$D \in \mathbb{R}^{n \times d}$

$n$: num of points

d: num of attributes (dimensionality)

"Curse of dimensionality"

hypercube:

2D: square

3D:

d - dir: hypercube

4D - hypercube
Hd: hypercube

hypersphere

...
2D:
line

3D: plane

\[ \mathbb{R}^d \]

\[ \text{normal vector} \]
\[ \text{(weight vector)} \]
\[ b \text{ - bias} \]
\[ \text{off set} \]

\[ d - \text{dim: hyperplane} \]

\[ y = \mathbf{w} \cdot \mathbf{x} + b \]

\[ \text{equation } \mathbf{y} = \mathbf{w} \cdot \mathbf{x} \]

\[ y - \mathbf{w} \cdot \mathbf{x} - b = 0 \]

\[ \begin{pmatrix} x \ T \ y \ end{pmatrix} \begin{pmatrix} -\mathbf{w} \ 1 \ end{pmatrix} + (-b) = 0 \]

\[ \begin{pmatrix} x \ y \ end{pmatrix} \begin{pmatrix} \mathbf{w} \ T \end{pmatrix} + (-b) = 0 \]
equation of a hyperplane in d-dim

$$\mathbf{x}^\top \mathbf{w} + b = 0$$

$$\mathbf{x} \in \mathbb{R}^d, \quad \mathbf{w} \in \mathbb{R}^d, \quad b \in \mathbb{R}$$

Set of all points $$\mathbf{x} \in \mathbb{R}^d$$ that satisfy this equation

$$x_1 w_1 + x_2 w_2 + \ldots + x_d w_d + b = 0$$

$$b$$: offser?

pick the $$(x_1, 0, 0, \ldots, 0)$$

$$x_1 w_1 = -b$$

$$x_1 = -\frac{b}{w_1}$$

in general

$$x_i = -\frac{b}{\omega_i}$$

intersection or offset along the i'th axis

$$\text{Vol}(H_d(l)) = l^d$$

$$\text{Vol}(S_2(r)) = \frac{1}{2} \pi r^2$$

$$\text{Vol}(S_1(r)) = \frac{1}{2} \pi r^3$$
\[
\text{vol}(S_d(r)) = \frac{r^d}{\Gamma\left(\frac{d}{2} + 1\right)} \Gamma\left(\frac{d}{2}\right)\\
k_d = \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d}{2} + 1\right)}\right)^{\frac{1}{d}}\\
\Gamma\left(\frac{d}{2} + 1\right) =\begin{cases} 
\frac{(d/2)!!}{\sqrt{\pi}} & \text{if } d \text{ is even} \\
\sqrt{\pi} \left(\frac{d!!}{2^{(d+1)/2}}\right) & \text{if } d \text{ is odd}
\end{cases}\\
(d/2)!! = \left(\frac{d}{2}\right)\left(\frac{d-2}{2}\right)\cdots\\
d!! = d(d-2)(d-4)\cdots
\]

\[
\text{vol}(S_d(r)) = \lim_{d \to \infty} k_d r^d
\]

\[
\gamma = 1 \implies \gamma \lim_{d \to \infty} k_d
\]

\[
\text{unit hypersphere...}
\]
In high dim a unit hypersphere has 0 volume --- "empty center"

\[
\text{Vol}(S_d(1)) = \frac{k_d y^d}{(2\pi)^{d/2}}
\]

2D: \[\frac{\pi y}{4 y^2} = \frac{\pi}{4} \approx 0.79\]

circle contains 79% of the vol

"Corners" contain 25% of the vol

3D: \[\frac{4}{3} \pi y^3 = \frac{4}{3} \cdot \frac{\pi}{6} = \frac{11}{6} \approx 58%\]
3D: \[ \frac{\frac{4}{3} \pi r^3}{8 r^3} = \frac{4}{3} \cdot \frac{1}{2} = \frac{11}{6} \approx 58\%. \]

56% of points are in corners.

d-dim: 2 corners
middle "hypersphere" is empty.

2D

all 9 points "migrate" to the corners
Vol of the thin $\varepsilon$-width shell

Vol of the outer hypersphere

\[
K_d r^d - K_d (r-\varepsilon)^d
\]

\[
\frac{r^d}{\varepsilon^d}
\]

\[
= 1 - \left(\frac{r-\varepsilon}{r}\right)^d
\]

as $d \to \infty$ \quad $\varepsilon > 0$

ratio $\to 1$

no matter how small $\varepsilon$ is, the thin shell contains 60% of the vol.

$\Rightarrow$ all the points lie in their thin shell.

1) all the points are on the boundary
2) $\varepsilon \to 0$ in the 2nd corners
3) center is empty

\[
\varepsilon_2 = (0,1)
\]

\[
\varepsilon_1 = (1,1) = \frac{\varepsilon_2}{2}
\]
\[
\cos \theta = \frac{\mathbf{e}_1 \cdot \mathbf{e}_2}{\| \mathbf{e}_1 \| \| \mathbf{e}_2 \|} = \frac{1}{\sqrt{2} \cdot 1} = \frac{1}{\sqrt{2}} = \sin^{-1} 45^\circ
\]

Angle between \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \):

\[
\frac{1}{\sin \theta}
\]

for \( \mathbf{x} \):

\[
\cos \theta(\mathbf{x}, \mathbf{e}_1) = \frac{1}{\sqrt{d}}
\]

as \( d \to \infty \), what happens to \( \cos \theta \):

\[
\cos \theta(1) : \frac{1}{\sqrt{d}} \to 0
\]

\[
\to 0 = 90^\circ
\]
\[ \Rightarrow \theta = 90^\circ \]

As \( d \to \infty \), the \( \mathbf{1} \) is oriented to \( \mathbf{e}_1 \)
\( \mathbf{2} \) to \( \mathbf{e}_2 \)
\( \ldots \)
\( \mathbf{d} \) to \( \mathbf{e}_d \)

Extra dimension

```
```

```
```

```
```

```
```
In $d$-dim, all of these $2^{d-1}$ directions are orthogonal to all of the $d$ original axes and they are also mutually orthogonal.

$d < \infty$

$d + 2$

Exponentially more "hidden axes".

Objects in high dim can vanish in lower dimensions.

Multivariate normal in $d$-dim

Standard multivariate normal
\[ \mu = 0 \]
\[ \Sigma = I = \text{identity} \]

**1D:**

\[ f(x|0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \]

Peak density \( \sigma = 0 \)

\[ f(x) = f(0) = \frac{1}{\sqrt{2\pi}} \]

If \( \frac{f(x)}{f(0)} \geq \alpha \)

\[ f(x|0, I) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{x^2}{2}} = \left( \frac{1}{(\sqrt{2\pi})^d} \right) e^{-\frac{||x||^2}{2}} \]
Peak: \( f(\hat{\alpha}) = \frac{1}{(\sqrt{2\pi})^d} \)

\[
\frac{f(\hat{\alpha})}{f(\hat{\beta})} \geq \alpha
\]

\[
P\left( \frac{f(\hat{\alpha})}{f(\hat{\beta})} \geq \alpha \right) = \text{what is the probability of being in a fraction of the peak density}
\]

what happens for small \(\alpha > 0\)

\(\alpha = 0.01\)

\(76\%\)

\[
P\left( \frac{f(\hat{\alpha})}{f(\hat{\beta})} = \frac{1}{(\sqrt{2\pi})^d} e^{-\frac{||x||^2}{2}} \right) \geq \alpha
\]
\[ 
\begin{align*} 
\frac{\|x\|^2}{2} & \geq \lambda \\
-\frac{\|x\|^2}{2} & \geq -2 \ln \lambda \\
\Rightarrow \quad \frac{\|x\|^2}{2} & \leq -2 \ln \lambda \\

P \left( \sum_{i=1}^{d} x_i^2 \leq -2 \ln \lambda \right) & \text{ follows a } \chi^2 \text{ distribution with } d \text{ degrees of freedom} \\
\chi & = 0.5 \\

P \left( \sum x_i^2 \leq -2 \ln 0.5 \right) & \\

\text{as } d \rightarrow \infty, \text{ for a fixed } \alpha, \text{ es. } \alpha = 0.5
\[ p\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{d} x_i^2 \right) \leq -2 \ln \alpha \Rightarrow 0 \]

\[ \alpha = 0.5 \]

\[ \Rightarrow \infty \]

\[ \beta : \beta = 1 \]

\[ d - \text{dimension} \]

\[ p\left( \sum_{i=1}^{d} x_i^2 \right) \leq -2 \ln \alpha = 0 \]

Empty

All the 'mass' or points are in the tail regions