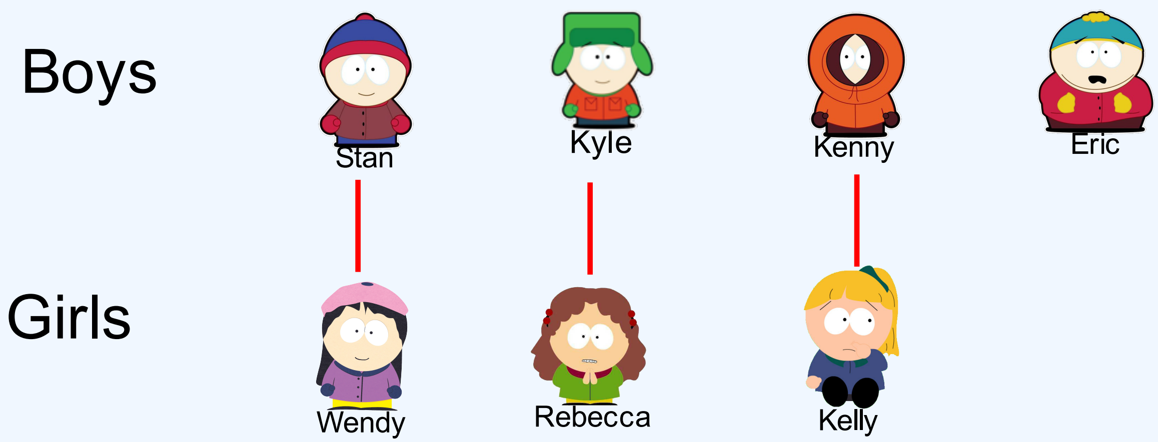
Fair division

Lirong Xia

Rensselaer

Fall, 2016

Last class: two-sided 1-1 stable matching



- Men-proposing deferred acceptance algorithm (DA)
  - outputs the men-optimal stable matching
  - runs in polynomial time
  - strategy-proof on men's side
- No matching mechanism is both stable and strategy-proof

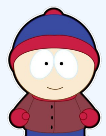
Today: FAIR division

- Fairness conditions
- Allocation of indivisible goods
  - serial dictatorship
  - Top trading cycle
- Allocation of divisible goods (cake cutting)
  - discrete procedures
  - continuous procedures

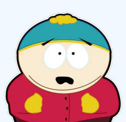
Example

Agents

Houses


Stan


Kyle


Eric

Formal setting

- Agents A = $\{1,\ldots,n\}$
- Goods G: finite or infinite
- Preferences: represented by utility functions
  - agent j, $u_j : G \rightarrow \mathrm{R}$
- Outcomes = Allocations
  - $g : G \rightarrow A$
  - $g^{-1} : A \rightarrow 2^G$
- Difference with matching in the last class
  - 1-1 vs 1-many
  - Goods do not have preferences

Efficiency criteria

- Pareto dominance: an allocation g Pareto dominates another allocation g', if
  - all agents are not worse off under g
  - some agents are strictly better off
- Pareto optimality
  – allocations that are not Pareto dominated
- Maximizes social welfare
  – utilitarian
  – egalitarian

Fairness criteria

- Given an allocation g, agent j_1 envies agent j_2 if $u_{j_1}(g^{-1}(j_2)) > u_{j_1}(g^{-1}(j_1))$
- An allocation satisfies envy-freeness, if
  - no agent envies another agent
  - c.f. stable matching
- An allocation satisfies proportionality, if
  - for all j, $u_j(g^{-1}(j)) \geq u_j(G)/n$
- Envy-freeness implies proportionality
  - proportionality does not imply envy-freeness

Why not…

- Consider fairness in other social choice problems
  - voting: does not apply
  - matching: when all agents have the same preferences
  - auction: satisfied by the 2nd price auction
- Use the agent-proposing DA in resource allocation (creating random preferences for the goods)
  - stableness is no longer necessary
  - sometimes not 1-1
  - for 1-1 cases, other mechanisms may have better properties

Allocation of indivisible goods

- House allocation
  - 1 agent 1 good
- Housing market
  - 1 agent 1 good
  - each agent originally owns a good
- 1 agent multiple goods (not discussed today)

House allocation

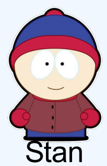
- The same as two sided 1-1 matching except that the houses do not have preferences
- The serial dictatorship (SD) mechanism
  - given an order over the agents, w.l.o.g. $a_1 \rightarrow \ldots \rightarrow a_n$
  - in step j, let agent j choose her favorite good that is still available
  - can be either centralized or distributed
  - computation is easy

Characterization of SD

- Theorem. Serial dictatorships are the only deterministic mechanisms that satisfy
  - strategy-proofness
  - Pareto optimality
  - neutrality
  - non-bossy
    - An agent cannot change the assignment selected by a mechanism by changing his report without changing his own assigned item
- Random serial dictatorship

Why not agent-proposing DA

- Agent-proposing DA satisfies
  - strategy-proofness
  - Pareto optimality
- May fail neutrality



: h1>h2 h1: S>K

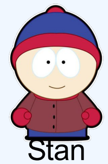
: h1>h2 h2: K>S

- How about non-bossy?
  - No
- Agent-proposing DA when all goods have the same preferences = serial dictatorship

Housing market

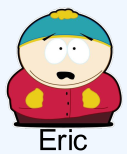
- Agent j initially owns h_j
- Agents cannot misreport h_j, but can misreport her preferences
- A mechanism f satisfies participation
  - if no agent j prefers h_j to her currently assigned item
- An assignment is in the core
  - if no subset of agents can do better by trading the goods that they own in the beginning among themselves
  - stronger than Pareto-optimality

Example: core allocation



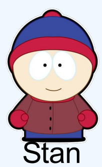
: h1>h2>h3, owns h3

: h3>h2>h1, owns h1



: h3>h1>h2, owns h2

Not in the core

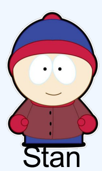


: h2

: h3

: h1

In the core



: h1

: h3

: h2

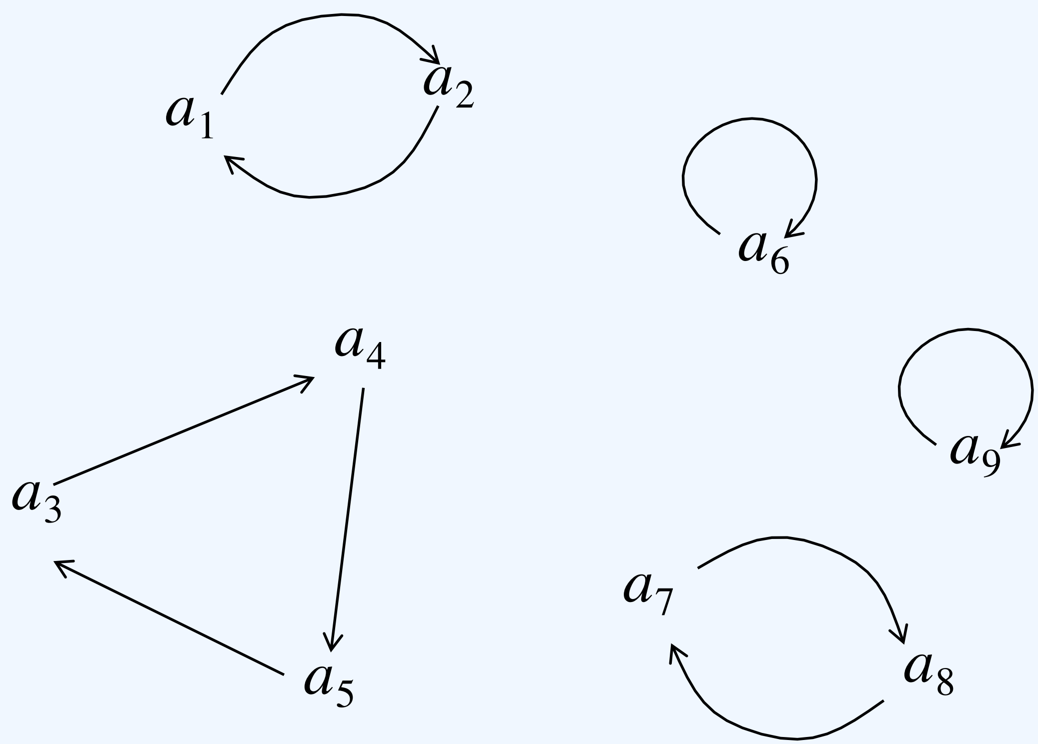
The top trading cycles (TTC) mechanism

- Start with: agent j owns h_j
- In each round
  - built a graph where there is an edge from each available agent to the owner of her most-preferred house
  - identify all cycles; in each cycle, let the agent j gets the house of the next agent in the cycle; these will be their final allocation
  - remove all agents in these cycles

Example

a_1: h_2>… | a_2: h_1>… | a_3: h_4>… | a_4: h_5>… | a_5: h_3>… | a_6: h_4>h_3>h_6>…

a_7: h_4>h_5>h_6>h_3>h_8>… | a_8: h_7>… | a_9: h_6>h_4>h_7>h_3>h_9>…



Properties of TTC

- Theorem. The TTC mechanism
  - is strategy-proof
  - is Pareto optimal
  - satisfies participation
  - selects an assignment in the core
    - the core has a unique assignment
  - can be computed in $O(n^2)$ time
- Why not using TTC in 1-1 matching?
  - not stable
- Why not using TTC in house allocation (using random initial allocation)?
  - not neutral

DA vs SD vs TTC

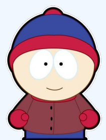
- All satisfy
  - strategy-proofness
  - Pareto optimality
  - easy-to-compute
- DA
  - stableness
- SD
  - neutrality
- TTC
  - chooses the core assignment

Multi-type resource allocation

- Each good is characterized by multiple issues
  - e.g. each presentation is characterized by topic and time
- Paper allocation
  - we have used SD to allocate the topic
  - we will use SD with reverse order for time
- Potential research project

Example 2

Agents



Stan

Kyle

Eric

Kenny

One divisible good

Allocation of one divisible good

- The set of goods is [0,1]

- Each utility function satisfies
  - Non-negativity: $u_j(B) \geq 0$ for all $B \subseteq [0, 1]$
  - Normalization: $u_j(\varnothing) = 0$ and $u_j([0, 1]) = 1$
  - Additivity: $u_j(B \cup B') = u_j(B) + u_j(B')$ for disjoint B, $B' \subseteq [0, 1]$
  - is continuous
- Also known as cake cutting

  - discrete mechanisms: as protocols
  - continuous mechanisms: use moving knives

2 agents: cut-and-choose

- Dates back to at least the Hebrew Bible [Brams&Taylor, 1999, p. 53]
- The cut-and-choose mechanism
  - 1st step: One player cuts the cake in two pieces (which she considers to be of equal value)
  - 2nd step: the other one chooses one of the pieces (the piece she prefers)
- Cut-and-choose satisfies
  - proportionality
  - envy-freeness
  - some operational criteria
    - each agent receive a continuous piece of cake
    - the number of cuts is minimum
    - is discrete

More than 2 agents: The Banach-Knaster Last-Diminisher Procedure

- In each round
  - the first agent cut a piece
  - the piece is passed around other agents, who can
    - pass
    - cut more
  - the piece is given to the last agent who cut
- Properties
  - proportionality
  - not envy-free
  - the number of cut may not be minimum
  - is discrete

The Dubins-Spanier Procedure

- A referee moves a knife slowly from left to right
- Any agent can say “stop”, cut off the piece and get it
- Properties
  - proportionality
  - not envy-free
  - minimum number of cuts (continuous pieces)
  - continuous mechanism

Envy-free procedures

- $n = 2$: cut-and-choose
- $n = 3$
  - The Selfridge-Conway Procedure
    - discrete, number of cuts is not minimum
  - The Stromquist Procedure
    - continuous, uses four simultaneous moving knives
- $n = 4$
  - no procedure produces continuous pieces is known
  - [Barbanel&Brams 04] uses a moving knife and may use up to 5 cuts
- $n \geq 5$
  - only procedures requiring an unbounded number of cuts are known [Brams&Taylor 1995]

Recap

- Indivisible goods
  - house allocation: serial dictatorship
  - housing market: Top trading cycle (TTC)
- Divisible goods (cake cutting)
  - $n = 2$: cut-and-choose
  - discrete and continuous procedures that satisfies proportionality
  - hard to design a procedure that satisfies envy-freeness

Next class: Judgment aggregation

	Action P	Action Q	Liable? ($P \wedge Q$)
Judge 1	Y	Y	Y
Judge 2	Y	N	N
Judge 3	N	Y	N
Majority	**Y**	**Y**	**N**