Part 1: Overview of Ordinary Differential Equations
Chapter 1

Basic Concepts and Problems

1.1 Problems Leading to Ordinary Differential Equations


1. Mechanical Vibrations. Let \( y(t) \) denote the displacement at time \( t \) of a block of mass \( m \) that is connected to a spring of stiffness \( k \), a damper of resistance \( c \), and an oscillator \( f(t) \) (Figure 1.1.1). If the system is released from position \( y_0 \) with a velocity \( y'_0 \) then its subsequent motion satisfies

\[
m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = f(t), \quad t > 0, \quad y(0) = y_0, \quad \frac{dy(0)}{dt} = y'_0.
\]

This is an example of an initial value problem (IVP) for a second-order linear ODE.

- The variable \( y(t) \) is called:
- The variable \( t \) is called:
- The ODE is second order because:
- It’s linear because:
- This is an IVP because:
2. **Ecology.** Consider a population of predators and prey living in an ecological niche. The predators survive by eating the prey and the prey exist on an independent source of food. A classical situation involves foxes and rabbits. Let $P(t)$ and $p(t)$, respectively, denote the populations of predators and prey at time $t$. Given the initial populations $P_0$ and $p_0$ of predators and prey, their subsequent populations satisfy the *Lotka-Volterra* equations

\[
\frac{dp}{dt} = p(a - aP), \quad t > 0, \quad p(0) = p_0,
\]

\[
\frac{dP}{dt} = P(-c + \gamma p), \quad t > 0, \quad P(0) = P_0,
\]

where $a$, $c$, $\alpha$, and $\gamma$ are positive constants corresponding to the prey’s natural growth rate, the predator’s natural death rate, the prey’s death rate upon coming into contact with predators, and the predator’s growth rate upon coming into contact with prey. This example involves an IVP for a system of two first-order nonlinear ODEs.

- The ODEs are *nonlinear* because:

- The system is *first order* because:

3. **Column Buckling.** The lateral displacement $y(t)$ at position $t$ of a clamped-hinged bar of length $l$ that is subjected to a load $P$ (Figure 1.1.2) may be approximated by the *Euler-Bernoulli* equations

\[
\frac{d^4y}{dt^4} + \lambda \frac{d^2y}{dt^2} = 0, \quad 0 < t < l,
\]

with the boundary conditions

\[
y(0) = \frac{dy(0)}{dt} = 0, \quad y(l) = \frac{d^2y(l)}{dt^2} = 0.
\]
The parameter $\lambda = P/EI$, where $EI$ is the flexural rigidity of the bar. This is an example of a boundary value problem (BVP) for a fourth-order linear ODE.

- This is a *boundary value* problem because:

Observe that $y(t) = 0$ is a solution of this problem for all values of $\lambda$ and $l$. A more interesting problem is to determine $y$ and those values of $\lambda$ and $l$ for which non-trivial ($y(t) \neq 0$) solutions exist. As such, this BVP is also a differential eigenvalue problem. Nontrivial solutions $y(t)$ are called *eigen functions* and their corresponding values of $\lambda$ are *eigenvalues*.

![Figure 1.1.2: Buckling of an elastic column.](image)

4. *Pendulum Oscillations* ([1], Section 1.3). The position $(x(t), y(t))$ at time $t$ of a particle of mass $m$ oscillating on a pendulum of length $l$ (Figure 1.1.3) is

$$
m \frac{d^2 x}{dt^2} = -T \sin \theta = -\frac{T}{l} x, \quad m \frac{d^2 y}{dt^2} = mg - T \cos \theta = mg - \frac{T}{l} y, \quad t > 0,
$$

where $g$ is the acceleration of gravity, $T(t)$ is the tension in the string, and $\theta(t)$ is the angle of the pendulum relative to the vertical at time $t$ (Figure 1.1.3). These equations are, however, insufficient to guarantee that the particle stays on the string. To ensure that this is so, we must supplement the ODEs by the algebraic constraint

$$
x^2 + y^2 = l^2.
$$

The two second-order differential equations and the constraint comprise a system of five *differential algebraic equations* (DAEs) for the unknowns $x(t)$, $y(t)$, and $T(t)$. Initial conditions specify $x(0)$, $dx(0)/dt$, $y(0)$, and $dy(0)/dt$, but not $T(0)$.
Of course, for this problem, it’s easy to eliminate the constraint by introducing the change of variables $x = l \sin \theta$, $y = l \cos \theta$. This would reduce the DAEs to the second-order ODE

$$\frac{d^2 \theta}{dt^2} = -\frac{g}{l} \sin \theta.$$ 

Such simplifications would not be possible with more difficult systems.

![Diagram of a simple pendulum](image)

Figure 1.1.3: Oscillations of a simple Pendulum.

IVPs will comprise our initial study (Part 2 of these notes). We’ll take up BVPs next (Part 3) and conclude with a study of DAEs (Part 4). In almost all cases, it will suffice to develop and analyze numerical methods for IVPs and BVPs that are written as first-order vector systems of ODEs in the explicit form

$$y'(t) = f(t, y), \quad t > 0,$$  \hspace{1cm} (1.1.1a)

where

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix}, \quad f(t, y) = \begin{bmatrix} f_1(t, y_1, \ldots, y_m) \\ f_2(t, y_1, \ldots, y_m) \\ \vdots \\ f_m(t, y_1, \ldots, y_m) \end{bmatrix},$$  \hspace{1cm} (1.1.1b)

and $y' := dy/dt$. Data for IVPs would be specified as

$$y(0) = y_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{bmatrix}.$$  \hspace{1cm} (1.1.1c)
Boundary data is more complex. The most general boundary conditions specify a nonlinear relationship between \( y(0) \) and \( y(l) \) of the form

\[
g(y(0), y(l)) = \begin{bmatrix} g_1(y(0), y(l)) \\ g_2(y(0), y(l)) \\ \vdots \\ g_m(y(0), y(l)) \end{bmatrix} = 0.
\] (1.1.1d)

In many cases the extension of a scalar to a vector IVP will be obvious and we can study methods for a scalar IVP

\[
y'(t) = f(t, y), \quad t > 0, \quad y(0) = y_0.
\] (1.1.2)

**Example 1.1.1.** Higher-order ODEs can be written as first-order systems. Consider the \( m \) th order equation

\[
z^{(m)} = g(t, z, z', \ldots, z^{(m-1)}),
\]

where \( z^{(i)} := \frac{d^i z}{dt^i} \), and introduce the new variables

\[
y_1 = z, \quad y_2 = z', \quad y_3 = z'', \quad \ldots, \quad y_m = z^{(m-1)}.\]

Then, we have a system in the form (1.1.1) with

\[
y_1' = y_2, \quad y_2' = y_3, \quad \ldots, \quad y_m' = g(t, y_1, y_2, \ldots, y_m).
\]

For an IVP, that data would prescribe as

\[
z^{(i)}(0) = c_i, \quad i = 0, 1, \ldots, m - 1,
\]

with the understanding that \( z^{(0)} = z \). Written in terms of \( y \), we have

\[
y_1(0) = c_0, \quad y_2(0) = c_1, \quad \ldots \quad y_m(0) = c_{m-1}.
\]

With DAEs and sometimes with IVPs and BVPs, it will be necessary to consider *implicit* differential systems

\[
F(t, y, y') = 0.
\] (1.1.3)

If the Jacobian \( \partial F / \partial y' \) is nonsingular then, by the implicit function theorem, (1.1.3) is equivalent to (1.1.1); however, it may be natural to solve (1.1.3) in its implicit form to maintain, e.g., sparsity. DAEs involve systems where \( \partial F / \partial y' \) is singular.
Example 1.1.2. Let us write the oscillating pendulum problem as a first-order system by letting

\[ y_1 = x, \quad y_2 = x', \quad y_3 = y, \quad y_4 = y'. \]

Then, we have

\[ y'_1 = y_2, \quad y'_2 = -\frac{T}{ml}y_1, \quad y'_3 = y_4, \quad y'_4 = g - \frac{T}{ml}y_3, \quad y_5 = T \]

with the constraint

\[ y_1^2 + y_3^2 = l^2. \]

This can be written in the form (1.1.3) with

\[ F(t, y, y') = \begin{bmatrix} y'_1 - y_2 \\ y'_2 + \frac{y_1 y_3}{ml} \\ y'_3 - y_4 \\ y'_4 - g + \frac{y_1 y_3}{ml} \\ y_1^2 + y_3^2 - l^2 \end{bmatrix} = 0. \]

The Jacobian

\[ F_y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

is clearly singular.

Problems

1. Letting \( y_1 = p \) and \( y_2 = P \), write the predator-prey model

\[ \frac{dp}{dt} = p(a - \alpha P), \quad t > 0, \quad p(0) = p_0, \]

\[ \frac{dP}{dt} = P(-c + \gamma p), \quad t > 0, \quad P(0) = P_0, \]

in the vector form (1.1.1).

2. Write the mechanical vibration problem

\[ m \frac{d^2 y}{dt^2} + c \frac{dy}{dt} + ky = f(t), \quad t > 0, \quad y(0) = y_0, \quad \frac{dy(0)}{dt} = y'_0, \]

in the form (1.1.1a-c)
3. Write the column buckling problem

\[
\frac{d^3y}{dt^3} + \lambda \frac{d^2y}{dt^2} = 0, \quad 0 < t < l, \quad y(0) = \frac{dy(0)}{dt} = 0, \quad y(l) = \frac{d^2y(l)}{dt^2} = 0,
\]

in the form (1.1.1a,b,d)

### 1.2 Existence, Uniqueness, and Stability of IVPs

Before considering numerical procedures for (1.1.1, 1.1.3), let us review some results from ODE theory that ensure the existence of unique solutions which depend continuously on the initial data. For simplicity, let us focus on a scalar IVP having the form (1.1.2).

**Definition 1.2.1.** A function \( f(t, y) \) satisfies a Lipschitz condition in a domain \( D \) if there exists a non-negative constant \( L \) such that

\[
|f(t, y) - f(t, z)| \leq L|y - z|, \quad \forall (t, y), (t, z) \in D. \tag{1.2.4}
\]

Satisfying a Lipschitz condition guarantees that solutions \( y(t) \) of (1.1.2) are unique as expressed by the following theorem.

**Theorem 1.2.1.** Let \( f(t, y) \) be continuous and satisfy a Lipschitz condition on

\[
\{(t, y) \mid 0 \leq t \leq T, -\infty < y < \infty\}.
\]

Then the IVP (1.1.2) has a unique continuously differentiable solution on \([0, T]\) for all \( y_0 \in (-\infty, \infty) \).

**Proof.** The proof appears in most books on ODE theory, e.g., [2].

**Remark 1.** A Lipschitz conditions guarantees unique solutions and is not needed for existence. For example, the IVP

\[
y' = 3y^{2/3}, \quad y(0) = 0,
\]

has the two solution \( y(t) = 0 \) and \( y(t) = t^3 \). This \( f(t, y) = 3y^{2/3} \) does not satisfy a Lipschitz condition.
Remark 2. Theorem 1.2.1 applies when \( f \) satisfies a Lipschitz condition on a compact, rather than an unbounded, domain \( D \) as long as the solution \( y(t) \) remains in \( D \) [2].

In addition to existence and uniqueness, we will want to know something about the stability of solutions of the IVP. In particular, we will usually be interested in the sensitivity of the solution to small changes in the data. Perturbations arise naturally in numerical computation due to discretization and round off errors. A formal study of sensitivity would lead us to the following notion of a well-posed system.

**Definition 1.2.2.** The IVP (1.1.2) is well-posed if there exists positive constants \( k \) and \( \dot{c} \) such that, for any \( \epsilon \leq \dot{c} \), the perturbed IVP

\[
\dot{z} = f(t, z) + \delta(t), \quad z(0) = y_0 + \epsilon_0,
\]

satisfies

\[
|y(t) - z(t)| \leq k\epsilon
\]

whenever \( |\epsilon_0| < \epsilon \) and \( |\delta(t)| < \epsilon \) for \( t \in [0, T] \).

Again, a Lipschitz condition ensures that we are dealing with a well-posed IVP.

**Theorem 1.2.2.** If \( f(t, y) \) satisfies a Lipschitz condition on

\[
\{(t, y) \mid 0 \leq t \leq T, -\infty < y < \infty\}
\]

then \( y' = f(t, y) \) is well posed on \([0, T]\) with respect to all initial data.

**Proof.** Let

\[
\zeta(t) = z(t) - y(t)
\]

and subtract (1.1.2) from (1.2.5) to obtain

\[
\zeta'(t) = f(t, z) - f(t, y) + \delta(t), \quad \zeta(0) = \epsilon_0.
\]

Taking an absolute value and using the Lipschitz condition (1.2.4)

\[
|\zeta'(t)| \leq L|\zeta(t)| + |\delta(t)|, \quad |\zeta(0)| = \epsilon_0.
\]
We easily see that $|\zeta(t)|' \leq |\zeta'(t)|$ provided that $|\zeta(t)|'$ exists and it is fairly easy to show that $|\zeta(t)|'$ exists. Additionally, by assumption,

$$\max_{0 \leq t \leq T} |\delta(t)| < \epsilon, \quad \max_{0 \leq t \leq T} |\epsilon_0| < \epsilon,$$

so

$$|\zeta(t)|' \leq L|\zeta(t)| + \epsilon, \quad |\zeta(0)| < \epsilon.$$

Multiply by the integrating factor $e^{-Lt}$ to obtain

$$(e^{-Lt}|\zeta(t)|)' \leq e^{-Lt}, \quad |\zeta(0)| < \epsilon.$$  

Integrating

$$|\zeta(t)| \leq \frac{\epsilon}{L(L+1)e^{Lt} - 1}.$$  

Since the denominator is smallest when $t = 0$

$$|z(t) - y(t)| \leq \frac{\epsilon}{L^2} = k\epsilon, \quad \forall t \in [0, T];$$

thus, $y' = f(t, y)$ is well posed on $[0, T]$.  

The notion of a well-posed system is related to the more common notion of stability as indicated by the following definition.

**Definition 1.2.3.** Consider the differential equation $y' = f(t, y)$ and (without loss of generality) let the origin $(0, 0)$ be an equilibrium point, i.e., $f(0, 0) = 0$. Then, the origin is:

- **stable** if a perturbation of the initial condition $|y(0)| < \epsilon$ grows no larger than $\epsilon$ for subsequent times, i.e., if $|y(t)| < \epsilon$ for $t > 0$.

- **asymptotically stable** if it is stable and $|y(0)| < \epsilon$ implies that $\lim_{t \to \infty} |y(t)| = 0$.

- **unstable** if it is not stable.

**Remark 3.** This definition could, like Definition 1.2.2, also involve perturbations of $f(t, y)$. We have omitted these for simplicity.
An autonomous system is one where \( f(t, y) \) does not explicitly depend on \( t \), i.e., \( f(t, y) = f(y) \). If \((0, 0)\) is an equilibrium point then, in this case, \( f(0) = 0 \). Expanding the solution in a Taylor’s series in \( y \), we have

\[
y'(t) = f(y) = f(0) + f_y(0)y(t) + O(y^2).
\]

Since \( f(0) = 0 \),

\[
y'(t) = f_y(0)y(t) + O(y^2),
\]

where \( f_y = \frac{\partial f}{\partial y} \). Recall that a function \( g(y) \) is \( O(y^p) \) if there exists a constant \( C > 0 \) such that \( |g(y)| \leq Cy^p \) as \( y \to 0 \).

Letting \( \lambda = f_y(0) \), we see that the stability of an autonomous system is related to that of the simple linear IVP

\[
y' = \lambda y, \quad t > 0, \quad y(0) = \epsilon. \tag{1.2.7}
\]

Additional details on autonomous and non-autonomous systems appear in Birkhoff and Rota [2] or Hairer et al. [5], Section 1.13. The solution of the linear IVP (1.2.7) is

\[
y(t) = \epsilon e^{\lambda t},
\]

hence, (1.2.7) is

\[
\begin{align*}
\text{•} \quad \text{stable when } Re(\lambda) \leq 0, \\
\text{•} \quad \text{asymptotically stable when } Re(\lambda) < 0, \text{ and} \\
\text{•} \quad \text{unstable when } Re(\lambda) > 0.
\end{align*}
\]

These conclusions remain true for the original nonlinear autonomous problem when \( Re(\lambda) \neq 0 \) and \( y \) is small enough for the \( O(y^2) \) term to be negligible relative to \( \lambda y \). This cannot happen in the stable case \( (Re(\lambda) = 0) \); hence, it requires a more careful analysis.

Remark 4. The analysis of non-autonomous systems is similar [1].

Analyzing the stability of autonomous vector systems

\[
y' = f(y(t)) \tag{1.2.8}
\]
is more complicated. Once again, assume that \( y = 0 \) is an equilibrium point and expand \( f(0) \) in a series

\[
y' = f(0) + f_y(0)y + O(\|y\|^2) = f_y(0)y + O(\|y\|^2).
\]

We need a brief digression for a few definitions.

**Definition 1.2.4.** The Jacobian matrix of a vector-valued function \( f(y) \) with respect to \( y \) is the matrix

\[
f_y := \frac{\partial f}{\partial y} = \begin{bmatrix}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_m} \\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_m}
\end{bmatrix}.
\]  

(1.2.9)

**Definition 1.2.5.** The norm of a vector \( y \) is a scalar \( \|y\| \) such that

1. \( \|y\| \geq 0 \) and \( \|y\| = 0 \) if and only if \( y = 0 \),

2. \( \|\alpha y\| = |\alpha|\|y\| \) for any scalar \( \alpha \), and

3. \( \|x + y\| \leq \|x\| + \|y\| \).

We’ll identify specific vector norms when they are needed. For the moment, let’s return to our stability analysis and let \( A = f_y(0) \). Then, the stability of the nonlinear autonomous system (1.2.8) is related to that of the linear system

\[
y' = Ay, \quad y(0) = y_0, \quad \|y_0\| \leq \varepsilon.
\]  

(1.2.10a)

The solution of this system is

\[
y(t) = e^{At}y_0,
\]  

(1.2.10b)

where the matrix exponential is defined by the series

\[
e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots.
\]  

(1.2.10c)

Often, \( A \) can be diagonalized as

\[
T^{-1}AT = \Lambda = \begin{bmatrix}
\lambda_1 & & \\
& \lambda_2 & \\
& & \cdots \\
& & & \lambda_m
\end{bmatrix}
\]  

(1.2.11)

13
where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of $A$ and the columns of $T$ are the corresponding eigenvectors. In this case, we may easily verify that the solution of (1.2.10) is

$$y(t) = Te^{\Lambda t}T^{-1}y_0.$$  

Thus, (1.2.10) is stable if all of the eigenvalues have non-positive real parts, asymptotically stable if all of the eigenvalues have negative real parts, and unstable otherwise.

Unfortunately, not all matrices are diagonalizable. However, $A$ can always be reduced to the Jordan canonical form

$$T^{-1}AT = \begin{bmatrix}
\Lambda_1 & & & \\
& \Lambda_2 & & \\
& & \ddots & \\
& & & \Lambda_i
\end{bmatrix}$$

(1.2.12a)

where each Jordan block has the form

$$\Lambda_i = \begin{bmatrix}
\lambda_i & 1 & & & \\
& \lambda_i & & \\
& & \ddots & 1 & \\
& & & \lambda_i
\end{bmatrix}$$

(1.2.12b)

The dimension of the Jordan block $\Lambda_i$ corresponds to the multiplicity of the eigenvalue $\lambda_i$. Thus, if $\lambda_i$ is simple, the block is a scalar. With this, it is relatively easy to show [2] that $y = 0$ is

- stable when either $Re(\lambda_i) < 0$ or $Re(\lambda_i) = 0$ and $\lambda_i$ is simple, $i = 1, 2, \ldots, m$,
- asymptotically stable when $Re(\lambda_i) < 0$, $i = 1, 2, \ldots, m$, and
- unstable otherwise.

**Example 1.2.1.** Consider the predator-prey model of Section 1.1

$$y_1' = y_1(a - \alpha y_2), \quad y_2' = y_2(-c + \gamma y_1),$$

where $a$, $\alpha$, $c$, and $\gamma$ are positive constants. This autonomous problem has two equilibrium points:

$$\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
c/\gamma \\
\alpha/\alpha
\end{bmatrix}.$$
The point $[0, 0]^T$ corresponds to extinction of predators and prey, whereas $[c/\gamma, a/\alpha]^T$ implies a co-existence of both species. The stability of $[0, 0]^T$ may be analyzed by using the linear system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$  

Since the system matrix is diagonal, its eigenvalues are $\lambda_1 = a$, $\lambda_2 = -c$. Thus, $\lambda_1 > 0$ and $[0, 0]^T$ is unstable.

In order to analyze the stability of the second equilibrium point, we introduce the change of variables

$$z_1 = y_1 - \frac{c}{\gamma}, \quad z_2 = y_2 - \frac{a}{\alpha}$$

to move the equilibrium point to the origin. Substituting this transformation into the predator-prey equations gives

$$z_1' = (z_1 + \frac{c}{\gamma})[a - \alpha(z_2 + \frac{a}{\alpha})] = -\alpha z_2 \left(\frac{c}{\gamma} + z_1\right),$$

$$z_2' = (z_2 + \frac{a}{\alpha})[-c + \gamma(z_1 + \frac{c}{\gamma})] = \gamma z_1 \left(\frac{a}{\alpha} + z_2\right).$$

The linearized system is now

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' = \begin{bmatrix} 0 & -\alpha c/\gamma \\ a\gamma/\alpha & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$  

The eigenvalues of this system are $\lambda_{1,2} = \pm i \sqrt{c/a}$. Since the eigenvalues are imaginary, the linear system is stable; however, the stability of the nonlinear system is undetermined without additional analysis.

**Problems**

1. Suppose $\partial f/\partial y$ is continuous on a closed convex domain $D$. Show that $f(t, y)$ satisfies a Lipschitz condition on $D$ with

$$L = \max_{(t, y) \in D} |\frac{\partial f(t, y)}{\partial y}|.$$  

    (Hint: use the Mean Value Theorem.)

2. Are the following IVPs well posed ([4], p. 23)? Explain.

$$y' = \sqrt{1 - y^2}, \quad y(0) = 0,$$

$$y' = \sqrt{y^2 - 1}, \quad y(0) = 2.$$
Bibliography


