To show other $L'$ is $NP$-complete:

1) $L \in NP$
2) $L' \in NPC$, $L \leq^P L'$

Independent set: $\{I_S\}$
- a set $I$ such that, $I \subseteq V$
- $\forall x, y \in I$, $(x, y) \notin E$
- optimization: maximum $I_S$

Vertex cover: $\{VC\}$
- $C \subseteq V$
- $\forall (x, y) \in E$, either $x \in C$ or $y \in C$
Optimization: Minimum Vertex Cover

1) VC ∈ NP

- Given any set \( X \subseteq V \)
- **Verification**
  - Travers all true edge \((u,v) \in E\)
  - Check that either \( u \) or \( v \) ∈ \( X \)
- \(|X| \leq b \rightarrow \text{Yes}
- \( \leftarrow \) else No.

2) \( IS \rightarrow VC \)

- Given \( G \), \( I \subseteq V \), is an independent set
- **Transformation**
  - \( C = V - \overline{I} \) is a vertex cover

\( \Rightarrow \) \( I \) be an independent set, show \( C = V - I \) is a vertex cover

Consider \((x,y) \in E\)
- Show that either \( x \in C \) or \( y \in C \)

**Proof by Contradiction**
- Assume that \( x \notin C \) and \( y \notin C \)
- \( \Rightarrow x \in I \) and \( y \in I \)
Proof by Contradiction

\[ \Rightarrow x \in I \text{ and } y \in I \]
\[ \emptyset \leq \text{Contradiction} \]

\[ \leq \]

If \( C \) is a cover, show that \( I = V - C \) is an independent set.

\[ \exists (x, y) \in E, \text{ either } x \in C \text{ or } y \in C \]

Show that \( I \) is an independent set:

Pick \( x \in I \) and \( y \in I \)

\[ \text{Assume } (x, y) \in E \]

1. \( x \in C \)
   \[ \Rightarrow \quad \exists x \notin I \]
   \[ \checkmark \]

or

2. \( y \in C \)
   \[ \Rightarrow \quad y \notin I \]
   \[ \checkmark \]

Therefore, assumption is wrong.

or \( (x, y) \notin I \)

QED

Diagram:

\[ I = \{4, 5, 6, 7\} \]
\[ C = \{1, 2, 3\} \]
IS $\rightarrow$ clique, $\implies$ clique $\in$ NPC

1) $\text{clique} \in \text{NP}$

\[ C \subseteq V \]

$O(1c^2)$ check that $\forall x, y \in C, (x, y) \in E$ which is a complete subgraph

$= O(1|E|)$

2) $\text{IS} \rightarrow \text{clique}$

$G = (V, E)$ $\forall I$ in $G$ is a clique in $G$

$I \subseteq V$ is an independent set in $G$

\[ G = (V, E) \quad \forall I \text{ is a clique in } G \]

$I = \{v_1, v_2, v_3\}$

$ \Rightarrow \quad \forall x, y \in I \quad (x, y) \notin E \quad \iff \quad (x, y) \in E'$

$\iff I$ is a clique in $G$
\[ \Rightarrow I \text{ is a clique in } \overline{G} \]

\[ L \text{ is NPC} \]

\[ \begin{align*}
1) \quad & \text{hard problem on the worst case input} \\
2) \quad & \text{what about average case?} \\
3) \quad & \text{what about "practical" case?}
\end{align*} \]

\[ \text{better also will help.} \]

\[ 1) \quad \text{good algo that can potentially take super-polynomial time} \]

\[ 2) \quad \text{just design some algo that is poly time} \]

\[ \text{local optima} \quad \text{but there is no guarantee on the quality of the solution} \]

\[ 3) \quad \text{Approximation algorithms} \]

\[ \text{we will guarantee the quality} \]

\[ \alpha = \frac{A(I)}{\text{opt}(I)} \geq 1 \]

\[ \Rightarrow A \text{ is the approximation factor} \]
factor

maximization \[ \alpha = \frac{\text{Opt}(F)}{A(F)} \geq 1 \]

A is the proposed algorithm

Vertex Cover

Given \( G \)

1) Poly time?
2) \( \alpha \)?

1) Find a maximal matching of edges \( M \subseteq E \)
2) \( C = \{ \text{all the vertices that are endpoints of an edge in } M \} \)

Matching: a subset of edges with no endpoints in common

Algorithm for maximal matching

while \( E \neq \emptyset \)
1) Pick a random edge \((x,y)\)
2) Add to \( M \)
3) Delete all edges from \( x \) and from \( y \)

\( O(|V|+|E|) \)

\( C = \{ 1,2,3,5,9 \} \)

\( M = \{ (1,5), (3,7), (2,6) \} \)

Maximal matching

\( M \) has local maxima

\( M \) needs not be globally maximal

\( \emptyset \neq 3 \)
\[ C = \{ (1, 2, 3), 3, 2 \} \]

\[ |C| = 6 \]

Optimal Cover = 3 \{ 1, 3, 5 \}

\[ \alpha = \frac{6}{3} = 2 \]

Prove that the method has \( \alpha \leq 2 \)

\[ \text{OPT}(\pm) \text{ is generally unknown} \]

\[ \alpha = \frac{A(I)}{\text{OPT}(\pm)} \leq \frac{A(I)}{\text{LB}} \]

\[ \frac{\text{LB}}{\text{OPT}(I)} \text{ lower bound} \]

\# of edges in \( M \) is a lower bound on min vertex cover

\[ \frac{\text{LB}}{\text{OPT}(I)} \]

For any edge \((x, y) \in M\)

\[ x < r \]
For any edge \((x, y) \in E\) when \(x \in V\) or \(y \in V\)

Since all edges in \(M\) are "independent"

\[|C| \geq |M|\]

\(C = \{\text{set of all vertices spanned by } M\}\)

Is this a vertex cover?

\[|C| = 2|M|\]

\[\alpha \leq \frac{\lambda(E)}{\lambda(M)} = \frac{2|M|}{|M|} = 2\]

\(\forall y \in \{x, y\} \notin M, \text{ and } x \notin M \text{ or } y \notin M\)

\(\Rightarrow \) we have a contradiction

\(\Rightarrow M \) is not maximal \(\Box\)

TSP approximation

\(\Rightarrow\) if metric TSP: \(\alpha \leq 2\)
metric TSP: \( d \leq 2 \)

General TSP

airline prices for flying cities

Not metric

\[ d_{ij} + d_{jk} \geq d_{ik} \]

triangle inequality

\[ d_{ij} + d_{jk} = d_{ik} \]

No poly-time approximation possible (unless \( P = NP \))

hard to approximate

metric TSP pair-wise

given a distance between all vertices

\[ G, V \longrightarrow d_{ij} = \text{distance between } i \text{ & } j \]

find a tour, i.e. a path that starts at \( V_0 \),

visits every vertex once & returns to \( V_0 \)

minimize the total distance

distance are in a metric space

\[
\begin{align*}
1) & \quad d(x, y) \geq 0 \\
2) & \quad d(x, x) = 0 \\
3) & \quad d(x, y) = d(y, x)
\end{align*}
\]
OPT (I) ?
Unknown.

Lower bound on a TSP path.

MST is a lower bound on TSP.

Since paths $\leq$ trees

1) $\text{Cost(MST)} \leq \text{Cost(TSP path)}$

Lower bound.

Approx

2) Convert MST into an $\overline{\text{TSP}}$ solution
   a) Create a tour with each edge visited twice
Cost = \( 2 \cdot \text{Cost}(\text{AST}) \)

b) Convert this double tour into a path, then connect \( v_0 \) to \( v_0 \).

poly time we can into a path.

\[ x = \frac{A(I)}{\text{OPT}(I)} \leq \frac{A(I)}{\text{LB}} = \frac{2 \text{AST}}{\text{LB}} \]