Chapter 1

Introduction

1.1 Problems Leading to Partial Differential Equations

Many problems in science and engineering are modeled and analyzed using systems of partial differential equations. Realistic models involve multi-dimensional behavior, non-linearity, and irregular boundaries which are too complicated to be solved by analytic techniques. Numerical techniques provide useful approximations and now provide the dominant approach to addressing partial differential equations. Let’s begin by examining examples of physical situations that involve partial differential equations.

Example 1.1.1. Consider the problem of determining the temperature $T$ in a long prismatical rod of length $L$ for times $t > 0$ given that its temperature is known at the instant that the experiment is started ($t = 0$) and that some information, e.g., the temperature or the heat flux, is prescribed at the boundaries $x = 0, L$ (Figure 1.1.1). The temperature in the rod may be determined as a function of the spatial coordinate $x$ and the time $t$ by solving the one-dimensional heat conduction equation

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right), \quad 0 < x < L, \quad t > 0,$$  \hspace{1cm} (1.1.1a)

subject to the initial conditions

$$T(x, 0) = T^0(x), \quad 0 \leq x \leq L,$$  \hspace{1cm} (1.1.1b)

and appropriate boundary data, such as

$$T(0, t) = T_L, \quad \frac{\partial T(L, t)}{\partial x} = 0.$$  \hspace{1cm} (1.1.1c)
The parameters $\rho$, $c$, and $k$ are, respectively, the density, specific heat, and conductivity of the rod. Equation (1.1.1c) implies that the left end of the rod is maintained at the temperature $T_L$ while the right end is insulated.

Example 1.1.2. The problem of finding the deflection $u(x, t)$ of a taut string of length $L$, as shown in Figure 1.1.2, given its initial deflection and velocity is governed by

$$\frac{\rho}{\rho} \frac{\partial^2 u}{\partial t^2} = T^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

(1.1.2a)

$$u(x, 0) = u^0(x), \quad \frac{\partial u(x, 0)}{\partial t} = v^0(x), \quad 0 \leq x \leq L,$$

(1.1.2b)

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0.$$  

(1.1.2c)

Here, $T$ is the tension in the string, $\rho$ is the string’s density per unit length, $u^0(x)$ is the initial deflection, and $v^0(x)$ is the initial velocity.

Example 1.1.3. The problem of determining the stresses in a prismatical shaft that is subjected to a torque $T$ (Figure 1.1.3) is governed by

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2, \quad (x, y) \in \Omega,$$

(1.1.3a)

$$\psi = 0, \quad (x, y) \in \partial \Omega,$$

(1.1.3b)
The variable \( \psi(x, y) \) is a stress function. Once it has been determined from (1.1.3a,b), the angle of twist per unit length \( \Theta \) and the components of the shear stress \( \tau_{xx} \) and \( \tau_{yz} \) may be determined from (1.1.3c). The parameter \( G \) is the shear modulus of the rod, \( \Omega \) is the cross-sectional area of the rod, and \( \partial \Omega \) denotes its boundary. Observe that this problem is in equilibrium and, unlike those of Examples 1.1.1 and 1.1.2, it does not evolve in time.

**Example 1.1.4.** The two-dimensional flow of a compressible inviscid fluid is governed by the Euler equations

\[
\rho_t + m_x + n_y = 0, \quad (1.1.4a)
\]

\[
m_t + \left( \frac{m^2}{\rho} + p \right)_x + \left( \frac{mn}{\rho} \right)_y = 0, \quad (1.1.4b)
\]

\[
n_t + \left( \frac{mn}{\rho} \right)_x + \left( \frac{n^2}{\rho} + p \right)_y = 0, \quad (1.1.4c)
\]

\[
e_t + \left[ (\epsilon + p) \frac{m}{\rho} \right]_x + \left[ (\epsilon + p) \frac{n}{\rho} \right]_y = 0. \quad (1.1.4d)
\]
Subscripts denote partial differentiation and $\rho$, $m$ and $n$, $e$, and $p$ are the fluid density, the $x$ and $y$ components of the momentum per unit volume, the total energy per unit volume, and the pressure, respectively. The $x$ and $y$ components of the velocity vector, $u$ and $v$, are related to the components of the momentum vector by

$$m = \rho u, \quad n = \rho v.$$  \hspace{1cm} (1.1.5)

Finally, the pressure must be related to $\rho$, $m$, $n$, and $e$ by an equation of state, which, for a perfect gas, is

$$p = (\gamma - 1)|e - (m^2 + n^2)/2\rho|,$$  \hspace{1cm} (1.1.6)

where $\gamma$ is a constant. Equations (1.1.4a), (1.1.4b), (1.1.4c), and (1.1.4d) express the physical conditions that the mass, the $x$ component of momentum, the $y$ component of momentum, and the energy of the fluid, respectively, are conserved during its motion. A typical problem involving the flow around an airfoil is shown in Figure 1.1.4. The boundary conditions for this example would state that the momentum vector must be tangent to the airfoil and that flow conditions far from the airfoil are uniform and prescribed.

### 1.2 Second-Order Partial Differential Equations

Mathematicians classify partial differential systems according to the order of their highest derivatives, the number of independent variables, and the number of dependent variables. Other properties, such as linearity, are also used for characterization. Thus, Examples 1.1.1, 1.1.2, and 1.1.3 are second-order, linear, scalar problems in two variables while Example 1.1.4 is a first-order, nonlinear, vector system in the three variables $x$, $y$, and
We’ll consider first-order problems in Section 1.3 and examine second-order problems here. In particular, we’ll focus on a linear, scalar, partial differential equation in two variables having the form

\[ \mathcal{L}[u] \equiv au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \]  

(1.2.1)

where \( a, b, \ldots, g \) are continuous functions of \( x \) and \( y \) and subscripts denote partial differentiation, e.g., \( u_x \equiv \partial u / \partial x \).

Such second-order equations are classified into three types depending on the roots

\[ \lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - ac}}{a} \]  

(1.2.2a)

of the characteristic equation

\[ a\lambda^2 + 2b\lambda + c = 0. \]  

(1.2.2b)

In particular, (1.2.1) is

- **hyperbolic** if \( \lambda_1, \lambda_2 \) are real and distinct, i.e., if \( b^2 - ac > 0 \),
- **elliptic** if \( \lambda_1, \lambda_2 \) are complex, i.e., if \( b^2 - ac < 0 \), and
- **parabolic** if \( \lambda_1 = \lambda_2 \), i.e., if \( b^2 - ac = 0 \).

**Example 1.2.1.** A problem can be of a different type in different spatial regions. The equation

\[ xu_{xx} + u_{yy} = 0, \]  

(1.2.3)

has \( a = x, b = 0, \) and \( c = 1 \); thus, \( b^2 - ac = -x \) and (1.2.3) is hyperbolic in the left-half plane \( (x < 0) \), elliptic in the right-half plane \( (x > 0) \), and parabolic on the line \( x = 0 \).

There are canonical partial differential equations of each type that are important for theoretical reasons and when developing practical numerical methods. The canonical hyperbolic equation is (1.2.1) with \( a = 1, c = -1, b = d = e = f = g = 0 \), or

\[ u_{xx} - u_{yy} = 0. \]  

(1.2.4)
This equation is called the \textit{one-dimensional wave equation}. Usually one of the independent variables, say \( y \), corresponds to time. Initial and/or boundary conditions have to be specified for the solution to be uniquely determined. An \textit{initial value} or \textit{Cauchy} problem for the wave equation consists of finding \( u \) satisfying (1.2.4) on \( y > 0, \ -\infty < x < \infty \), given the initial conditions

\[
  u(x, 0) = \phi(x), \quad u_y(x, 0) = \psi(x). \quad (1.2.5a)
\]

An \textit{initial-boundary value} problem for the wave equation consists of determining \( u \) satisfying (1.2.4) on \( y > 0, \ 0 < x < 1 \), given the initial conditions (1.2.5a) and the boundary conditions

\[
  \alpha_0 u(0, t) + \beta_0 u_x(0, t) = \gamma_0, \quad \alpha_1 u(1, t) + \beta_1 u_x(1, t) = \gamma_1. \quad (1.2.5b)
\]

A boundary condition \((1.2.5b)\) is called \textit{Dirichlet} if \( \beta_j = 0 \) for \( j = 0 \) or \( 1 \); it is called \textit{Neumann} if \( \alpha_j = 0 \); and it is called \textit{Robin} if \( \alpha_j \) and \( \beta_j \) are nonzero.

The canonical parabolic problem is the \textit{one-dimensional heat conduction equation} which has \( a = 1, \ e = -1, \ b = c = d = f = g = 0 \) in (1.2.1), or

\[
  u_{xx} - u_y = 0. \quad (1.2.6)
\]

Again, \( y \) frequently corresponds to time. The initial value problem for the heat equation is to find \( u \) satisfying (1.2.6) on \( y > 0, \ -\infty < x < \infty \), given the initial condition

\[
  u(x, 0) = \phi(x). \quad (1.2.7a)
\]

An initial-boundary value problem for the heat equation is to find \( u \) satisfying (1.2.6) on \( y > 0, \ 0 < x < 1 \), given the initial condition \((1.2.7a)\) and the boundary conditions

\[
  \alpha_0 u(0, t) + \beta_0 u_x(0, t) = \gamma_0, \quad \alpha_1 u(1, t) + \beta_1 u_x(1, t) = \gamma_1. \quad (1.2.7b)
\]

The canonical elliptic problem is the two-dimensional \textit{Poisson equation} where \( a = 1, \ c = 1, \ b = d = e = f = 0 \) in (1.2.1) to produce

\[
  u_{xx} + u_{yy} = g(x, y). \quad (1.2.8)
\]
If the function \( g(x, y) \equiv 0 \) then (1.2.8) is called *Laplace’s equation*. There are no well posed (stable) initial or initial-boundary value problems for Poisson’s or Laplace’s equation. There are only boundary value problems, which consist of determining \( u(x, y) \) for \( (x, y) \in \Omega \) satisfying (1.2.8) given

\[
\alpha u + \beta u_n = \gamma, \quad (x, y) \in \partial \Omega,
\]

where \( n \) is a unit outward normal vector to \( \partial \Omega \) and \( \alpha, \beta, \) and \( \gamma \) are functions of \( x \) and \( y \).

In analogy with the boundary conditions for the wave and heat equations, if \( \beta \equiv 0 \) the problem is called a Dirichlet problem, if \( \alpha \equiv 0 \) the problem is called a Neumann problem, and if neither \( \alpha \) nor \( \beta \) are zero the problem is called a Robin problem.

Exact solutions to each of the canonical problems are known in certain circumstances and the classical technique for obtaining them is by using Fourier series. Fourier series will also play a key role in analyzing finite difference approximations of the canonical problems, so let us illustrate Fourier’s method for a simple heat conduction problem. We’re not going to delve into this subject, so consult an elementary differential equations text such as Boyce and DiPrima [1] for additional information.

**Example 1.2.2.** Consider the Dirichlet problem for the heat conduction equation

\[
\begin{align*}
    u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0, \\
    u(x, 0) &= \phi(x), \quad 0 \leq x \leq 1, \\
    u(0, t) &= u(1, t) = 0.
\end{align*}
\]  

(As noted, \( y \) in (1.2.6) frequently corresponds to time. In order to emphasize this, we have replaced the symbol \( y \) with a \( t \).)

The homogeneous boundary conditions suggest the use of the Fourier sine series

\[
    u(x, t) = \sum_{k=1}^{\infty} a_k(t) \sin k\pi x,
\]

which satisfies (1.2.10c) for all \( k > 0 \). Substitution of (1.2.11) into (1.2.10a) yields

\[
    \sum_{k=1}^{\infty} [\dot{a}_k + (k\pi)^2 a_k] \sin k\pi x = 0,
\]
where a superimposed dot denotes time differentiation. The term in brackets must vanish if the above system is to be satisfied for all values of \( k \) and \( x \); thus,

\[
\dot{a}_k + (k\pi)^2a_k = 0, \quad \forall \ k > 0,
\]

or

\[
a_k(t) = a_k^0 e^{-k^2\pi^2t}, \quad \forall \ k > 0.
\]

Using this result in (1.2.11) gives

\[
u(x,t) = \sum_{k=1}^{\infty} a_k^0 e^{-k^2\pi^2t} \sin k\pi x.
\]

(1.2.12a)

The constants \( a_k^0, \ k > 0 \), are determined from the initial conditions (1.2.10b); thus, evaluating (1.2.12a) at \( t = 0 \) yields

\[
u(x,0) = \phi(x) = \sum_{k=1}^{\infty} a_k^0 \sin k\pi x
\]

Multiplying the above equation by \( \sin l\pi x \), integrating on \((0,1)\), interchanging limiting processes, and using the orthogonality properties of the sine function yields

\[
\int_0^1 \phi(x) \sin l\pi x dx = \sum_{k=1}^{\infty} a_k^0 \int_0^1 \sin k\pi x \sin l\pi x dx = \frac{a_l^0}{2} \begin{cases} 1, & \text{if } k = l \\ 0, & \text{otherwise} \end{cases}.
\]

Thus,

\[
a_k^0 = 2 \int_0^1 \phi(x) \sin k\pi x dx, \quad \forall \ k > 0.
\]

(1.2.12b)

Even without a specific initial function \( \phi(x) \) we are able to discern several properties of the solution. For example, assuming that (1.2.12a) converges, we see that (i) \( u(x,t) \) is a smooth function of \( x \) and \( t \) even if \( \phi(x) \) isn’t and (ii) \( u(x,t) \) is a decreasing function of \( t \). Incidentally, (1.2.12) converges in the \( L^2 \) norm (cf. (1.2.14)) under rather general conditions [3].

To be more specific, suppose that

\[
\phi(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2 \\ 2(1-x), & 1/2 \leq x \leq 1 \end{cases}.
\]

Then, (1.2.12b) implies

\[
a_k^0 = 2 \left[ \int_0^{1/2} 2x \sin k\pi x dx + \int_{1/2}^1 2(1-x) \sin k\pi x dx \right] = \frac{8}{(k\pi)^2}
\]
and (1.2.12a) becomes

\[ u(x,t) = \sum_{k=1}^{\infty} \frac{8}{(k\pi)^2} e^{-k^2\pi^2 t} \sin k\pi x. \]  

(1.2.13)

With this initial data, (1.2.13) is formally converging as \(O(1/k^2)\). In general, the Fourier series converges as \(O(1/k^{\sigma+2})\) if \(\phi(x) \in C^\sigma\). Thus, even (some) discontinuous data leads to convergent Fourier series \([3]\).

It will not be possible to obtain Fourier series or other explicit solutions to practical (variable-coefficient or nonlinear) problems. Nevertheless, we can obtain information about the solution without much effort. Let us, for example, obtain an estimate of the behavior of the solution in the \(L^2\) norm, which, for (1.2.10), is defined as

\[ ||u(\cdot,t)||_2 = \left( \int_0^1 u(x,t)^2 dx \right)^{1/2}. \]

(1.2.14)

Multiplying (1.2.10a) by \(u\) and integrating on \((0,1)\), we find

\[ \int_0^1 uu_t dx = \int_0^1 uu_{xx} dx. \]

This may be written as

\[ \frac{1}{2} \frac{d}{dt} \int_0^1 u^2 dx = \int_0^1 (uu_x)_x dx - \int_0^1 u_x^2 dx \]

or, using (1.2.14)

\[ \frac{1}{2} \frac{d}{dt} ||u(\cdot,t)||_2^2 = uu_x|_0^1 - \int_0^1 u_x^2 dx. \]

Using the boundary conditions (1.2.10c)

\[ \frac{d}{dt} ||u(\cdot,t)||_2^2 = -2 \int_0^1 u_x^2 dx. \]

Since the integral on the right is non-positive, the \(L^2\) norm of the solution is a decreasing function of \(t\). The decrease is due to the presence of the \(u_{xx}\) term in (1.2.10a), which we call dissipative.

**Problems**

1. ([1], Section 10.7.) Find the Fourier-series solution of Laplace’s equation

\[ u_{xx} + u_{yy} = 0 \]
on the rectangle $0 < x < a$, $0 < y < b$, satisfying the boundary conditions

$$ u(0, y) = 0, \quad u(a, y) = 0, \quad 0 < y < b, $$

$$ u(x, 0) = 0, \quad u(x, b) = g(x), \quad 0 < x < a. $$

Additionally, find the solution in the particular case when

$$ g(x) = \begin{cases} 
  x, & 0 < x \leq a/2 \\
  a - x, & a/2 < x < a 
\end{cases}.$$

2. ([5], Section 8.) Find where the following partial differential equations are hyperbolic, parabolic, and elliptic:

$$ x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0, $$

$$ x^2 \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} + u = 0, $$

$$ x \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0. $$

3. Under suitable assumptions [2] the equations of two-dimensional, compressible, steady, inviscid flow ((1.1.4) with all time derivatives set to zero) can be reduced to the simpler system

$$ (a^2 - u^2) u_x - uv(u_y + v_x) + (a^2 - v^2) v_y = 0, $$

$$ v_x - u_y = 0. $$

Once again, $u$ and $v$ are the Cartesian coordinates of the velocity vector and $a$ is the speed of sound

$$ (a/a_0)^2 = 1 - \frac{\gamma - 1}{2} \frac{u^2 + v^2}{a_0^2}, $$

with $a_0$ being the speed of sound when $u = v = 0$ and $\gamma > 1$ being a constant.

Introducing a potential function $\phi(x, y)$ such that

$$ u = \phi_x, \quad v = \phi_y. $$
we satisfy the second partial differential equation and “reduce” the first to the 
transonic full-potential equation

\[(a^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (a^2 - \phi_y^2)\phi_{yy} = 0.\]

Although this is a complicated nonlinear second-order differential equation, it can
still be classified as hyperbolic, parabolic, or elliptic. However, with nonlinear
equations, the type may depend on the unknown solution. Determine the type of
the transonic full-potential equation. Express your answer in terms of the Mach
number

\[M^2 = \frac{u^2 + v^2}{a^2} = \frac{\phi_x^2 + \phi_y^2}{a^2}.\]

4. Show that the \(L^2\) norm of the solution of Burgers’ equation

\[u_t + uu_x = \sigma u_{xx}, \quad 0 < x < 1, \quad t > 0,\]

where \(\sigma\) is a positive constant, is a decreasing function of time when the initial and
boundary conditions are

\[u(x, 0) = \phi(x), \quad 0 \leq x \leq 1,\]

\[u(0, t) = u(1, t) = 0, \quad t > 0.\]

1.3 Hyperbolic Conservation Laws: Characteristics, Shock Waves, and Rankine-Hugoniot Conditions

A conservation laws states that the total amount of some quantity remains unchanged
during the evolution of the solution according to the partial differential equation. In physical processes without dissipation, these quantities might be the total mass, momentum, and energy. In this introductory chapter, let us confine our attention to conservation
laws in one space dimension which typically have the form

\[\frac{d}{dt} \int_\alpha^\beta u \, dx = -f(u)|_\alpha^\beta = -f(u(\beta, t)) + f(u(\alpha, t)), \quad (1.3.1)\]
where

\[
\mathbf{u}(x, t) = \begin{bmatrix} u_1(x, t) \\ u_2(x, t) \\ \vdots \\ u_m(x, t) \end{bmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} f_1(\mathbf{u}) \\ f_2(\mathbf{u}) \\ \vdots \\ f_m(\mathbf{u}) \end{bmatrix}.
\] (1.3.2)

are density and flux vectors, respectively. Equation (1.3.1) expresses the fact that the rate of change of \( \mathbf{u} \) within the “volume” \( \alpha \leq x \leq \beta \) is equal to the change in its flux through the boundaries \( x = \alpha, \beta \).

If \( \mathbf{f} \) and \( \mathbf{u} \) are smooth functions, then (1.3.1) can be written as

\[
\int_{\alpha}^{\beta} [\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x] \, dx = 0.
\]

Since this result should hold for all possible “control volumes” \( (\alpha, \beta) \), the integrand must vanish, i.e.,

\[
\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0. \quad (1.3.3)
\]

**Example 1.3.1.** The Euler equations for one-dimensional compressible inviscid flows may be obtained by setting all \( y \) derivatives and \( n \) to zero in equations (1.1.4) to obtain

\[
\rho_t + m_x = 0, \quad (1.3.4a)
\]

\[
m_t + \left( \frac{m^2}{\rho} + p \right)_x = 0, \quad (1.3.4b)
\]

\[
e_t + \left[ (e + p) \frac{m}{\rho} \right]_x = 0. \quad (1.3.4c)
\]

The pressure is determined by an equation of state which we take in the rather general form \( p = p(\rho, m, e) \). Recall that equations (1.3.4a), (1.3.4b), and (1.3.4c) express the facts that the mass, momentum, and energy of the fluid are neither created nor destroyed and are, hence, conserved. We readily see that the system (1.3.4) has the form of (1.3.3) with

\[
\mathbf{u} = \begin{bmatrix} \rho \\ m \\ e \end{bmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} m \\ \frac{m^2}{\rho} + p \\ (e + p) \frac{m}{\rho} \end{bmatrix}.
\] (1.3.5)
Example 1.3.2. The deflection of a taut string has the form

\[ u_{tt} = c^2 u_{xx}, \]  

(1.3.6)

where \( c^2 = T/\rho \) with \( T \) being the tension and \( \rho \) being the linear density of the string (cf. (1.1.2)).

Equation (1.3.6) can be written as a first-order system in a variety of ways. Perhaps the most common approach is to let

\[ u_1 = u_t, \quad u_2 = cu_x. \]  

(1.3.7)

Differentiating with respect to \( t \) while using (1.3.6) and (1.3.7) yields

\[ (u_1)_t = u_{tt} = c^2 u_{xx} = c(u_2)_x, \quad (u_2)_t = cu_{xt} = cu_{tx} = c(u_1)_x. \]

Thus, the one-dimensional wave equation has the form of (1.3.3) with

\[ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad f(u) = \begin{bmatrix} -cu_2 \\ -cu_1 \end{bmatrix}. \]

For the remainder of this introductory section, we'll confine our attention to the scalar conservation law

\[ u_t + f(u)_x = 0 \]  

(1.3.8)

and return to vector systems of conservation laws in Chapter 6. To begin, let

\[ a(u) = \frac{df(u)}{du}, \]  

(1.3.9a)

and write (1.3.8) in the convective form

\[ u_t + a(u)u_x = 0. \]  

(1.3.9b)

Regard \( u(x, t) \) as a function of two variables and compute its total (directional) derivative

\[ du = u_t dt + u_x dx = (u_t + \frac{dx}{dt} u_x) dt. \]  

(1.3.10)
Let us evaluate \((1.3.10)\) in the direction given by
\[
\frac{dx}{dt} = a(u). \tag{1.3.11}
\]

Then, using \((1.3.9b)\) we see that
\[
du = (u_t + au_x)dt = 0.
\]

Thus, \(u(x,t)\) is constant along the curve \(dx/dt = a\). This curve is called a characteristic of the differential equation \((1.3.8)\). Characteristics can be used to construct solutions of initial or initial boundary value problems. Let us begin with an initial value problem for \((1.3.8)\) on \(-\infty < x < \infty, t > 0\), with the initial condition
\[
u(x,0) = \phi(x), \quad -\infty < x < \infty. \tag{1.3.12}
\]

Since \(u\) is constant along the characteristic curves, it must have the same value that it had initially. Thus, \(u = \phi(x_0) \equiv \phi_0\) along the characteristic that passes through \((x_0,0)\). This characteristic satisfies the ordinary initial value problem
\[
\frac{dx}{dt} = a(\phi_0) \equiv a_0, \quad t > 0, \quad x(0) = x_0. \tag{1.3.13}
\]

Integrating, we see that the characteristic is the straight line
\[
x = x_0 + a_0 t. \tag{1.3.14}
\]

This procedure can be repeated to trace other characteristics and thereby construct the solution. The solution of \((1.3.8)\) is implicitly given by the formula
\[
u(x,t) = \phi(x - a(\phi_0)t); \tag{1.3.15}
\]
however, this formula is rather obscure. Let us clarify the situation with some examples.

**Example 1.3.3.** The simplest case occurs when \(a\) is a constant and \(f(u) = au\). All of the characteristics are parallel straight lines with slope \(1/a\). The solution of the initial value problem \((1.3.8, 1.3.12)\) is \(u(x,t) = \phi(x - at)\) and is, as shown in Figure 1.3.1, a wave that maintains its shape and travels with speed \(a\).
1.3. Hyperbolic Conservation Laws

Figure 1.3.1: Characteristic curves and solution of the initial value problem (1.3.8, 1.3.12) when $a$ is a constant.

Figure 1.3.2: Characteristic curves for two initial points $x_0$ and $x_1$ for Burgers’ equation. The characteristics intersect at a point $P$. 
Example 1.3.4. Setting \( a(u) = u \) and \( f(u) = u^2/2 \) in (1.3.8, 1.3.9a) yields the inviscid Burgers’ equation

\[
  u_t + \frac{1}{2}(u^2)_x = 0. \tag{1.3.16}
\]

Again, consider an initial value problem having the initial condition (1.3.12), so the characteristic is given by (1.3.14) with \( a_0 = u(x_0, 0) = \phi(x_0) \), i.e.,

\[
x = x_0 + \phi(x_0)t. \tag{1.3.17}
\]

The characteristics are straight lines, but they are not parallel as they were in Example 1.3.3. Their slope depends on the value of the initial data; thus, the characteristic passing through the point \((x_0, 0)\) has slope \(1/\phi(x_0)\). The fact that the characteristics are not parallel introduces a difficulty that was not present in the linear problem of Example 1.3.3. Consider characteristics passing through \((x_0, 0)\) and \((x_1, 0)\) and suppose that \(\phi(x_0) > \phi(x_1)\) for \(x_1 > x_0\). Since the slope of the characteristic passing through \((x_0, 0)\) is less than the slope of the one passing through \((x_1, 0)\), the two characteristics will intersect at a point, say, \(P\) as shown in Figure 1.3.2. The solution would appear to be multivalued at points such as \(P\).

In order to help clarify matters, let’s examine the specific choice of \(\phi\) given by Lax [4]

\[
  \phi(x) = \begin{cases} 
  1, & \text{if } x < 0 \\
  1 - x, & \text{if } 0 \leq x < 1 \\
  0, & \text{if } 1 \leq x 
  \end{cases}. \tag{1.3.18}
\]

Using (1.3.17), we see that the characteristic passing through the point \((x_0, 0)\) satisfies

\[
x = \begin{cases} 
  x_0 + t, & \text{if } x_0 < 0 \\
  x_0 + (1 - x_0)t, & \text{if } 0 \leq x < 1 \\
  x_0, & \text{if } 1 \leq x
  \end{cases}. \tag{1.3.19}
\]

Several characteristics are shown in Figure 1.3.3. The characteristics first intersect at \(t = 1\). After that, the solution would presumably be multivalued, as shown in Figure 1.3.4.

It’s, of course, quite possible for multivalued solutions to exist; however, (i) they are not observed in physical situations and (ii) they do not satisfy (1.3.8) in any classical sense. Discontinuous solutions are often observed in nature once characteristics of the
corresponding conservation law model have intersected. They also do not satisfy (1.3.8), but they might satisfy the integral form of the conservation law (1.3.1). We examine the simplest case when two classical solutions satisfying (1.3.8) are separated by a single smooth curve \( x = \xi(t) \) across which \( u(x,t) \) is discontinuous. For each \( t > 0 \) we assume that \( \alpha < \xi(t) < \beta \) and let superscripts - and + denote conditions immediately to the left and right, respectively, of \( x = \xi(t) \). Then, using (1.3.1), we have

\[
\frac{d}{dt} \left( \int_{\alpha}^{\beta} u \, dx \right) = \frac{d}{dt} \left( \int_{\alpha}^{\xi^-} u \, dx + \int_{\xi^+}^{\beta} u \, dx \right) = -f(u)_{\mid^\beta^\alpha}.
\]

or, differentiating the integrals

\[
\int_{\alpha}^{\xi^-} u \, dx + u^- \xi^- + \int_{\xi^+}^{\beta} u \, dx - u^+ \xi^+ = -f(u)_{\mid^\beta^\alpha}.
\]

The solution on either side of the discontinuity was assumed to be smooth, so (1.3.8) holds in \((\alpha, \xi^-)\) and \((\xi^+, \beta)\) and can be used to replace the integrals. Additionally, since \( \xi \) is smooth, \( \dot{\xi}^- = \dot{\xi}^+ = \dot{\xi} \). Thus, we have

\[
-f(u)_{\mid^\xi^-} + u^- \xi^- - f(u)_{\mid^\xi^+} + u^+ \xi^+ = -f(u)_{\mid^\beta^\alpha},
\]

or

\[
\dot{\xi}(u^+ - u^-) = f(u^+) - f(u^-). \tag{1.3.20}
\]

Let

\[
[q] \equiv q^+ - q^-
\]

denote the jump in a quantity \( q \) and write (1.3.20) as

\[
[u]\dot{\xi} = [f(u)]. \tag{1.3.22}
\]

Equation (1.3.22) is called the Rankine-Hugoniot jump condition and the discontinuity is called a shock wave. We can use the Rankine-Hugoniot condition to find a discontinuous solution of Example 1.3.4.

\textit{Example 1.3.5}. For \( t < 1 \), the discontinuous solution of (1.3.16, 1.3.18) is as given in Example 1.3.4. For \( t \geq 1 \), we hypothesize the existence of a single shock wave, passing
Figure 1.3.3: Characteristics for Burgers’ equation (1.3.16) with initial data given by (1.3.18).

Figure 1.3.4: Multivalued solution of Burgers’ equation (1.3.16) with initial data given by (1.3.18). The solution $u(x, t)$ is shown as a function of $x$ for $t = 0$, $1/2$, $1$, and $3/2$. 
through (1, 1) in the \((x, t)\)-plane. As shown in Figure 1.35, the solution of Example 1.3.4 can be used to infer that \(u^- = 1\) and \(u^+ = 0\). Thus, \(f(u^-) = (u^-)^2/2 = 1/2\) and \(f(u^+) = (u^+)^2/2 = 0\). Using (1.322), the velocity of the shock wave is

\[
\dot{\xi} = \frac{1}{2}.
\]

Integrating, we find the shock location as

\[
\xi = \frac{1}{2}t + c.
\]

Since the shock passes through (1, 1), the constant of integration \(c = 1/2\), and

\[
\xi = \frac{1}{2}(t + 1). \tag{1.323}
\]

The characteristics and shock wave are shown in Figure 1.35 and the solution \(u(x, t)\) is shown as a function of \(x\) for several times in Figure 1.36.

Let us consider another problem for Burgers’ equation with different initial conditions that will illustrate another structure that arises in the solution of nonlinear hyperbolic systems.

**Example 1.3.6.** Consider Burgers’ equation (1.3.16) subject to the initial conditions

\[
\phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
x, & \text{if } 0 \leq x < 1 \\
1, & \text{if } 1 \leq x
\end{cases}. \tag{1.324}
\]

Using (1.3.17) and (1.3.24), we see that the characteristic passing through \((x_0, 0)\) satisfies

\[
x = \begin{cases} 
x_0, & \text{if } x < 0 \\
x_0(1 + t), & \text{if } 0 \leq x < 1 \\
x_0 + t, & \text{if } 1 \leq x
\end{cases}. \tag{1.325}
\]

These characteristics, shown in Figure 1.37, may be used to verify that the solution, shown in Figure 1.38, is continuous. Additional considerations and difficulties with nonlinear hyperbolic systems are discussed in Lax [4].

Thus far, we have only considered initial value problems for (1.3.8). Initial-boundary problems are also possible; however, boundary conditions cannot be prescribed arbitrarily.
Figure 1.3.5: Characteristics and shock discontinuity for Example 1.3.5.

Figure 1.3.6: Solution $u(x, t)$ of Example 1.3.5 as a function of $x$ at $t = 0, 1/2, 1,$ and $3/2$. The solution is discontinuous for $t > 1$. 
Figure 1.3.7: Characteristics for Example 1.3.6.

Figure 1.3.8: Solution $u(x, t)$ of Example 1.3.6 as a function of $x$ at $t = 0, 1/2, 1, \text{ and } 3/2$. 
and are determined by the characteristic directions (1.3.11). Let us begin with the linear
problem
\[ u_t + au_x = 0, \quad 0 < x < 1, \quad t > 0, \] (1.3.26a)
\[ u(x, 0) = \phi(x), \quad 0 \leq x \leq 1. \] (1.3.26b)

Suppose that \( a > 0 \). As shown in Figure 1.3.9, the solution at a point \((x_1, t_1)\) where
\( x_1 > at_1 \) is determined by the initial condition at \((x_1 - at_1, 0)\). On the other hand, the
solution at a point \((x_2, t_2)\) where \( x_2 < at_2 \) is not determined by the initial condition
and requires a boundary value to be prescribed at \((0, t_2 - x_2/a)\). Therefore, the proper
boundary condition to apply is
\[ u(0, t) = \psi(t), \quad t > 0. \] (1.3.26c)

The solution of the initial-boundary value problem (1.3.26a) is given by
\[ u(x, t) = \begin{cases} \psi(t - x/a), & \text{if } 0 < x < at \\ \phi(x - at), & \text{if } at \leq x < 1 \end{cases}. \] (1.3.27)

There are no boundary conditions at \( x = 1 \). However, if \( a < 0 \), the situation would be
reversed and boundary conditions would be needed at \( x = 1 \) and not at \( x = 0 \).

Determining the proper boundary conditions for nonlinear problems can be quite
complicated. For example, consider solving (1.3.9b) on \( 0 < x < 1, \ t > 0 \) with \( a(u(0, t)) > \)
0 and $a(u(1,t)) < 0$. As shown on the left of Figure 1.3.10, such a problem would need boundary conditions at both $x = 0$ and $x = 1$. On the other hand, a problem with $a(u(0,t)) < 0$ and $a(u(1,t)) > 0$ (Figure 1.3.10, right) would not require any boundary conditions.

![Figure 1.3.10](image.png)

Figure 1.3.10: Characteristics for a nonlinear initial-boundary value problem with $a(u(0,t)) > 0$ and $a(u(1,t)) < 0$ (left) and with $a(u(0,t)) < 0$ and $a(u(1,t)) > 0$ (right).

**Problems**

1. A Riemann problem is a Cauchy (initial value) problem with piecewise constant initial data. Consider the Riemann problem for the inviscid Burgers’ equation

$$u_t + \frac{1}{2}(u^2)_x = 0$$

$$u(x,0) = \begin{cases} u_L, & \text{if } x < 0 \\ u_R, & \text{if } x \geq 0 \end{cases}.$$  

where $u_L$ and $u_R$ are constants. Solve this problem in the two cases when $u_L > u_R$ and when $u_L < u_R$. Sketch several characteristic curves and sketch the solution as a function of $x$ at a few times. As a hint, you may want to first solve a problem with continuous initial data, e.g.,

$$u(x,0) = \begin{cases} u_L, & \text{if } x < -\varepsilon \\ \frac{u_L}{\varepsilon}(\varepsilon - x) + \frac{u_R}{\varepsilon}(\varepsilon + x), & \text{if } -\varepsilon < x \leq \varepsilon \\ u_R, & \text{if } \varepsilon < x \end{cases}$$

and take the limit as $\varepsilon \to 0$. You may also consult Lax [4].
2. Use the method of characteristics to construct a solution of the inhomogeneous initial value problem

\[ u_t + au_x = b(x), \quad -\infty < x < \infty, \quad t > 0, \]

\[ u(x, 0) = \phi(x), \quad -\infty < x < \infty, \]

where \( a \) is a constant and \( b(x) \) is smooth.

3. A scalar conservation law having cylindrical symmetry has the form

\[ \frac{d}{dt} \int_{\alpha}^{\beta} ur \, dr = -f(u)r|_{\alpha}^{\beta}, \]

where \( r \) is a radial coordinate. If solutions are smooth this may be written as the first-order hyperbolic partial differential equation

\[ u_t + \frac{1}{r}[f(u)r]_r = 0. \]

Solve an initial-boundary value problem for this equation on \( r > 0, \, t > 0 \) when \( f(u) = au \) with \( a \) a positive constant. Assume the initial conditions prescribe

\[ u(r, 0) = \phi(r), \quad r > 0, \]

where \( \phi(r) \) is smooth. What boundary conditions, if any, must be prescribed?

4. Consider a linear vector systems of \( m \) conservation laws of the form

\[ u_t + A u_x = 0 \]

where \( A \) is a constant \( m \times m \) matrix. Let \( P \) be the \( m \times m \) matrix that reduces \( A \) to the canonical form

\[ P^{-1}AP = \Lambda. \]

If \( A \) has \( m \) distinct real eigenvalues then \( \Lambda \) is the diagonal matrix

\[ \Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} \]

and the partial differential system is called hyperbolic. (Additionally, if \( \Lambda \) has \( m \) complex eigenvalues the system is called elliptic. These extend the notions of hyperbolicity and ellipticity that were introduced in Section 1.2.)
4.1. Assume that the given partial differential system is hyperbolic and introduce the transformation

$$u = Pv.$$ 

Show that \(v\) satisfies

$$v_t + \Lambda v_x = 0.$$ 

Thus, the transformation has uncoupled the system (except possibly for the initial and/or boundary conditions). It now has the scalar form

$$(v_i)_t + \lambda_i (v_i)_x = 0, \quad i = 1, 2, \cdots, m.$$ 

This system may be analyzed using characteristics in the same manner as for the scalar case.

4.2. Consider the wave equation written as a first-order system of two equations with

$$A = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix}.$$ 

Find \(P\), \(A\), and the diagonal system satisfied by \(v\). Determine the characteristics and, hence, \(v\). Knowing \(v\), determine \(u(x, t)\) such that it satisfies the initial data

$$u(x, 0) = \begin{bmatrix} u_1^0(x) \\ u_2^0(x) \end{bmatrix}.$$
Bibliography


