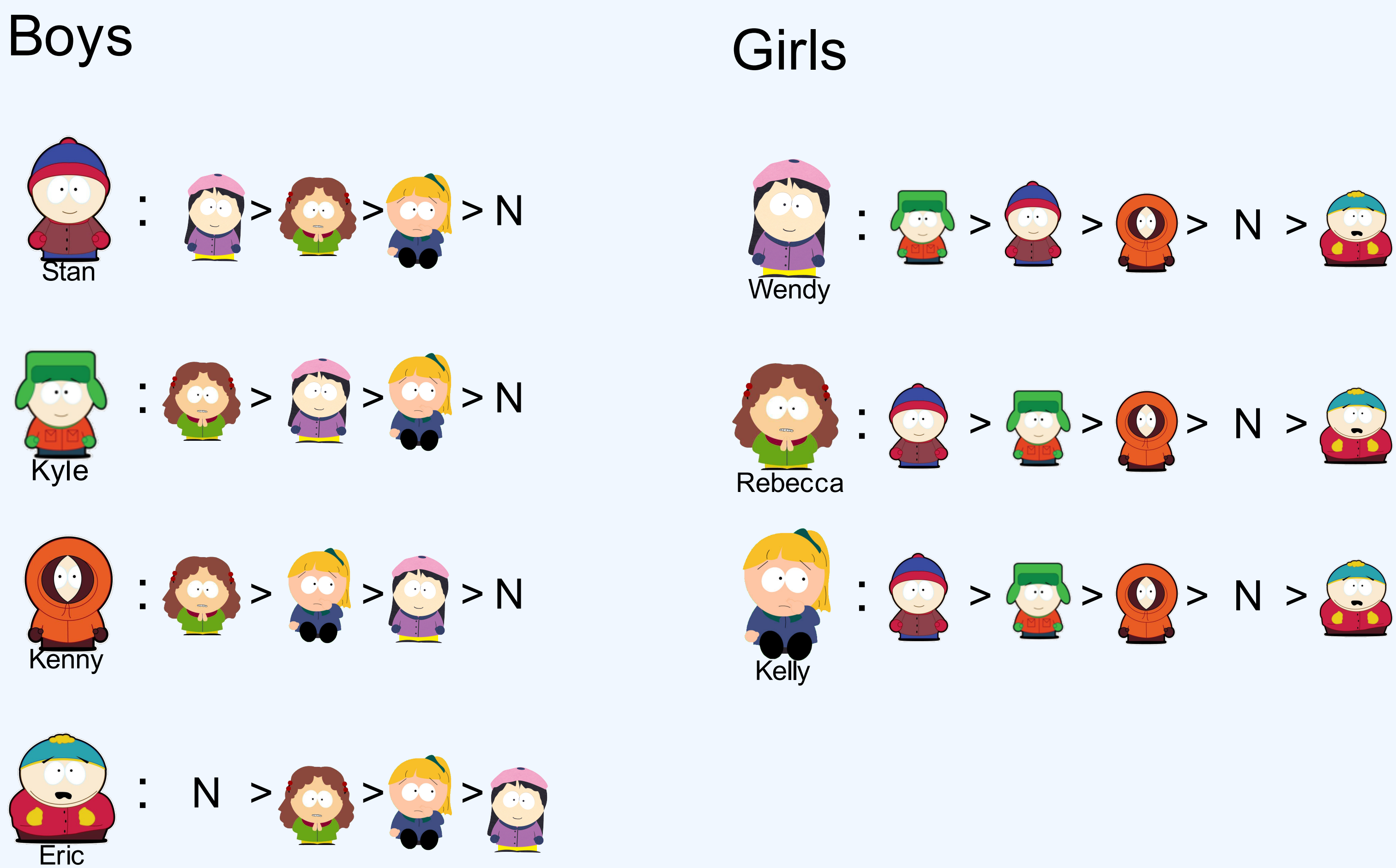
Good matching?

- Does a matching always exist?
  - apparently yes
- Which matching is the best?
  - utilitarian: maximizes “total satisfaction”
  - egalitarian: maximizes minimum satisfaction
  - but how to define utility?

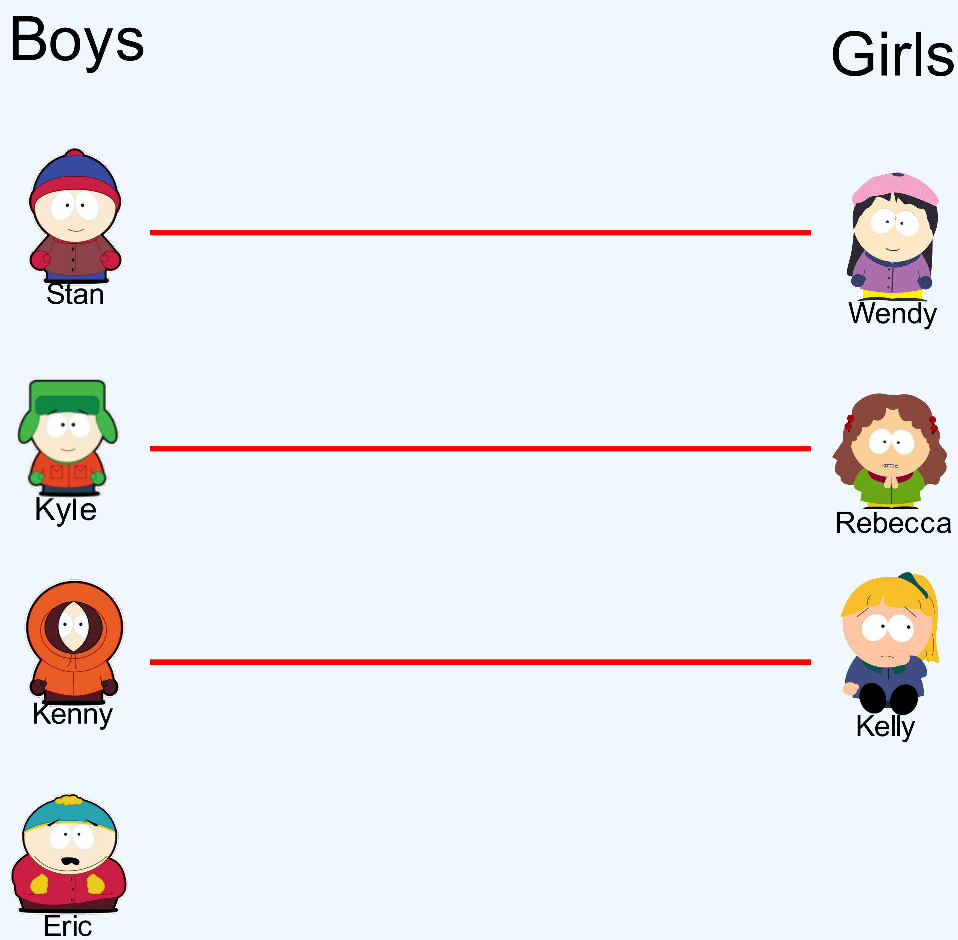
Stable matchings

- Given a matching M, (b,g) is a blocking pair if
  - $g >_b M(b)$
  - $b >_g M(g)$
  - ignore the condition for nobody
- A matching is stable, if there is no blocking pair
  - no (boy,girl) pair wants to deviate from their currently matches

Example

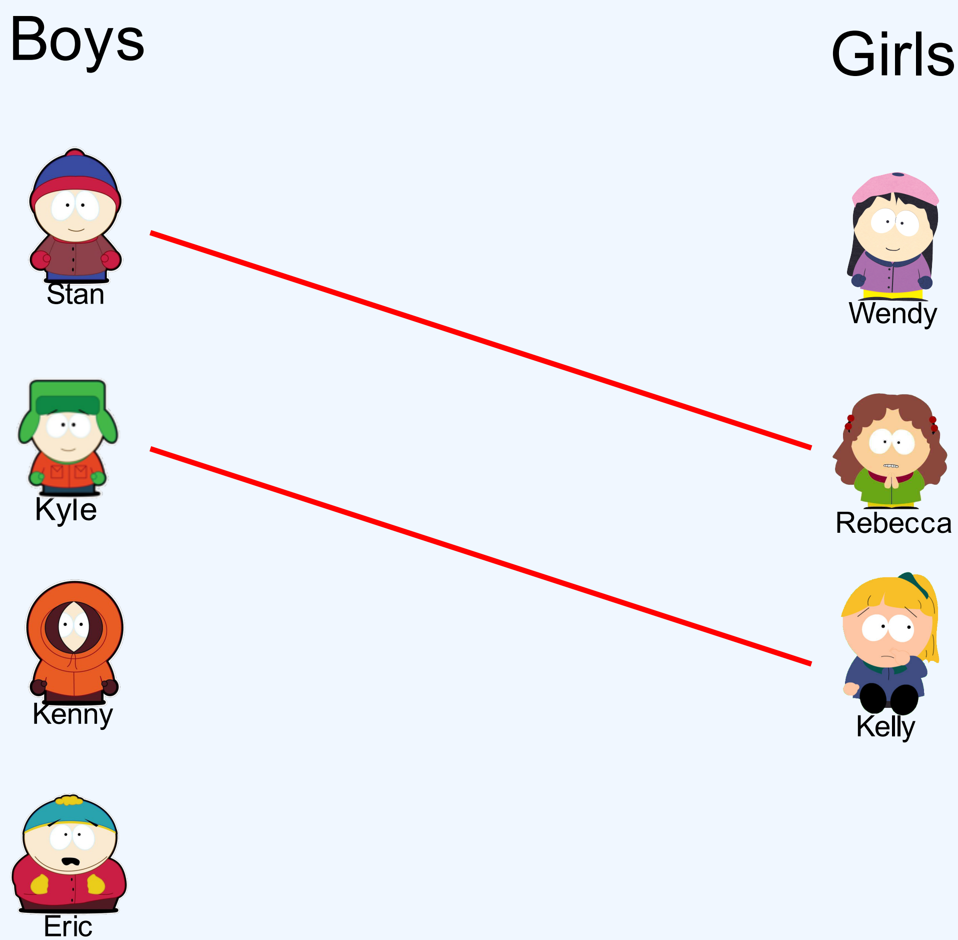

Boys
Girls
Stan
: > > > N
Kyle
: > > > N
Kenny
: > > > N
Eric
: N > > >
Wendy
: > > > N >
Rebecca
: > > > N >
Kelly
: > > > N >

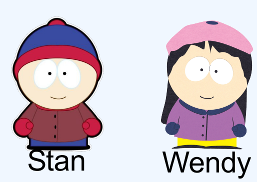
A stable matching



no link = matched to “nobody”

An unstable matching



Blocking pair: (

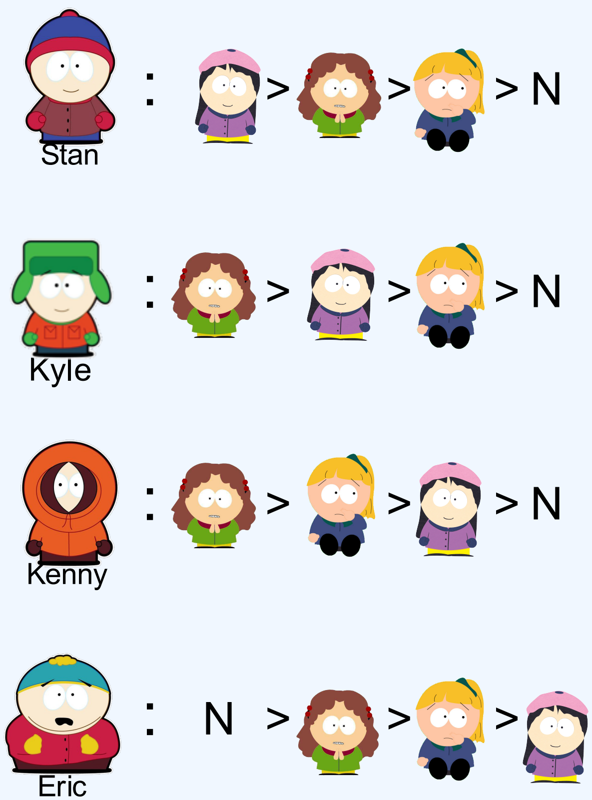
)

Does a stable matching always exist?

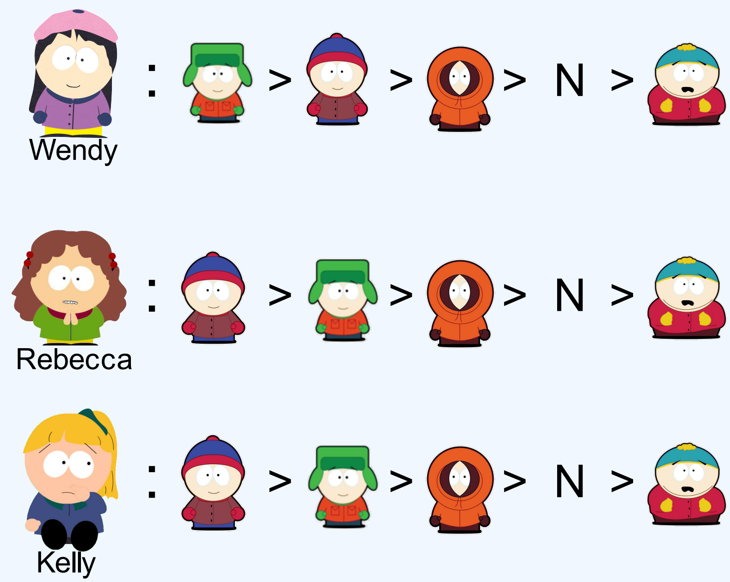
- Yes: Gale-Shapley's deferred acceptance algorithm (DA)
- Men-proposing DA: each girl starts with being matched to "nobody"
  - each boy proposes to his top-ranked girl (or "nobody") who has not rejected him before
  - each girl rejects all but her most-preferred proposal
  - until no boy can make more proposals
- In the algorithm
  - Boys are getting worse
  - Girls are getting better

Men-proposing DA (on blackboard)

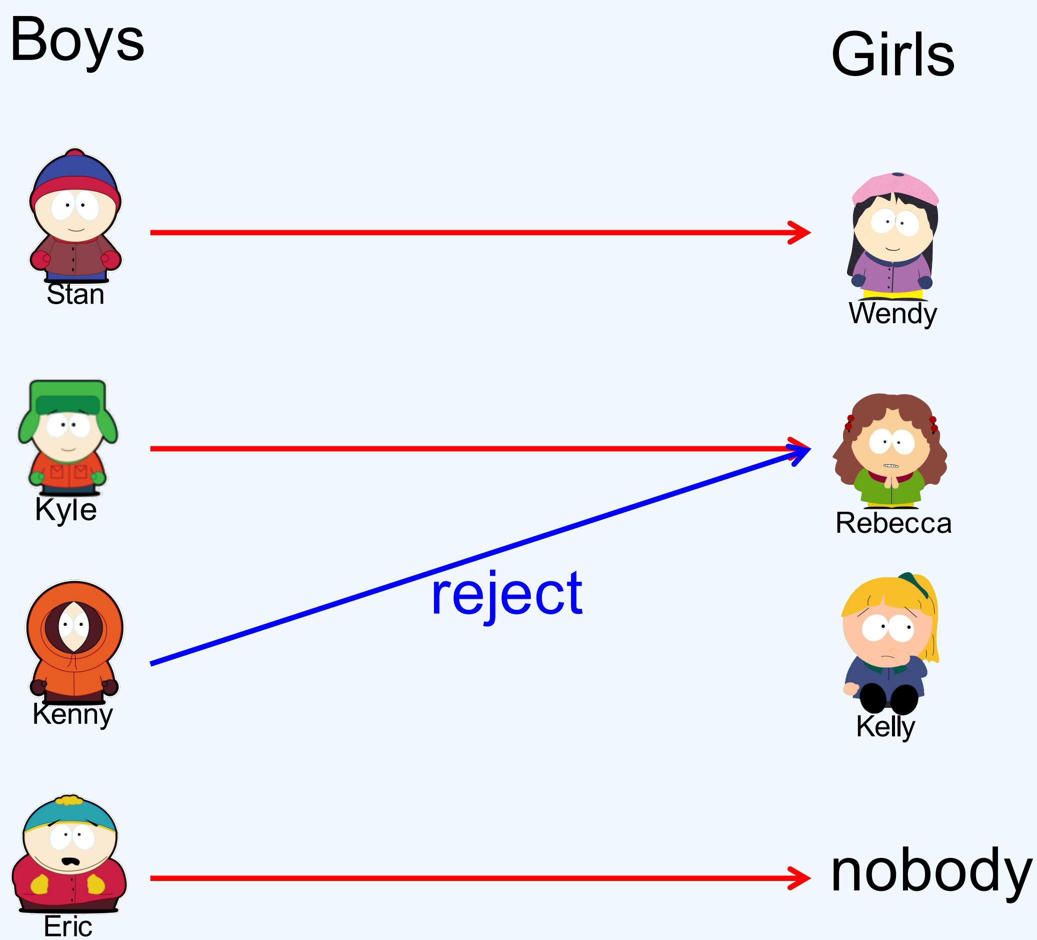
Boys



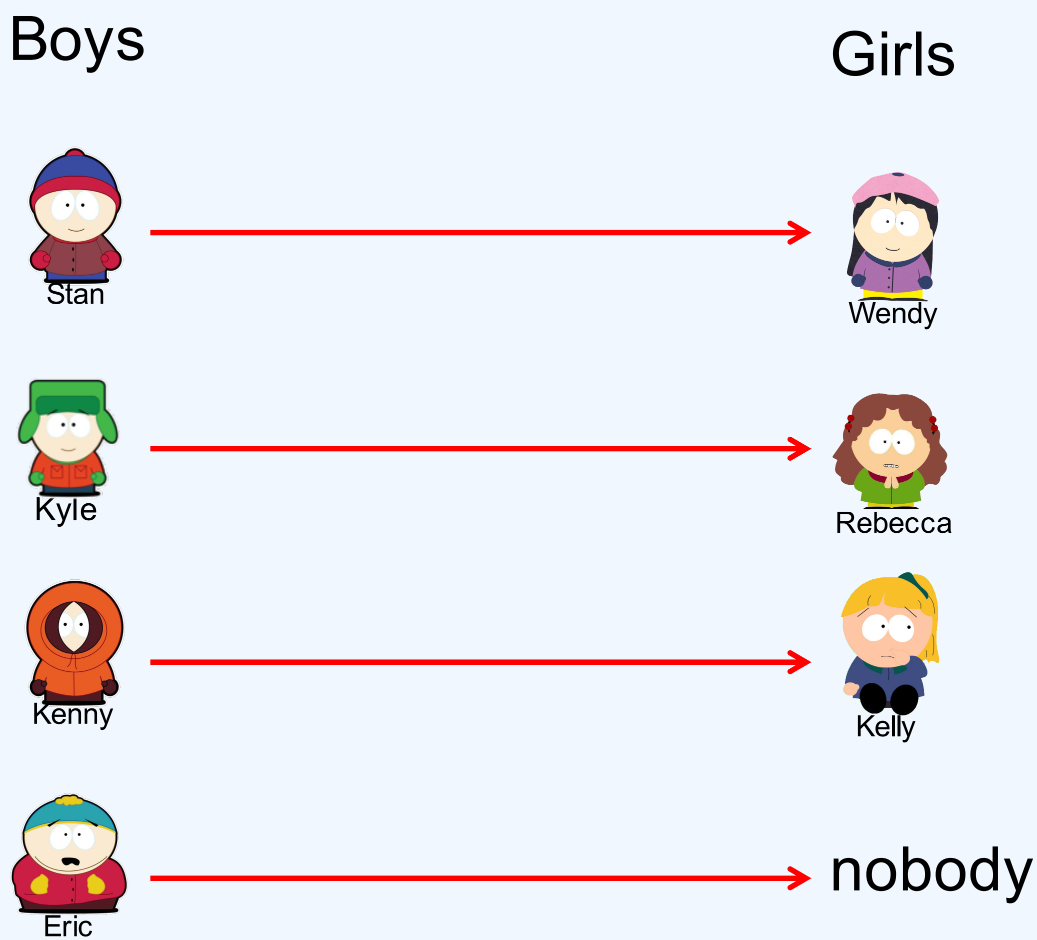
Girls



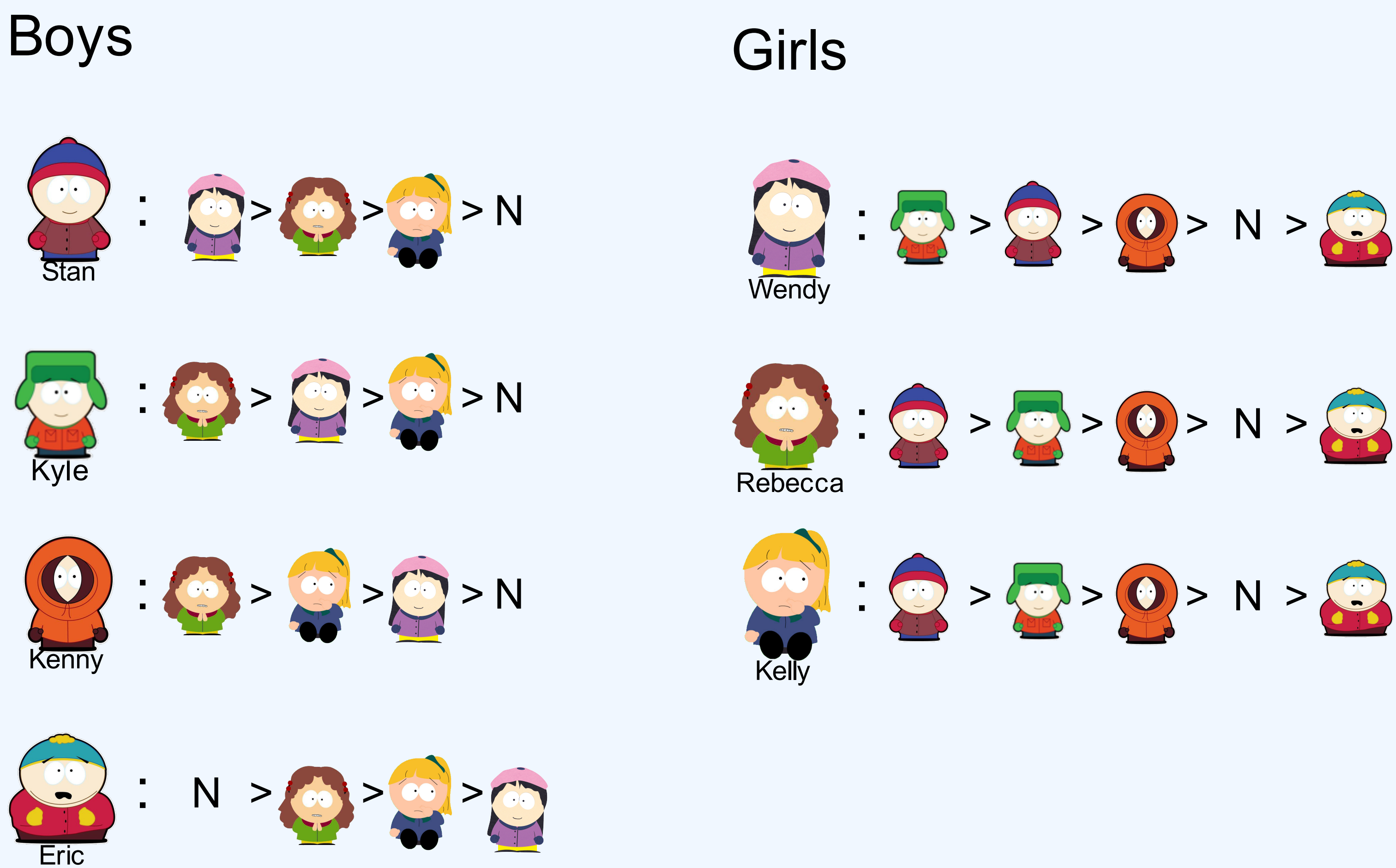
Round 1


Boys
Girls
Stan
Wendy
Kyle
Rebecca
reject
Kenny
Kelly
Eric
nobody

Round 2



Women-proposing DA (on blackboard)



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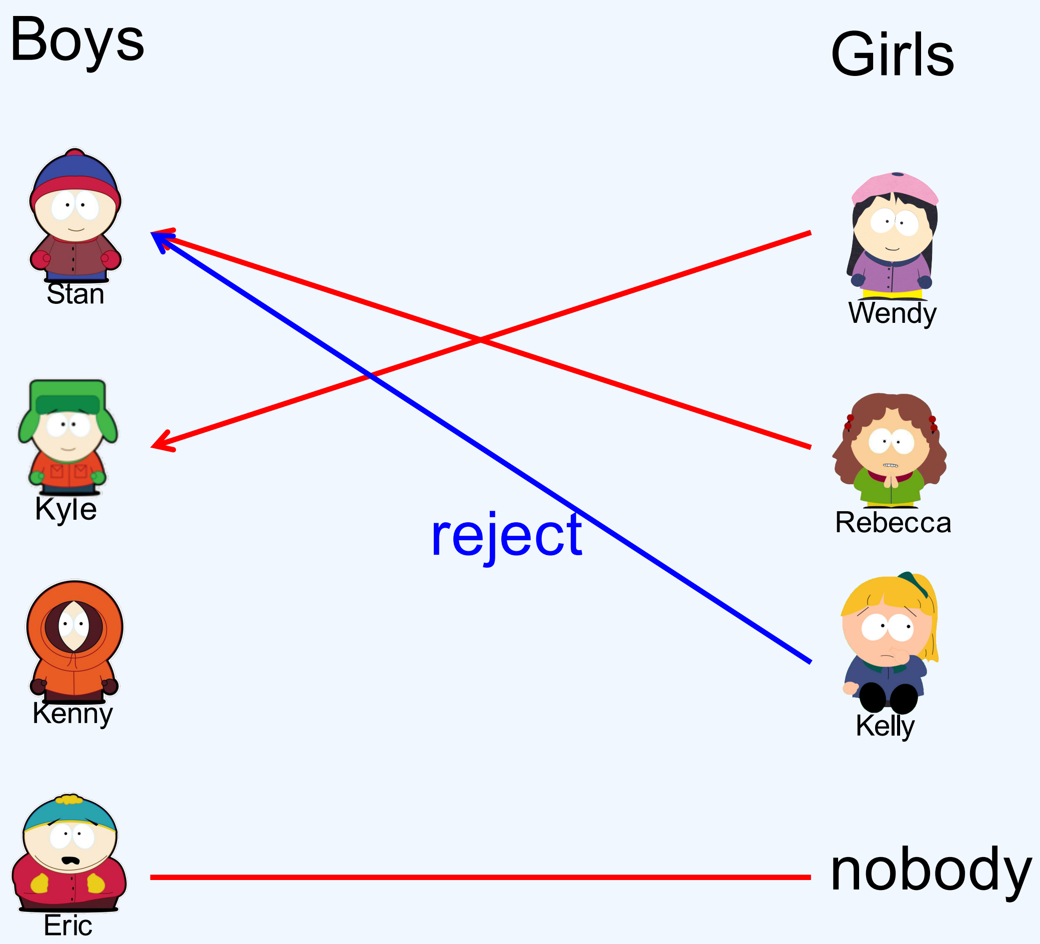
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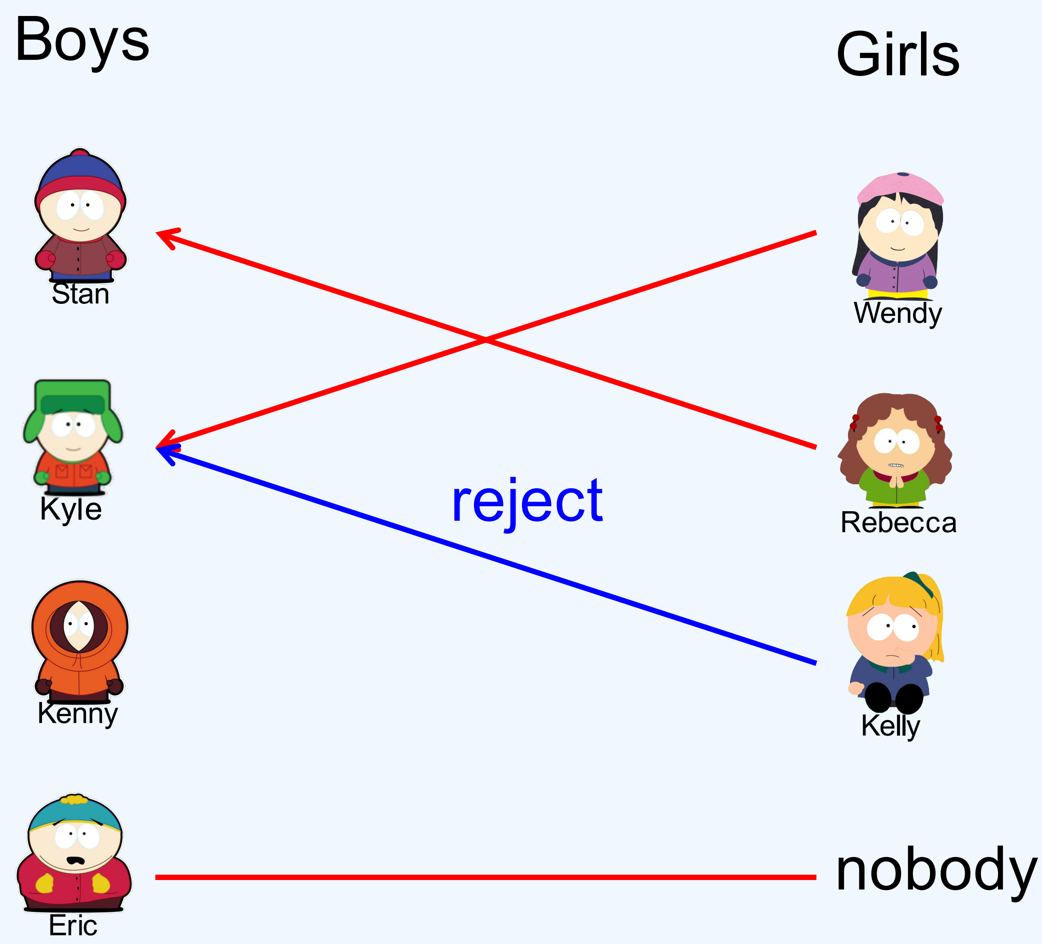
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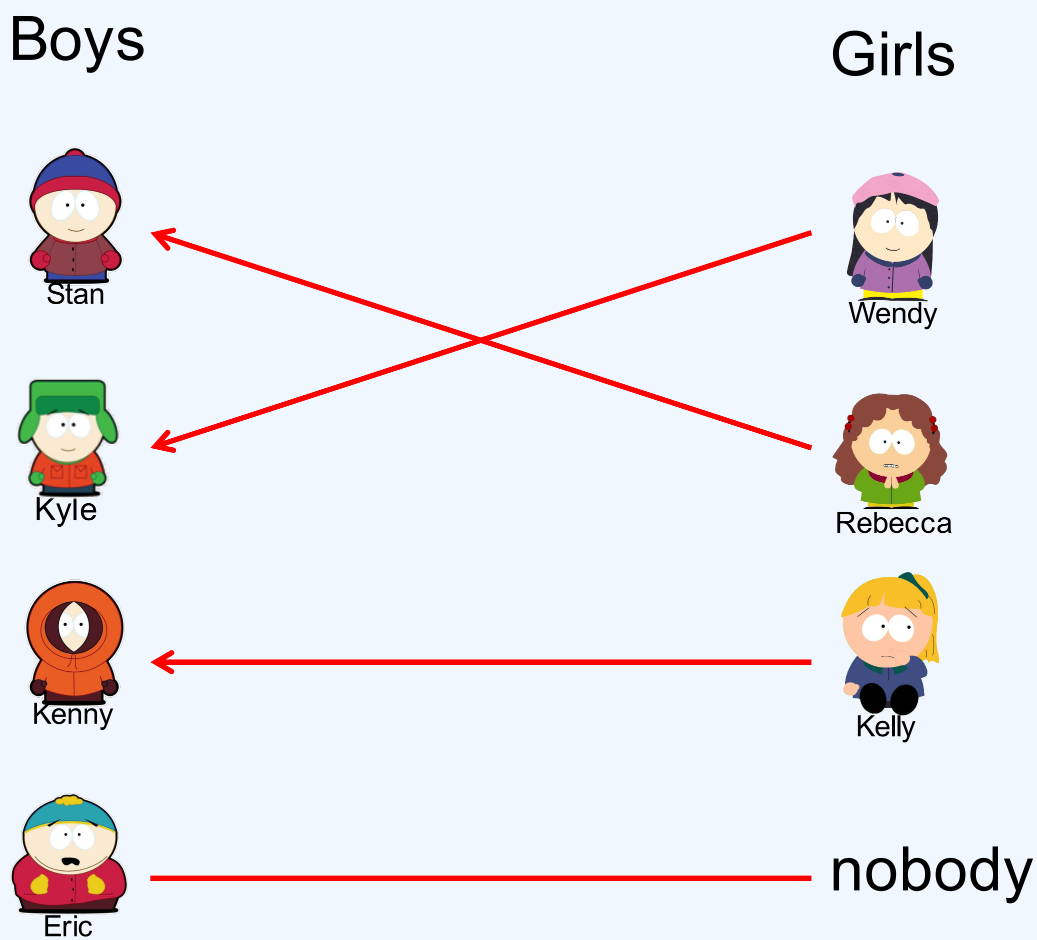
reject
nobody

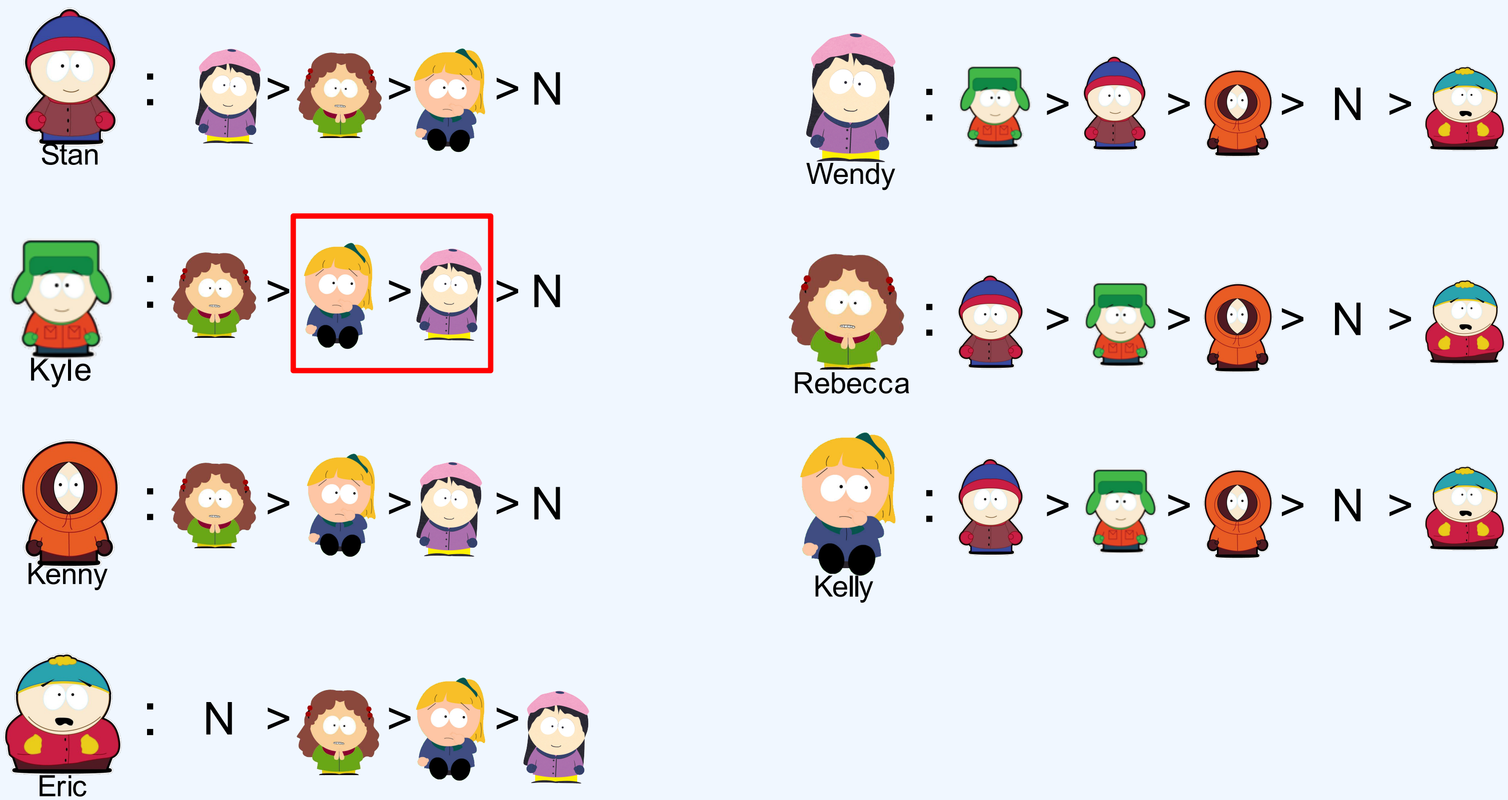
Round 3



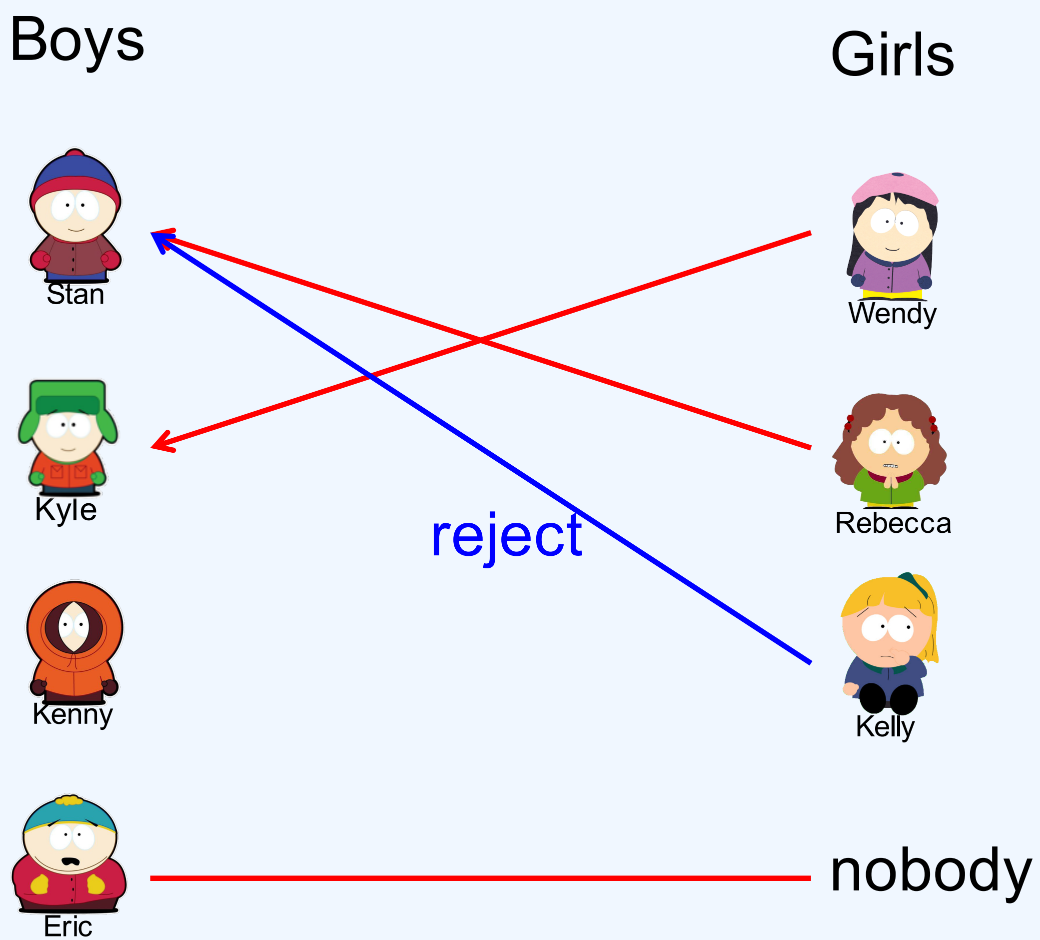
Women-proposing DA with slightly different preferences

Boys

Girls



Round 1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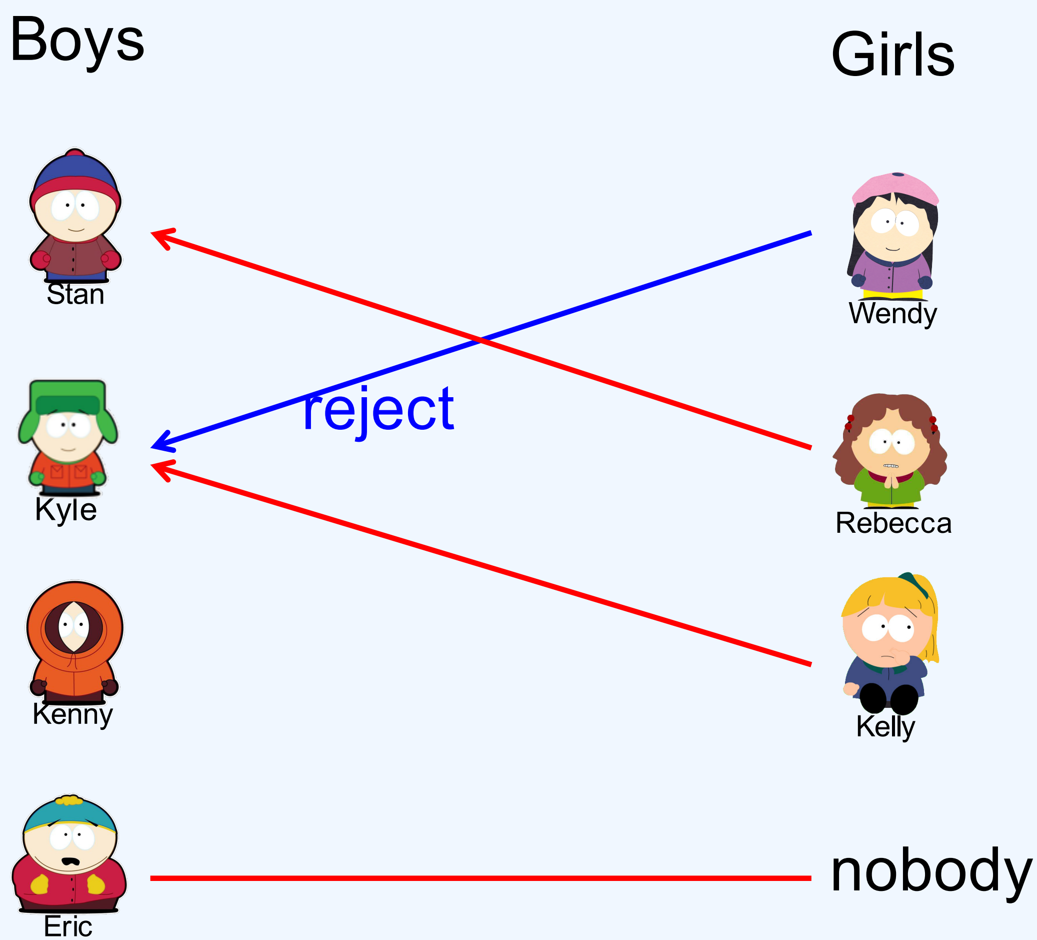
nobody

Round 2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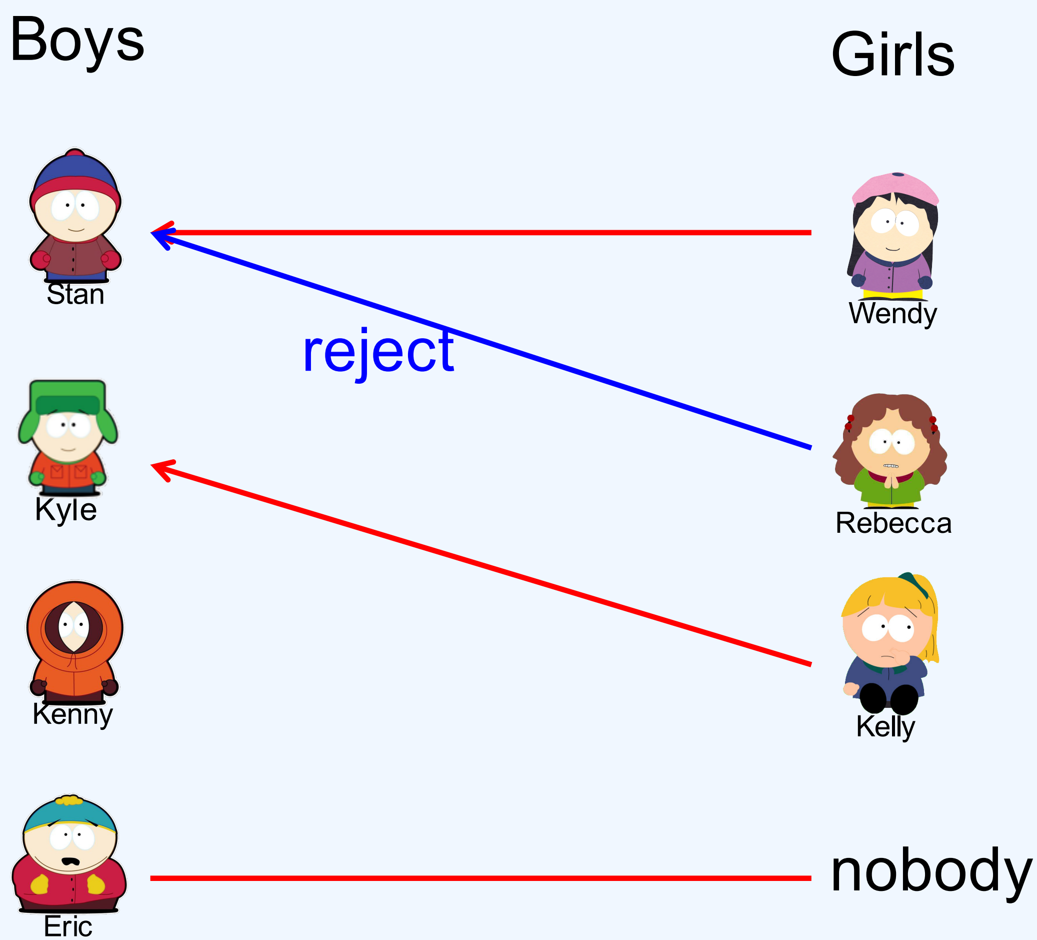
Round 3


Boys
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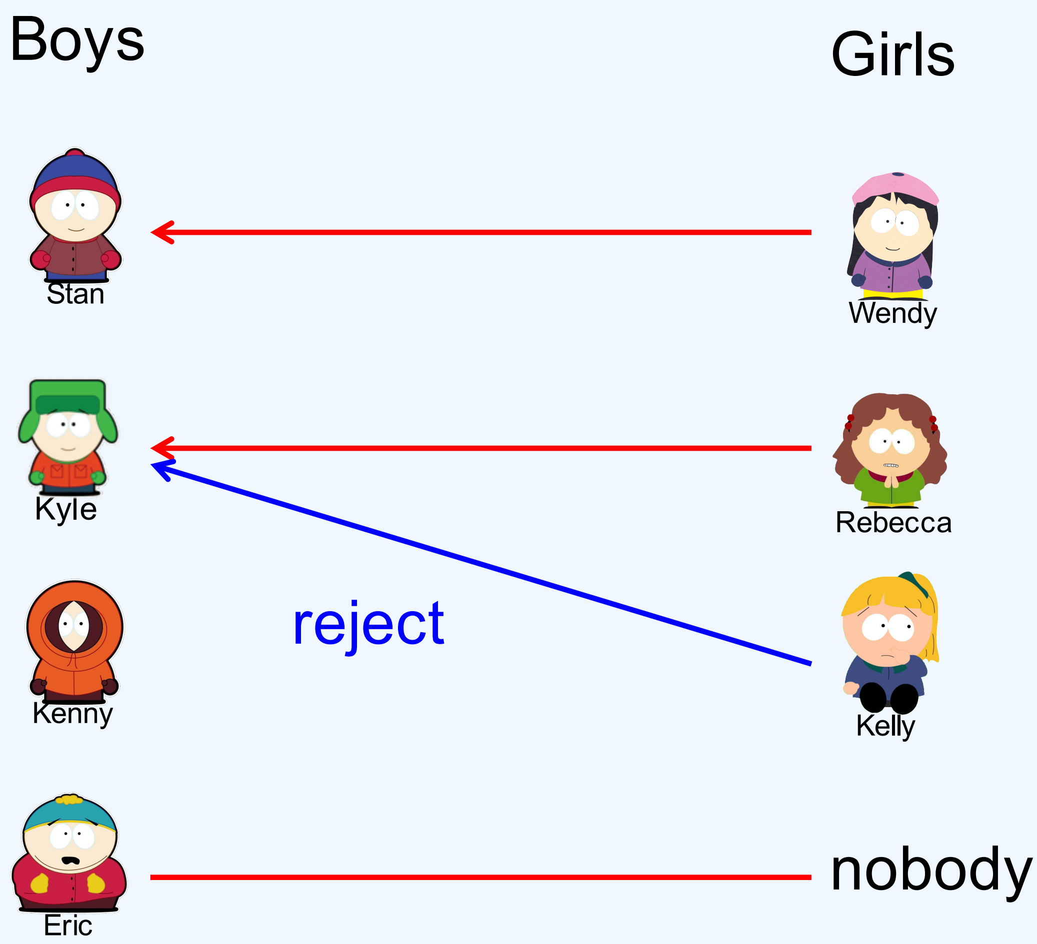
Eric
nobody

Round 4


Boys
Girls
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Wendy
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reject
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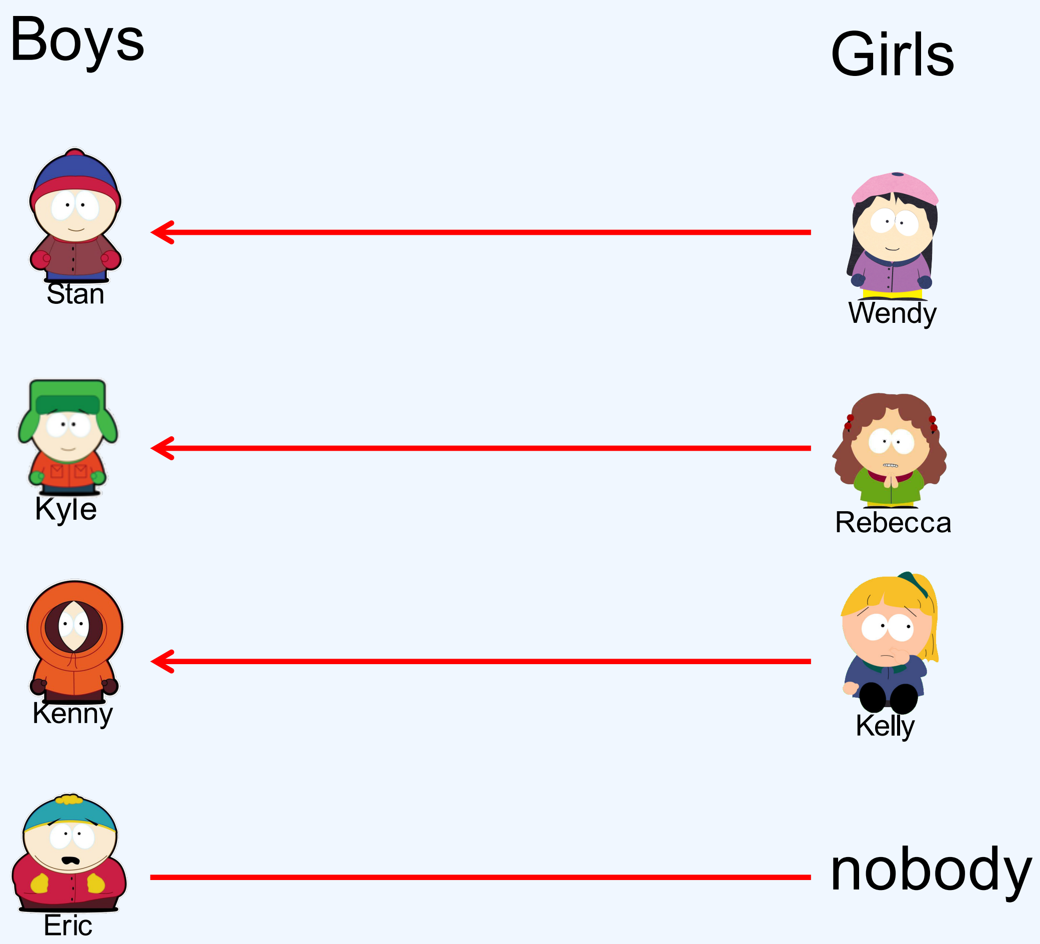
Eric
nobody

Round 5



Properties of men-proposing DA

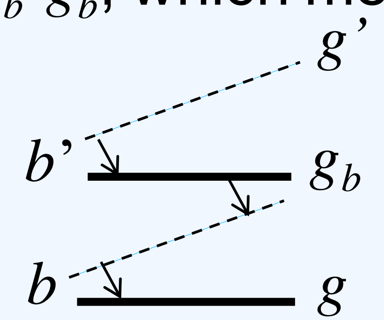
- Can be computed efficiently
- Outputs a stable matching
  - The “best” stable matching for boys, called men-optimal matching
  - and the worst stable matching for girls
- Strategy-proof for boys

The men-optimal matching

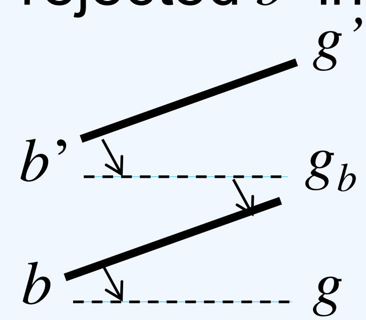
- For each boy b, let g_b denote his most favorable girl matched to him in any stable matching
- A matching is men-optimal if each boy b is matched to g_b
- Seems too strong, but…

Men-proposing DA is men-optimal

- Theorem. The output of men-proposing DA is men-optimal
- Proof: by contradiction
  - suppose b is the first boy not matched to $g \neq g_b$ in the execution of DA,
  - let M be an arbitrary matching where b is matched to g_b
  - Suppose b' is the boy whom g_b chose to reject b, and M(b')=g'
  - $g' >_{b'} g_b$, which means that g' rejected b' in a previous round



DA



M

Strategy-proofness for boys

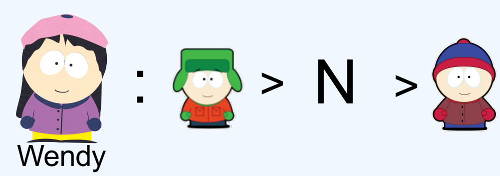
- Theorem. Truth-reporting is a dominant strategy for boys in men-proposing DA

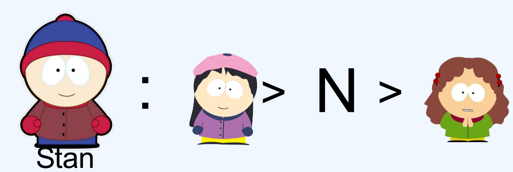
No matching mechanism is strategy-proof and stable

- Proof.

Boys

Girls

- If (S,W) and (K,R) then 

- If (S,R) and (K,W) then 

Recap: two-sided 1-1 matching

- Men-proposing deferred acceptance algorithm (DA)
  - outputs the men-optimal stable matching
  - runs in polynomial time
  - strategy-proof on men's side

Next class: Fair division

- Indivisible goods: one-sided 1-1 or 1-many matching (papers, apartments, etc.)
- Divisible goods: cake cutting