# Introduction to Social Choice

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# Keep in mind

➢ Good science
  - What question does it answer?

➢ Good engineering
  - What problem does it solve?

# Last class

- How to model agents' preferences?
- Order theory
  - linear orders
  - weak orders
  - partial orders
- Utility theory
  - preferences over lotteries
  - risk attitudes: aversion, neutrality, seeking

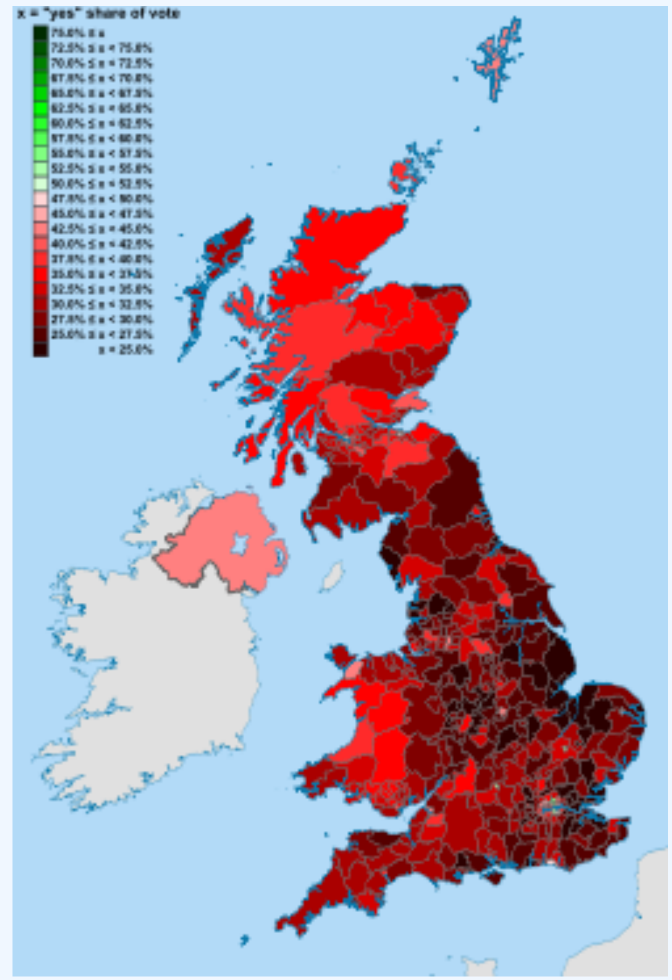
# Today

- Q: What problem does it solve?
- A: Aggregating agents' preferences and make a joint decision by voting

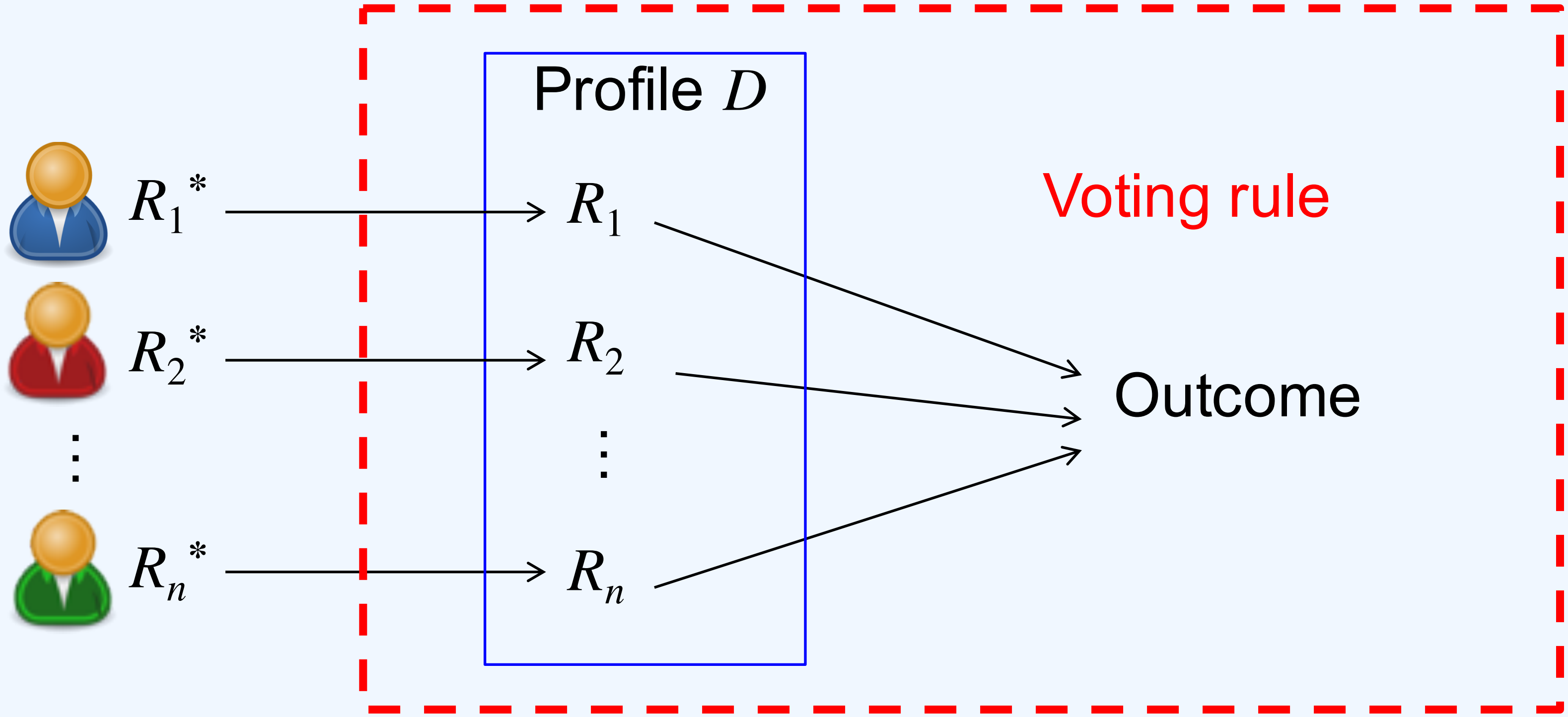
# Change the world: 2011 UK Referendum

- The second nationwide referendum in UK history
  - The first was in 1975
- Member of Parliament election:

  Plurality rule ➔ Alternative vote rule
- 68% No vs. 32% Yes
- In 10/440 districts more voters said yes
  - 6 in London, Oxford, Cambridge, Edinburgh Central, and Glasgow Kelvin
- Why change?
- Why failed?
- Which voting rule is the best?



# Social choice: Voting

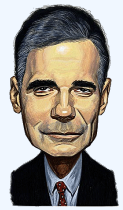
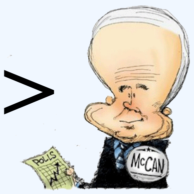
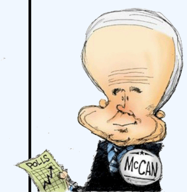
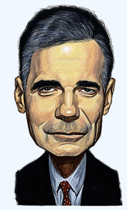


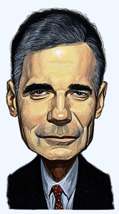
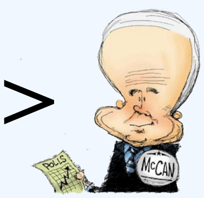
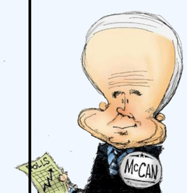
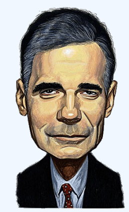

- Agents: $n$ voters, $N=\{1,\ldots,n\}$
- Alternatives: $m$ candidates, $A=\{a_1,\ldots,a_m\}$ or $\{a, b, c, d,\ldots\}$
- Outcomes:
  - winners (alternatives): $O=A$. Social choice function
  - rankings over alternatives: $O$=Rankings($A$). Social welfare function
- Preferences: $R_j^*$ and $R_j$ are full rankings over $A$
- Voting rule: a function that maps each profile to an outcome

# Popular voting rules

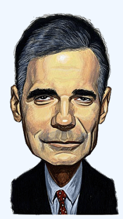
(a.k.a. what people have done in the past two centuries)

# The Borda rule

$P$={  >  >  ×4,  >  >  ×3

 >  >  ×2,  >  >  ×2 }

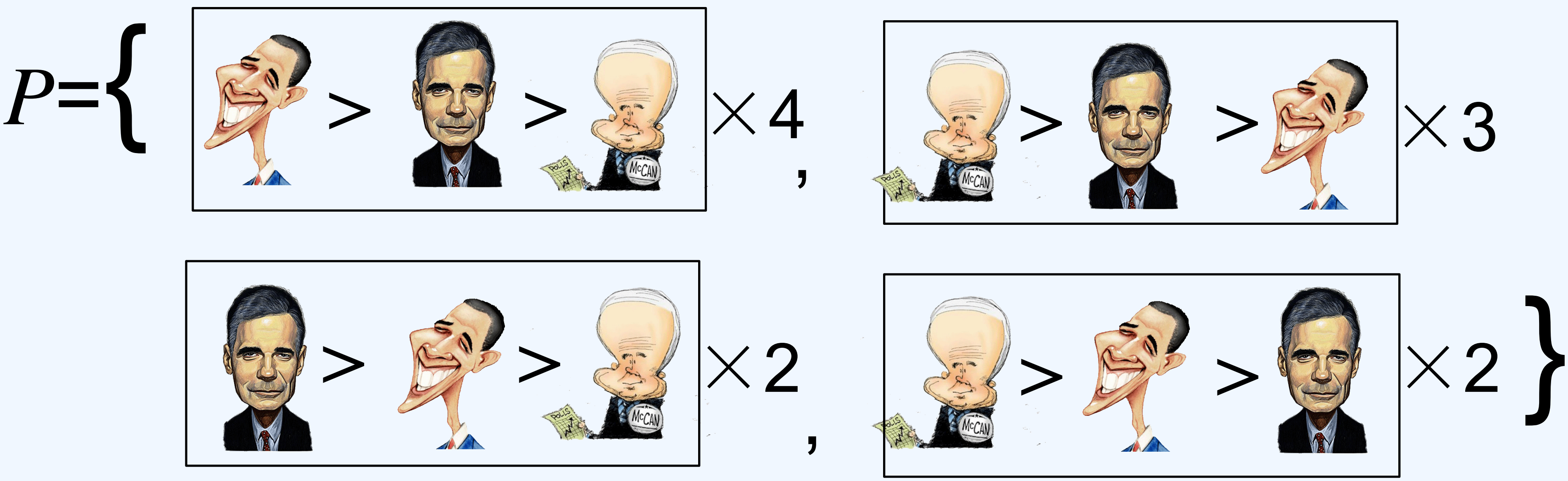
Borda($P$)= 

Borda scores  : 2×4+4=12  : 2*2+7=11  : 2*5=10

# Positional scoring rules

- Characterized by a score vector $s_1,\dots,s_m$ in non-increasing order
- For each vote $R$, the alternative ranked in the $i$-th position gets $s_i$ points
- The alternative with the most total points is the winner
- Special cases
  - Borda: score vector $(m\text{-}1, m\text{-}2, \dots, 0)$ [French academy of science 1784-1800, Slovenia, Naru]
  - $k$-approval: score vector $(\underbrace{1\dots1}_{k}, 0\dots0)$
  - Plurality: score vector $(1, 0\dots0)$ [UK, US]
  - Veto: score vector $(1\dots1, 0)$

# Example




Borda

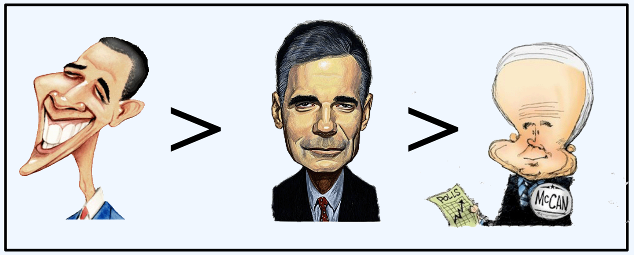
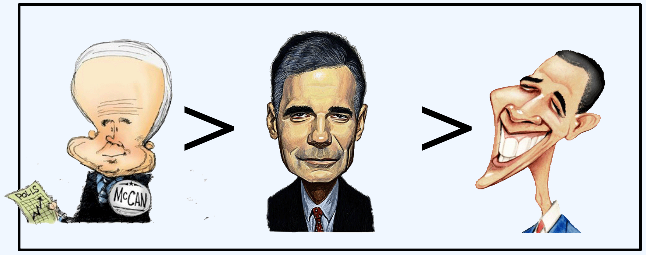
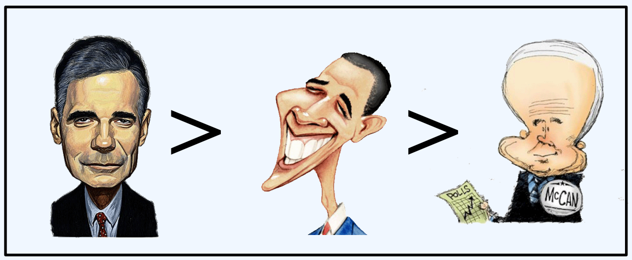
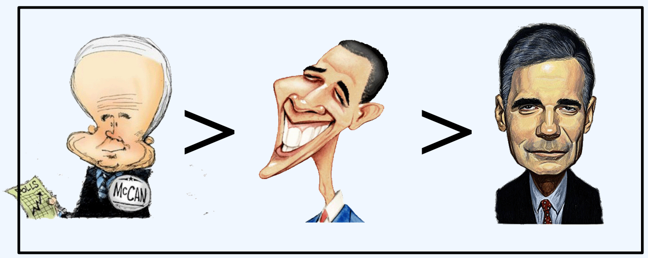
Plurality
(1- approval)

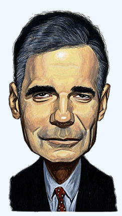
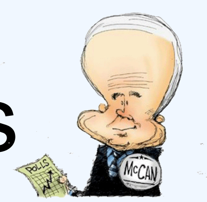
Veto
(2-approval)

# Plurality with runoff

- The election has two rounds
  - First round, all alternatives except the two with the highest plurality scores drop out
  - Second round, the alternative preferred by more voters wins
- [used in France, Iran, North Carolina State]

# Example: Plurality with runoff

$P=\{$  $\times 4$,  $\times 3$,  $\times 2$,  $\times 2$ $\}$

- First round:  drops out
- Second round:  defeats 

Different from Plurality!

# Single transferable vote (STV)

- ➢ Also called instant run-off voting or alternative vote
- ➢ The election has $m$-1 rounds, in each round,
  - The alternative with the lowest plurality score drops out, and is removed from all votes
  - The last-remaining alternative is the winner
- ➢ [used in Australia and Ireland]

| $a > b > c > d$ | $d > a > b > c$ | $c > d > a > b$ | $b > c > d > a$ |
|---|---|---|---|
| 10 | 7 | 6 | 3 |


$a$

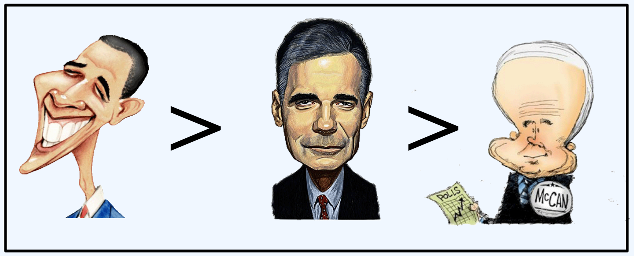
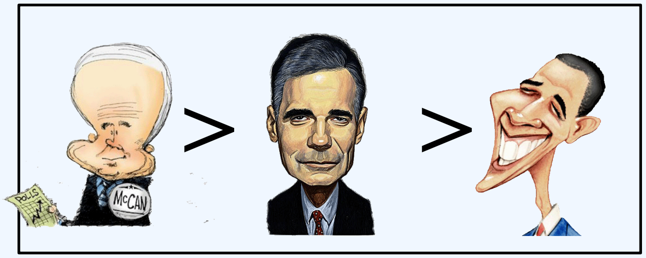
# Other multi-round voting rules

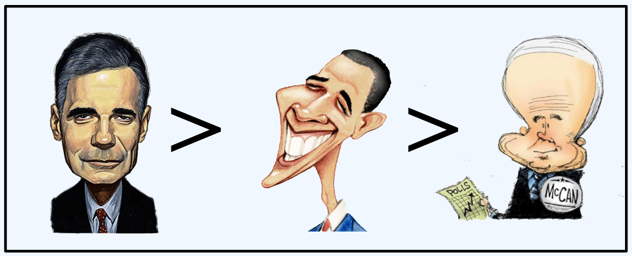
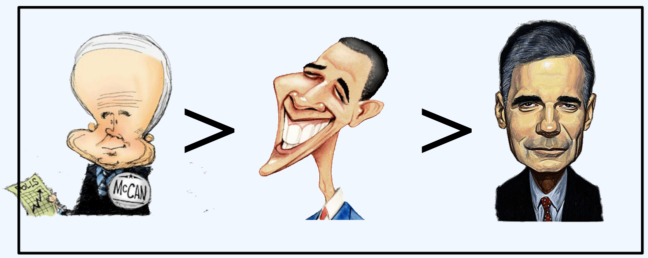
- Baldwin's rule
  - Borda+STV: in each round we eliminate one alternative with the lowest Borda score
  - break ties when necessary
- Nanson's rule
  - Borda with multiple runoff: in each round we eliminate all alternatives whose Borda scores are below the average
  - [Marquette, Michigan, U. of Melbourne, U. of Adelaide]

# The Copeland rule

- The Copeland score of an alternative is its total "pairwise wins"
  - the number of positive outgoing edges in the WMG
- The winner is the alternative with the highest Copeland score
- WMG-based

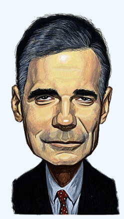
# Example: Copeland

P={  ×4,  ×3,

 ×2,  ×2 }

Copeland score:

 : 2

 : 1

 : 0

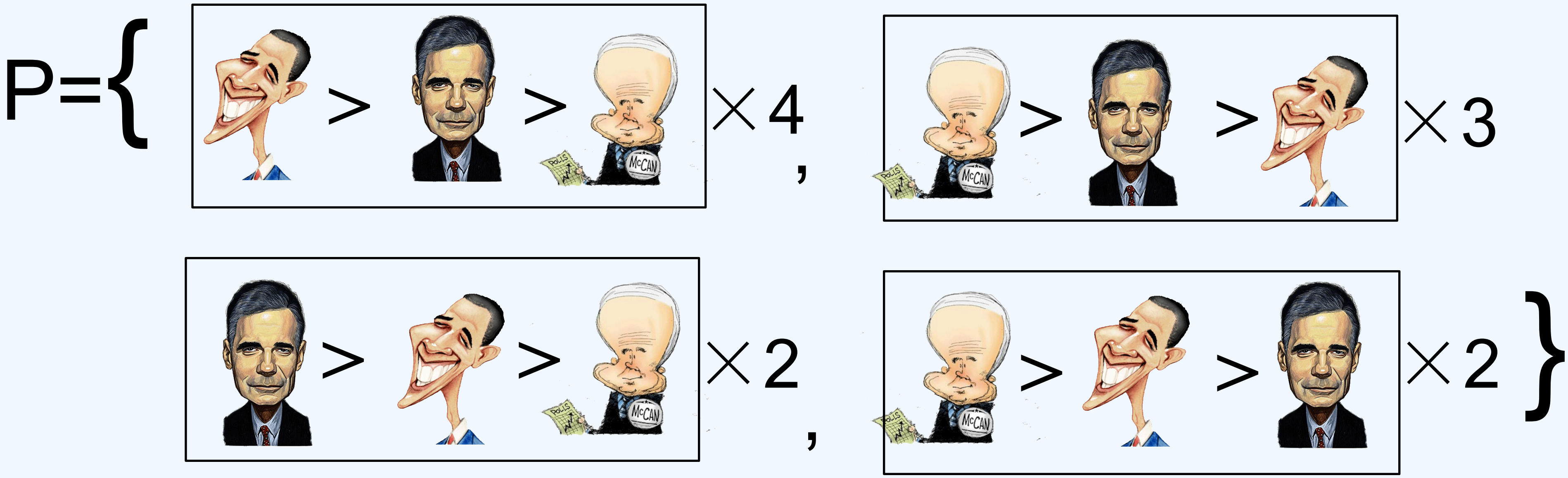
# The maximin rule

- A.k.a. Simpson or minimax
- The maximin score of an alternative $a$ is

  $$\text{MS}_P(a)=\text{min}_b\,(\#\{a > b \text{ in } P\}-\#\{b > a \text{ in } P\})$$

  - the smallest pairwise defeats
- The winner is the alternative with the highest maximin score
- WMG-based

# Example: maximin

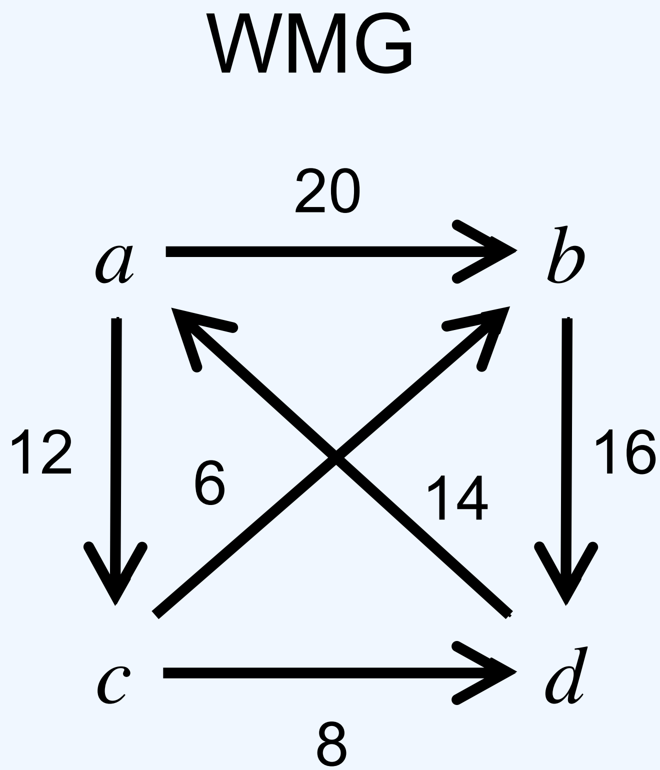


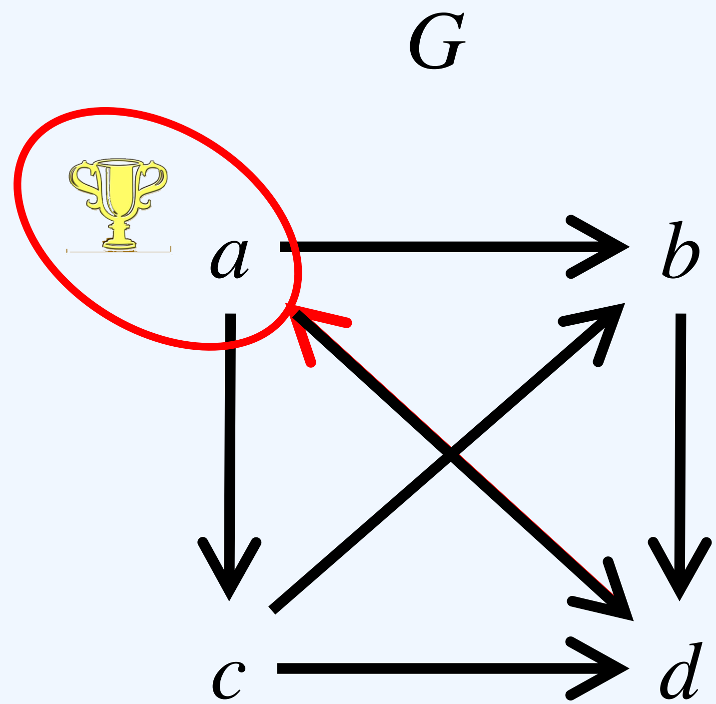

Maximin score:

# Ranked pairs

- Given the WMG
- Starting with an empty graph $G$, adding edges to $G$ in multiple rounds
  - In each round, choose the remaining edge with the highest weight
  - Add it to $G$ if this does not introduce cycles
  - Otherwise discard it
- The alternative at the top of $G$ is the winner

# Example: ranked pairs




Q1: Is there always an alternative at the "top" of $G$?

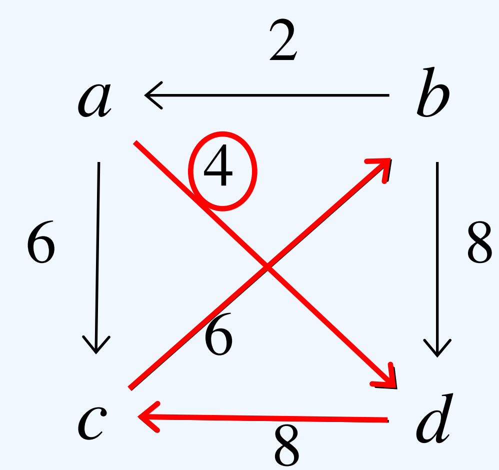
Q2: Does it suffice to only consider positive edges?

# The Schulze Rule

- In the WMG of a profile, the strength
  - of a path is the smallest weight on its edges
  - of a pair of alternatives $(a,b)$, denoted by $S(a,b)$, is the largest strength of paths from $a$ to $b$
- The Schulze winners are the alternatives $a$ such that
  - for all alternatives $a'$, $S(a, a') \geq S(a', a)$
  - $S(a,b)=S(a,c)=S(a,d)=6$

    Strength($a \rightarrow d \rightarrow c \rightarrow b$)=4

    $>2=S(b,a)=S(c,a)=S(d,a)$
  - The (unique) winner is $a$



# Ranked pairs and Schulze

- Ranked pairs [Tideman 1987] and Schulze [Schulze 1997]
  - Both satisfy anonymity, Condorcet consistency, monotonicity, immunity to clones, etc
  - Neither satisfy participation and consistency (these are not compatible with Condorcet consistency)
- Schulze rule has been used in elections at Wikimedia Foundation, the Pirate Party of Sweden and Germany, the Debian project, and the Gento Project

# The Bucklin Rule

- An alternative $a$'s Bucklin score
  - smallest $k$ such that for the majority of agents, $a$ is ranked within top $k$
- Simplified Bucklin
  - Winners are the agents with the smallest Bucklin score

# Kemeny's rule

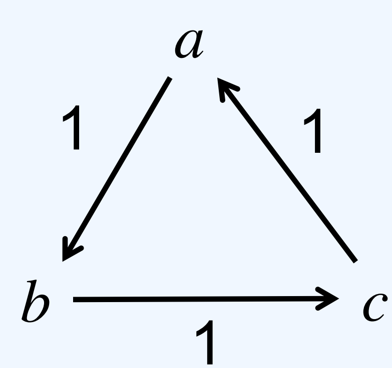
- Kendall tau distance
  - $\mathrm{K}(R,W)$= # {different pairwise comparisons}

$$\mathrm{K}(\, b \succ c \succ a \,,\, a \succ b \succ c\,) = ?$$

- Kemeny($D$)=argmin$_W$ $\mathrm{K}(D,W)$=argmin$_W$ $\Sigma_{R\in D}\mathrm{K}(R,W)$
- For single winner, choose the top-ranked alternative in Kemeny($D$)
- [reveals the truth]

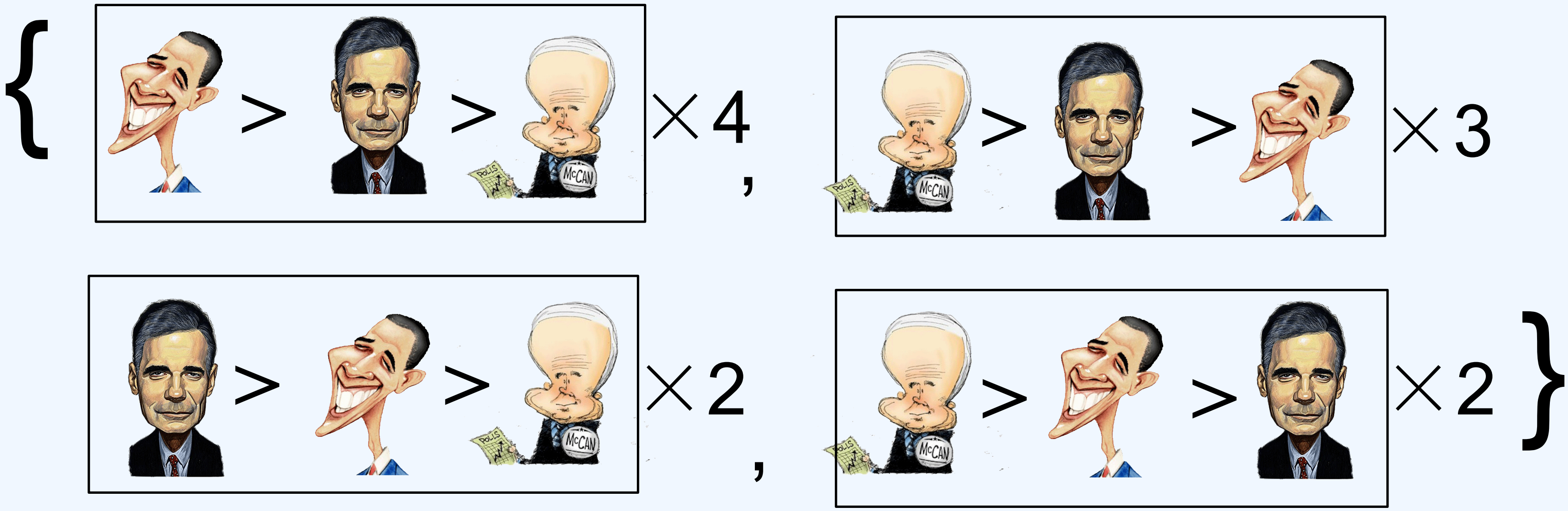
# Weighted majority graph

- Given a profile $P$, the weighted majority graph WMG($P$) is a weighted directed complete graph ($V$,$E$,$w$) where
  - $V = A$
  - for every pair of alternatives $(a, b)$
  - $w(a\rightarrow b)$ = #{$a > b$ in $P$} - #{$b > a$ in $P$}
  - $w(a\rightarrow b)$ = -$w(b\rightarrow a)$
- WMG (only showing positive edges} might be cyclic
  - Condorcet cycle: { $a$>$b$>$c$, $b$>$c$>$a$, $c$>$a$>$b$}

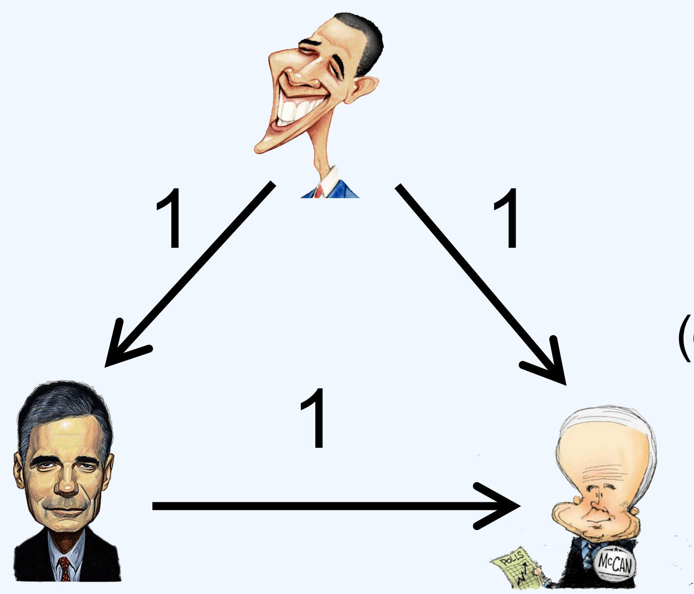


# Example: WMG

P=




WMG($P$) =




(only showing positive edges)

# WMG-based voting rules

- A voting rule $r$ is based on weighted majority graph, if for any profiles $P_1$, $P_2$,

$$[\text{WMG}(P_1)=\text{WMG}(P_2)] \Rightarrow [r(P_1)=r(P_2)]$$

- WMG-based rules can be redefined as a function that maps {WMGs} to {outcomes}
- Example: Borda is WMG-based
  - Proof: the Borda winner is the alternative with the highest sum over outgoing edges.

# Voting with Prefpy

- Implemented
  - All positional scoring rules
  - Bucklin, Copeland, maximin
  - not well-tested for weak orders
- Project ideas
  - implementation of STV, ranked pairs, Kemeny
    - all are NP-hard to compute
  - extends all rules to weak orders

# Popular criteria for voting rules

(a.k.a. what people have done in the past 60 years)

# How to evaluate and compare voting rules?

- No single numerical criteria
  - Utilitarian: the joint decision should maximize the total happiness of the agents
  - Egalitarian: the joint decision should maximize the worst agent's happiness
- Axioms: properties that a "good" voting rules should satisfy
  - measures various aspects of preference aggregation

# Fairness axioms

- Anonymity: names of the voters do not matter
  - Fairness for the voters
- Non-dictatorship: there is no dictator, whose top-ranked alternative is always the winner, no matter what the other votes are
  - Fairness for the voters
- Neutrality: names of the alternatives do not matter
  - Fairness for the alternatives

# A truth-revealing axiom

- Condorcet consistency: Given a profile, if there exists a Condorcet winner, then it must win
  - The Condorcet winner beats all other alternatives in pairwise comparisons
  - The Condorcet winner only has positive outgoing edges in the WMG
- Why this is truth-revealing?
  - why Condorcet winner is the truth?

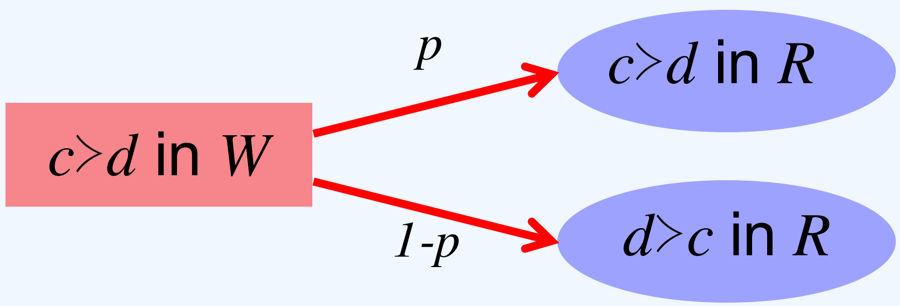
# The Condorcet Jury theorem [Condorcet 1785]

- Given
  - two alternatives $\{a,b\}$. $a$: liable, $b$: not liable
  - $0.5<p<1$,
- Suppose
  - given the ground truth ($a$ or $b$), each voter's preference is generated i.i.d., such that
    - w/p $p$, the same as the ground truth
    - w/p $1-p$, different from the ground truth
- Then, as $n\rightarrow\infty$, the probability for the majority of agents' preferences is the ground truth goes to 1
- "lays, among other things, the foundations of the ideology of the democratic regime" (Paroush 1998)

# Condorcet's model

## [Condorcet 1785]

- Given a "ground truth" ranking $W$ and $p$>$1/2$, generate each pairwise comparison in $R$ independently as follows (suppose $c \succ d$ in $W$)




$$\Pr(b \succ c \succ a \mid a \succ b \succ c) = ?\ p(1-p)^2$$

- Its MLE is Kemeny's rule [Young JEP-95]

# Truth revealing

## Extended Condorcet Jury theorem

- Given
  - A ground truth ranking $W$
  - $0.5<p<1$,
- Suppose
  - each agent's preferences are generated i.i.d. according to Condorcet's model
- Then, as $n\to\infty$, with probability that $\to 1$
  - the randomly generated profile has a Condorcet winner
  - The Condorcet winner is ranked at the top of $W$
- If $r$ satisfies Condorcet criterion, then as $n\to\infty$, $r$ will reveal the "correct" winner with probability that $\to 1$.

# Other axioms

- Pareto optimality: For any profile $D$, there is no alternative $c$ such that every voter prefers $c$ to $r(D)$
- Consistency: For any profiles $D_1$ and $D_2$, if $r(D_1)=r(D_2)$, then $r(D_1 \cup D_2)=r(D_1)$
- Monotonicity: For any profile $D_1$,
  - if we obtain $D_2$ by only raising the position of $r(D_1)$ in one vote,
  - then $r(D_1)=r(D_2)$
  - In other words, raising the position of the winner won't hurt it

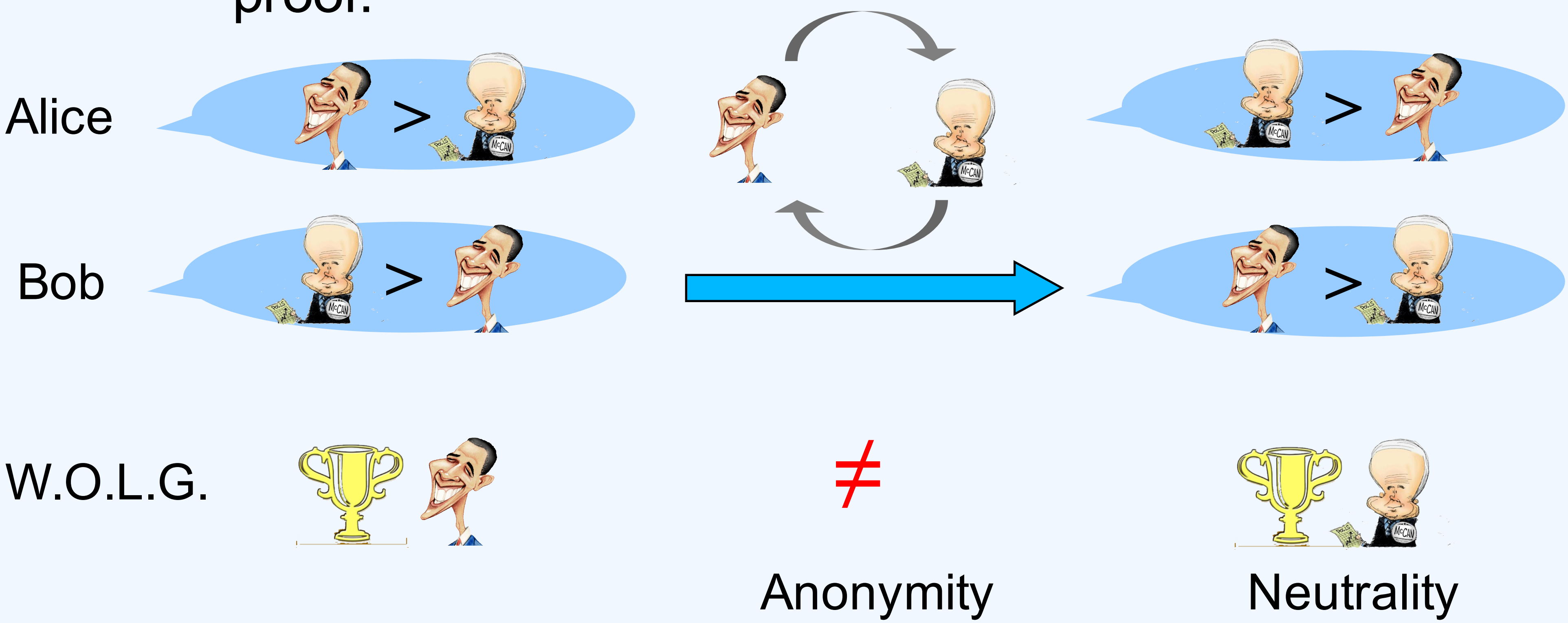
# Which axiom is more important?

|  | Condorcet criterion | Consistency | Anonymity/neutrality, non-dictatorship, monotonicity |
|---|---|---|---|
| Plurality | N | Y | Y |
| STV (alternative vote) | Y | N | Y |

- Some axioms are not compatible with others
- Which rule do you prefer?

# An easy fact

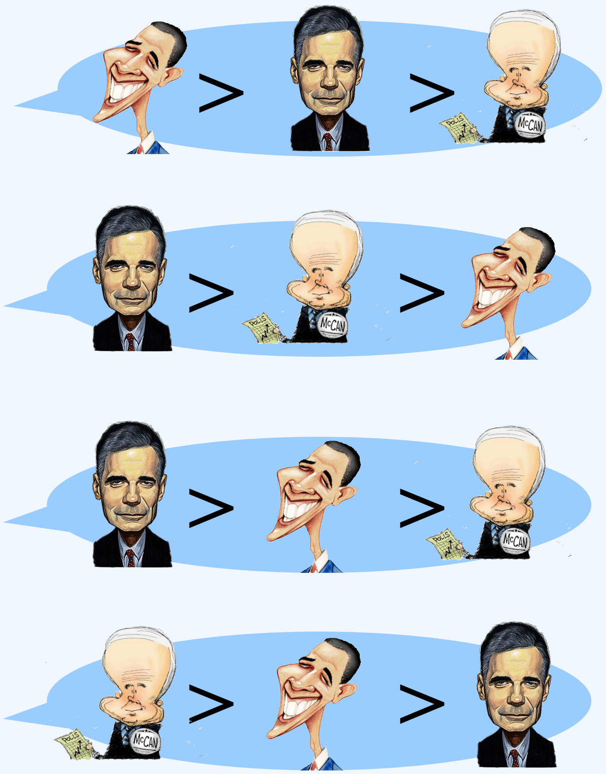
- Theorem. For voting rules that selects a single winner, anonymity is not compatible with neutrality
  - proof:



# Another easy fact [Fishburn APSR-74]

- **Theorem.** No positional scoring rule satisfies Condorcet criterion:
  - suppose $s_1 > s_2 > s_3$

3 Voters
2 Voters
1 Voter
1 Voter

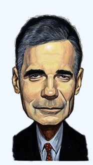


 is the Condorcet winner

CONTRADICTION

 : $3s_1 + 2s_2 + 2s_3$

$\wedge$

 : $3s_1 + 3s_2 + 1s_3$

# Arrow's impossibility theorem

- Recall: a social welfare function outputs a ranking over alternatives
- Arrow's impossibility theorem. No social welfare function satisfies the following four axioms
  - Non-dictatorship
  - Universal domain: agents can report any ranking
  - Unanimity: if $a$>b in all votes in $D$, then $a$>b in $r(D)$
  - Independence of irrelevant alternatives (IIA): for two profiles $D_1$= $(R_1,\ldots,R_n)$ and $D_2$=$(R_1',\ldots,R_n')$ and any pair of alternatives $a$ and $b$
    - if for all voter $j$, the pairwise comparison between $a$ and $b$ in $R_j$ is the same as that in $R_j'$
    - then the pairwise comparison between $a$ and $b$ are the same in $r(D_1)$ as in $r(D_2)$

# Other Not-So-Easy facts

- Gibbard-Satterthwaite theorem
  - Later in the “hard to manipulate” class
- Axiomatic characterization
  - Template: A voting rule satisfies axioms A1, A2, A2 ⇔ if it is rule X
  - If you believe in A1 A2 A3 are the most desirable properties then X is optimal
  - (unrestricted domain+unanimity+IIA) ⇔ dictatorships [Arrow]
  - (anonymity+neutrality+consistency+continuity) ⇔ positional scoring rules [Young SIAMAM-75]
  - (neutrality+consistency+Condorcet consistency) ⇔ Kemeny [Young&Levenglick SIAMAM-78]

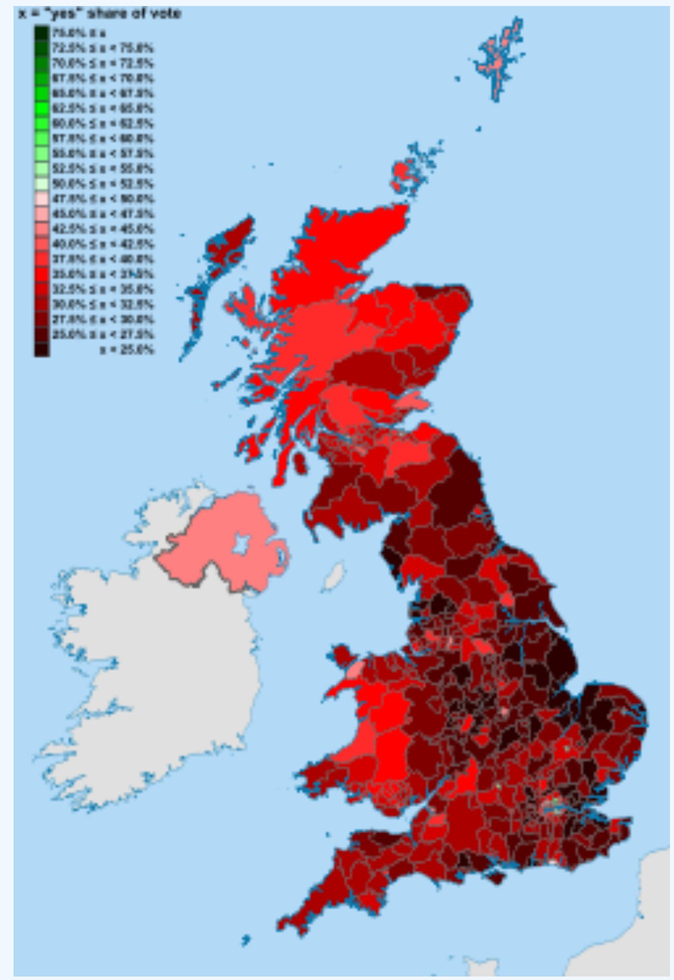
# Remembered all of these?

- Impressive! Now try a slightly larger tip of the iceberg at wiki

# Change the world: 2011 UK Referendum

- The second nationwide referendum in UK history
  - The first was in 1975
- Member of Parliament election:

  Plurality rule ➔ Alternative vote rule
- 68% No vs. 32% Yes
- Why people want to change?
- Why it was not successful?
- Which voting rule is the best?



# Wrap up

- Voting rules
  - positional scoring rules
  - multi-round elimination rules
  - WMG-based rules
  - A Ground-truth revealing rule (Kemeny's rule)
- Criteria (axioms) for "good" rules
  - Fairness axioms
  - A ground-truth-revealing axiom (Condorcet consistency)
  - Other axioms
- Evaluation
  - impossibility theorems
  - Axiomatic characterization