Dijkstra's Shortest Path

G is directed, with positive weights

Priority & data structure to select the next min node

PQ: binary tree

1) root has the least value (min)
2) every node has a value less than its children

\[ 9 \ 7 \ 10 \ 3 \ 1 \ 5 \ 4 \ 2 \ 6 \ 8 \]

Making the heap in linear time

- Each node \[ \log n \] work
- \[ n \] nodes
- \[ n \log n \] work

\[ k \geq n \]

\[ \log_{2} n \]

Total # steps/time: \[ \frac{\log_{2} n}{h \cdot (\log_{2} n - h)} \]
Total # steps/time:

$$\sum_{i=0}^{\infty} \gamma^i = \frac{1}{1-\gamma}$$

$|\gamma| < 1$

geometric series

$$\sum_{i=0}^{\infty} i \cdot \gamma^i = \frac{\gamma}{(1-\gamma)^2}$$

$$\sum_{h=0}^{\left\lceil \frac{\log_2 n}{\gamma} \right\rceil} h \cdot \left(\frac{1}{2}\right)^h = n \cdot \frac{h}{\left(1-h^2\right)^2} = n \cdot \frac{1}{2} / \frac{1}{4}$$

$$= \frac{2n}{4} = 2 \cdot n$$

Dijkstra's Algorithm

$|V|$ delete min operations

$|E|$ decrease key ops

$(|V| + |E|) \cdot \log(|V|)$

Bellman-Ford:

$O(|E| \cdot |V|)$

handle negative weights

(no negative cycle)

Exercises

which offset to choose?

$C$

$\omega(u,v) + C$
longest possible path is \( |V| - 1 \)

negative cycle: sum of the weights on edges in the cycle is negative

Dijkstra's

Expanding search region/frontier

\[ 1 \] we can always update the dist of a vertex
$d(x) = \min \{ d(x), d(u) + l(u, x) \}$

2. Since the max path len is $|V| - 1$, after this many steps, we will have the shortest paths.

Bellman-Ford $(G, s)$

$\forall x \in V, d(x) = \infty$

$d(s) = 0$

$O(|V|)$

$\left( \begin{array}{c}
O(|E|) \\
\text{for } |V| - 1 \text{ steps} \\
\text{for every edge } e = (u, v) \in E
\end{array} \right)$

Update weight for $v$

$d(v) = \min \{ d(v), d(u) + l(u, v) \}$

Total: $O(|E| \cdot |V|)$

Checking negative cycle

- $D \leftarrow \emptyset$
- For $|V| - 1$ times
  - $D \leftarrow \emptyset$
  - End

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1. Run Bellman-Ford (for |V| - 1) steps
   \[ d_{\text{Bellman-Ford}}(s, x) \neq d(s, x) \quad \forall x \in V. \]
2. Run one more step of BF algo.
   If any distance changes, then there is a negative cycle.

Can you prove this? We have |V| - 1 as the max length of a shorter path.

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**Greedy Algorithms**

Choose the best step with only local information.

**Minimum Spanning Tree**

Input: Undirected graph \( G = (V, E) \) with positive weights on edges (Connected)

Output: A tree \( T = (V_T, E_T) \)

\[ V_T = V \quad \text{Spanning condition} \]

\[ E_T \leq E \]
\[ w(T) = \sum_{e \in E_T} w(e) \]

Choose the minimum weight tree among all possible trees.

**Kruskal’s Algorithm for MST**

1. One time sorting of the edges by weight
2. Pick edge in increasing order and add to \( T \) (tree)

If \( G \) is disconnected, we run MST on each component.

\( \text{MST: Min spanning forest} \)
and add to \((T)\) \(\leftarrow\) tree

(reject an edge \(y\) if introduce a cycle)

\[
T = \emptyset
\]

\[
= \{ A-C, C-D, A-B, C-E, E-F \}
\]

\(|E| - 1\) edge

\(w(T) = 16\)

Why should this work?

1. Sorting \(|E|\) edges

2. Each edge once

\(|E|\) steps

we have to check for cycle in \(T\)

\(O(|E|)\) time

\(|E| = |V|\)

\(|\log|E| = 2 \log |V|\)

\(O(|E| \log |V|)\)
Naive approach:

Run a DFS on $T$ each time
at most $T$ has $|V| - 1$ edges
and it has $|V|$ nodes

$O(|E| + |V|) = O(|V|)$ time

Total time
$|E| \cdot O(|V|)$ time

$\#$ steps

$O(|E| \cdot |V|)$

$(|E| \log |V| + |E| |V|) = O(|E| \cdot |V|)$

Union-find data structure $\leftarrow$ fast cycle checking

$\log |V|$ time

Kruskal: $O(|E| \log |V|)$ time