Matching and Resource Allocation

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Nobel prize in Economics 2013

- "for the theory of stable allocations and the practice of market design."

Alvin E. Roth  
Lloyd Shapley
Two-sided one-one-one matching

Boys
- Stan
- Kyle
- Kenny
- Eric

Girls
- Wendy
- Rebecca
- Kelly

Applications: student/hospital, National Resident Matching Program
Formal setting

• Two groups: $B$ and $G$

• Preferences:
  – members in $B$: full ranking over $G \cup \{\text{nobody}\}$
  – members in $G$: full ranking over $B \cup \{\text{nobody}\}$

• Outcomes: a matching $M$: $B \cup G \rightarrow B \cup G \cup \{\text{nobody}\}$
  – $M(B) \subseteq G \cup \{\text{nobody}\}$
  – $M(G) \subseteq B \cup \{\text{nobody}\}$
  – $[M(a)=M(b) \neq \text{nobody}] \Rightarrow [a=b]$
  – $[M(a)=b] \Rightarrow [M(b)=a]$
Example of a matching

Boys

- Stan
- Kyle
- Kenny
- Eric

Girls

- Wendy
- Rebecca
- Kelly

nobody
Good matching?

• Does a matching always exist?
  – apparently yes

• Which matching is the best?
  – utilitarian: maximizes “total satisfaction”
  – egalitarian: maximizes minimum satisfaction
  – but how to define utility?
Given a matching $M$, $(b, g)$ is a **blocking pair** if
- $g >_b M(b)$
- $b >_g M(g)$
- ignore the condition for nobody

A matching is **stable**, if there is no blocking pair
- no (boy, girl) pair wants to deviate from their currently matches
Example

Boys

Stan

Kyle

Kenny

Eric

Girls

Wendy

Rebecca

Kelly
A stable matching

Boys

Girls

Stan

Wendy

Kyle

Rebecca

Kenny

Kelly

Eric

no link = matched to “nobody”
An unstable matching

Boys

- Stan
- Kyle
- Kenny
- Eric

Girls

- Wendy
- Rebecca
- Kelly

Blocking pair: (Stan, Wendy)
Does a stable matching always exist?

- Yes: Gale-Shapley’s deferred acceptance algorithm (DA)
- Men-proposing DA: each girl starts with being matched to “nobody”
  - each boy proposes to his top-ranked girl (or “nobody”) who has not rejected him before
  - each girl rejects all but her most-preferred proposal
  - until no boy can make more proposals
- In the algorithm
  - Boys are getting worse
  - Girls are getting better
Men-proposing DA (on blackboard)

Boys

- Stan
- Kyle
- Kenny
- Eric

Girls

- Wendy
- Rebecca
- Kelly
Round 1

Boys
- Stan
- Kyle
- Kenny
- Eric

Girls
- Wendy
- Rebecca
- Kelly
- nobody

Stan → Wendy
Kyle → Rebecca
Kenny → reject
Eric → nobody
Round 2

Boys

Stan
Kyle
Kenny
Eric

Girls

Wendy
Rebecca
Kelly
nobody
Women-proposing DA (on blackboard)

Boys

Stan

Kyle

Kenny

Eric

Girls

Wendy

Rebecca

Kelly

N

N

N

N

N

N
Round 1

Boys

Stan
Kyle
Kenny
Eric

Girls

Wendy
Rebecca
Kelly

reject

nobody
Round 2

Boys

Stan
Kyle
Kenny
Eric

Girls

Wendy
Rebecca
Kelly
nobody

reject
Round 3

Boys

- Stan
- Kyle
- Kenny
- Eric

Girls

- Wendy
- Rebecca
- Kelly

nobody
Women-proposing DA with slightly different preferences

Boys

Stan: N > Girls > Boys > N
Kyle: N > Girls > Boys > N
Kenny: N > Girls > Boys > N
Eric: N > Girls > Boys > N

Girls

Wendy: Girls > Boys > N
Rebecca: Boys > Girls > N
Kelly: Boys > Girls > N
Round 1

Boys
- Stan
- Kyle
- Kenny
- Eric

Girls
- Wendy
- Rebecca
- Kelly

reject

nobody
Round 2

Boys

- Stan
- Kyle
- Kenny
- Eric

Girls

- Wendy
- Rebecca
- Kelly
- nobody

Reject
Round 3

Boys

Stan

Kyle

Kenny

Eric

Girls

Wendy

Rebecca

Kelly

nobody

reject
Round 4

Boys

Stan

Kyle

Kenny

Eric

Girls

Wendy

Rebecca

Kelly

nobody

reject
Round 5

Boys
- Stan
- Kyle
- Kenny
- Eric

Girls
- Wendy
- Rebecca
- Kelly

nobody
Properties of men-proposing DA

• Can be computed efficiently
• Outputs a stable matching
  – The best stable matching for boys, called men-optimal matching
  – and the worst stable matching for girls
• Strategy-proof for boys
The men-optimal matching

- For each boy $b$, let $g_b$ denote his most favorable girl matched to him in any stable matching.
- A matching is men-optimal if each boy $b$ is matched to $g_b$.
- Seems too strong, but…
Men-proposing DA is men-optimal

- **Theorem.** The output of men-proposing DA is men-optimal
- **Proof:** by contradiction
  - suppose $b$ is the first boy not matched to $g \neq g_b$ in the execution of DA,
  - let $M$ be an arbitrary matching where $b$ is matched to $g_b$
  - Suppose $b'$ is the boy whom $g_b$ chose to reject $b$, and $M(b')=g'$
  - $g' >_{b'} g_b$, which means that $g'$ rejected $b'$ in a previous round

![Diagram](DA)  ![Diagram](M)
Strategy-proofness for boys

• **Theorem.** Truth-reporting is a dominant strategy for boys in men-proposing DA
No matching mechanism is strategy-proof and stable

- Proof.

If (S, W) and (K, R) then
If (S, R) and (K, W) then
Recap: two-sided 1-1 matching

- Men-proposing deferred acceptance algorithm (DA)
  - outputs the men-optimal stable matching
  - runs in polynomial time
  - strategy-proof on men’s side
Example

Agents

Houses

Stan

Kyle

Eric
Formal setting

- Agents $A = \{1, \ldots, n\}$
- Goods $G$: finite or infinite
- Preferences: represented by utility functions
  - agent $j$, $u_j : G \rightarrow \mathbb{R}$
- Outcomes = Allocations
  - $g : G \rightarrow A$
  - $g^{-1} : A \rightarrow 2^G$
- Difference with matching in the last class
  - 1-1 vs 1-many
  - Goods do not have preferences
Efficiency criteria

• Pareto dominance: an allocation $g$ Pareto dominates another allocation $g'$, if
  • all agents are not worse off under $g$
  • some agents are strictly better off

• Pareto optimality
  – allocations that are not Pareto dominated

• Maximizes social welfare
  – utilitarian
  – egalitarian
Fairness criteria

• Given an allocation $g$, agent $j_1$ envies agent $j_2$ if $u_{j_1}(g^{-1}(j_2)) > u_{j_1}(g^{-1}(j_1))$

• An allocation satisfies envy-freeness, if
  – no agent envies another agent
  – c.f. stable matching

• An allocation satisfies proportionality, if
  – for all $j$, $u_j(g^{-1}(j)) \geq u_j(G)/n$

• Envy-freeness implies proportionality
  – proportionality does not imply envy-freeness
Why not…

• Consider fairness in other social choice problems
  – voting: does not apply
  – matching: when all agents have the same preferences
  – auction: satisfied by the 2\textsuperscript{nd} price auction

• Use the agent-proposing DA in resource allocation
  (creating random preferences for the goods)
  – stableness is no longer necessary
  – sometimes not 1-1
  – for 1-1 cases, other mechanisms may have better properties
Allocation of indivisible goods

- House allocation
  - 1 agent 1 good

- Housing market
  - 1 agent 1 good
  - each agent originally owns a good

- 1 agent multiple goods (not discussed)
House allocation

- The same as two sided 1-1 matching except that the houses do not have preferences
- The serial dictatorship (SD) mechanism
  - given an order over the agents, w.l.o.g.
    
    \[ a_1 \rightarrow \ldots \rightarrow a_n \]
  - in step \( j \), let agent \( j \) choose her favorite good that is still available
  - can be either centralized or distributed
  - computation is easy
Characterization of SD

• **Theorem.** Serial dictatorships are the only deterministic mechanisms that satisfy
  – strategy-proofness
  – Pareto optimality
  – neutrality
  – non-bossy
    • An agent cannot change the assignment selected by a mechanism by changing his report without changing his own assigned item

• Random serial dictatorship
Why not agent-proposing DA

- Agent-proposing DA satisfies
  - strategy-proofness
  - Pareto optimality
- May fail neutrality

  ![Stan](image1) : h1 > h2  
  ![Kyle](image2) : h1 > h2

  h1: S > K  
  h2: K > S

- How about non-bossy?
  - No
- Agent-proposing DA when all goods have the same preferences = serial dictatorship
Housing market

- Agent $j$ initially owns $h_j$
- Agents cannot misreport $h_j$, but can misreport her preferences
- A mechanism $f$ satisfies participation
  - if no agent $j$ prefers $h_j$ to her currently assigned item
- An assignment is in the core
  - if no subset of agents can do better by trading the goods that they own in the beginning among themselves
  - stronger than Pareto-optimality
Example: core allocation

- Stan: h1 > h2 > h3, owns h3
- Kyle: h3 > h2 > h1, owns h1
- Eric: h3 > h1 > h2, owns h2

Not in the core:
- Stan: h2
- Kyle: h3
- Eric: h1

In the core:
- Stan: h1
- Kyle: h3
- Eric: h2
The top trading cycles (TTC) mechanism

- Start with: agent \( j \) owns \( h_j \)
- In each round
  - built a graph where there is an edge from each available agent to the owner of her most-preferred house
  - identify all cycles; in each cycle, let the agent \( j \) gets the house of the next agent in the cycle; these will be their final allocation
  - remove all agents in these cycles
Properties of TTC

- **Theorem.** The TTC mechanism
  - is strategy-proof
  - is Pareto optimal
  - satisfies participation
  - selects an assignment in the core
    - the core has a unique assignment
  - can be computed in $O(n^2)$ time

- Why not using TTC in 1-1 matching?
  - not stable

- Why not using TTC in house allocation (using random initial allocation)?
  - not neutral
DA vs SD vs TTC

• All satisfy
  – strategy-proofness
  – Pareto optimality
  – easy-to-compute

• DA
  – stableness

• SD
  – neutrality

• TTC
  – chooses the core assignment
Multi-type resource allocation

- Each good is characterized by multiple issues
  - e.g. each presentation is characterized by topic and time

- Paper allocation
  - we have used SD to allocate the topic
  - we will use SD with reverse order for time

- Potential research project