\( G = (V, E) \)

\( V : \text{ vertex set} \)
\quad \text{or node}

\( E : \text{ edges} \)
\quad \text{if } E \subseteq V \times V \)

Undirected graph
\((u, v) \in \text{ unordered}\)

Directed graph
\((u, v) \in \text{ ordered}\)

(edges are ordered)
\((u, v) \in E \rightarrow u \rightarrow v\)

Weighted graphs
\( w(u, v) = \omega(v) \in \mathbb{R} \)
\((u, v) \in E \rightarrow \text{ nod number} \)

Labeled graphs
\( L(u) = \text{ label for node } u \)

Simple graph: there are no self-loops

Multigraph: multiple edges between two nodes
Let's analyze the given graph and its properties.

**Definition:**

A graph $G = (V, E)$ consists of a set of vertices $V$ and a set of edges $E$.

**Vertices:**

$V = \{1, 2, \ldots, n\}$

**Edges:**

$E$: Adjacency matrix

**Adjacency Matrix:**

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 0 & 1 & 0 \\
2 & 0 & 1 & 1 \\
3 & 1 & 0 & 0 \\
4 & 1 & 1 & 0 \\
\end{array}
$$

**Degree of a vertex $V$:**

$d(u) =$ number of vertices adjacent to $u$

$(u, x) \in E$ if and only if $X$ is adjacent to $u$.

$d(u) = \# \text{of neighbors}$

**Properties:**

$|V| = n$

$d(u) \leq n - 1$
\[
\text{Space:} \quad \text{avg } d(u) = \Theta(1) \\
E = n \cdot \Theta(1) \\
= \Theta(n)
\]

**Adjacency matrix:** \( \Theta(n^2) \) **space**

regardless of sparsity

**For sparse graphs**

\( |E| = \Theta(n) \)

Adjacency list format

\[
2 \cdot |E| = \Theta(n) \\
2 \cdot \Theta(n) = \Theta(n^2) \\
\]

\[ N(3) = \text{neighbors of 3?} \]
Consider the graph $G$ and the set $N(3) = \text{neighbors of } 3$. The adjacency matrix $A$ for $G$ is given as:

$$A = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}$$

The degree of a vertex $u$, denoted $\deg(u)$, is defined as the sum of the entries in the $u$-th row (in-degree) and in the $u$-th column (out-degree) of the adjacency matrix. For example:

- In-degree of 1: $\deg(1) = 0 + 1 = 1$
- Out-degree of 1: $\deg(1) = 1 + 0 = 1$

The set of vertices reachable from $u$ via some path is denoted $\text{reachability}(u) = \{v \mid (u, v) \in E\}$. For instance, if $G$ is a directed graph:

- $\text{reachability}(1) = \{2, 3\}$
- $\text{reachability}(3) = \{1\}$

In a directed graph, an "arc" is an ordered pair $(u, v)$, which represents a directed edge from vertex $u$ to vertex $v$. The adjacency list for $G$ shows:

- Vertices: 1, 2, 3, 4
- Edges: 1: 2
- 2: 1, 3, 4
- 3: 4
- 4: 1

The adjacency matrix is asymmetric, indicating the directions of the edges.
Reachability: Whether u can reach v via some "path".

Walk: Any alternating sequence of vertex & edge

\[
\begin{align*}
2 & \xrightarrow{e_3} 4 \xrightarrow{e_4} 1 \xrightarrow{e_2} 2 \xrightarrow{e_1} 1 \\
2 & \rightarrow 4 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1
\end{align*}
\]

length (walk) = 6

# of hops

Trail: A walk with no repeated edge

\[
\begin{align*}
2 & \rightarrow 4 \rightarrow 1 \rightarrow 2 \rightarrow 1
\end{align*}
\]

Path: A repeated vertex (except for 1st & last vertex)

\[
1 \rightarrow 2 \rightarrow 1
\]

distance in a network

Shortest path between u, v
Airline Fares

AUS

SFO

$100

$200

$300

$400

$500

24h!

Shortest path
# of hops
min total cost

"Reachability" "Connectedness"

DFS tree (G)

DFS (v):
- Input: v ∈ V
- Output: all vertices reachable from v

\[
\text{Visit}(v) = \top
\]

for each \( u \in N(v) \)

DFS (u)

DFS (x)
**Connected Component**

1. \( S \subseteq V \) such that for any \( x, y \in S \),
   - \( x \) and \( y \) are connected i.e., there is a path from \( x \) to \( y \) or \( y \) to \( x \).

2. \( S \) is maximal (largest in terms of subset of \( V \))
   - No vertex can be added.
   - No vertex can be deleted.

**Maximal Subset \( S \)**
- Mutually reachable vertices.

**CC**: Connected Component \( (G = (V, E)) \):

1. \( \forall \, V \in V \)
   - \( \text{visited}(v) = F \)
   - \( d(v) = 0 \)
   - \( cnum = 0 \)

2. \( \forall \, u \in V \)
   - \( y \in V \)
   - \( \text{visited}(v) = F \)
   - \( d(v) = 0 \)
   - \( cnum < cnum + 1 \)

\[ \sum_{u \in V} d(u) = 2 |E| = 2m \]
Sum of the degree = twice # of edge in Undirected G.

Total time: $O(M + |E|)$

Linear time in graphs $\mathbb{N} + m$

Pre-post numbers (DFS numbering)

Pre: count 1st time we enter (push)
Post: count when we leave the vertex (pop)

If u is reachable from v:

You can prove that

$\begin{bmatrix}
\text{pre}(v) & \text{pre}(u), \text{push}(u) & \text{push}(v) \\
\end{bmatrix}$

Interval of u is contained in interval of v.
Interval of $u$ is contained in Interval of $v$ for undirected graphs.

Find a sequence of containment:

Is $u$ connected to $v$?

Find some $x$ such that:

$\text{Interval}(u) \subseteq \text{Int}(x)$

$\text{Int}(v) \subseteq \text{Int}(x)$