Chapter 2

Introduction to Finite Difference Methods

2.1 Constructing Difference Operators

Our initial study will involve the solution of time-dependent partial differential equations in one space variable. We’ll begin by introducing some elementary finite difference operators and examining their basic properties. Following this, in Sections 2.2 and 2.3, respectively, we’ll use these difference operators to solve simple wave propagation and heat conduction problems. With this plan, let us partition the upper half of the \((x, t)\)-plane into uniform cells of size \(\Delta x \times \Delta t\) as shown in Figure 2.1.1. The grid intersection with the Cartesian coordinates \(x_j = j\Delta x\) and \(t_n = n\Delta t\) is denoted as \((j, n)\) and the restriction of the solution of the partial differential equation to the grid is denoted as

\[
\quad u^n_j \equiv u(j\Delta x, n\Delta t). \quad (2.1.1)
\]

Finite difference approximations of \(u^n_j\) are denoted as \(U^n_j\) and are obtained by replacing derivatives of \(u\) at \(x_j\) and \(t_n\) with algebraic expressions involving \(u\) at points neighboring \((x_j, t_n)\). Difference approximations may be constructed in a variety of ways, but the use of Taylor’s formula is probably the simplest for our present purposes. To begin, let us use Taylor’s formula to express \(u^n_{j+1}\) in terms of \(u^n_j\) and its derivatives as

\[
\quad u^n_{j+1} = u^n_j + \left(\frac{\partial u^n}{\partial x}\right)_j \Delta x + \frac{1}{2!} \left(\frac{\partial^2 u^n}{\partial x^2}\right)_j \Delta x^2 + \ldots + \frac{1}{k!} \left(\frac{\partial^k u^n}{\partial x^k}\right)_j \Delta x^k + \frac{1}{(k+1)!} \left(\frac{\partial^{k+1} u^n}{\partial x^{k+1}}\right)_{j+\xi} \Delta x^{k+1}. \quad (2.1.2)
\]
Figure 2.1.1: A partition of the upper half of the \((x,t)\)-plane into uniform cells of size \(\Delta x \times \Delta t\).

The last term in (2.1.2), the remainder, involves the evaluation of \(\partial u^{k+1}/\partial x^{k+1}\) at \(x = (j + \xi)\Delta x\) and \(n\Delta t\), with \(\xi\) an unknown point on \((0, 1)\).

As an example, suppose that we retain the first two terms \((k = 1)\) in (2.1.2) and solve for \((\partial u/\partial x)_{j}^{n}\) to obtain

\[
(u_{x})_{j}^{n} = \frac{u_{j+1}^{n} - u_{j}^{n}}{\Delta x} - \frac{1}{2}(u_{xx})_{j+\xi}^{n} \Delta x. \tag{2.1.3}
\]

Neglecting the remainder term, we get the formula for the first forward finite difference approximation of \(u_{x}\) as

\[
(U_{x})_{j}^{n} = \frac{U_{j+1}^{n} - U_{j}^{n}}{\Delta x}. \tag{2.1.4a}
\]

The neglected remainder term in (2.1.3)

\[
\tau_{j}^{n} = (u_{x})_{j}^{n} - \frac{u_{j+1}^{n} - u_{j}^{n}}{\Delta x} = -\frac{1}{2}(u_{xx})_{j+\xi}^{n} \Delta x, \quad 0 < \xi < 1, \tag{2.1.4b}
\]

is called the local discretization error or local truncation error.

Backward finite difference approximations can be developed by expanding \(u_{j-1}^{n}\) in a
2.1. Difference Operators

Taylor’s series about \((x_j, t_n)\) to obtain

\[
u^n_{j-1} = u^n_j - \frac{\partial u}{\partial x} j^n \Delta x + \frac{1}{2!} \left( \frac{\partial^2 u}{\partial x^2} \right)^n_j \Delta x^2 + \ldots + \frac{(-1)^k}{k!} \left( \frac{\partial^k u}{\partial x^k} \right)^n_j \Delta x^k + \frac{(-1)^{k+1}}{(k+1)!} \left( \frac{\partial^{k+1} u}{\partial x^{k+1}} \right)^n_j \eta \Delta x^{k+1}, \quad 0 < \eta < 1. \tag{2.1.5}
\]

Retaining the first two terms in (2.1.5) and neglecting the remainder gives the first backward finite difference approximation of \((u_x)^n_j\) as

\[
(U_x)^n_j = \frac{U^n_j - U^n_{j-1}}{\Delta x}.
\tag{2.1.6a}
\]

The local discretization error is again obtained from the remainder term as

\[
\tau^n_j = \frac{1}{2} (u_{xx})^n_{j-\eta} \Delta x, \quad 0 < \eta < 1.
\tag{2.1.6b}
\]

The approximations of \((u_x)^n_j\) given by (2.1.4) and (2.1.6) are directional. A centered finite difference approximation of the first derivative can be obtained by retaining the first three terms \((k = 2)\) in (2.1.2) and (2.1.5) and subtracting the resulting expressions to get

\[
(U_x)^n_j = \frac{U^n_{j+1} - U^n_{j-1}}{2 \Delta x}.
\tag{2.1.7a}
\]

The local discretization error is obtained from the remainder terms as

\[
\tau^n_j = \frac{1}{6} (u_{xxx})^n_{j+\zeta} \Delta x^2, \quad -1 < \zeta < 1.
\tag{2.1.7b}
\]

The discretization error of the centered formula (2.1.7) is \(O(\Delta x^2)\) while those of the forward and backward formulas (2.1.4) and (2.1.6) are \(O(\Delta x)\). Since the centered formula converge at a faster rate than either of the two directional formulas, it would normally be preferred; however, we shall see examples (Section 2.2) where this is not the case.

Obviously, Taylor’s series can also be used to construct approximations of time derivatives. The first forward difference approximation of \(u_t\) at \((x_j, t_n)\) is

\[
(U_t)^n_j = \frac{U^n_{j+1} - U^n_j}{\Delta t}.
\tag{2.1.8a}
\]

The local discretization error is

\[
\tau^n_j = -\frac{1}{2} (u_{tt})^n_{j+\theta} \Delta t, \quad 0 < \theta < 1.
\tag{2.1.8b}
\]
The same approach can be used to construct approximations of higher derivatives. For example, a centered difference approximation of \((u_{xx})^n_j\) can be obtained by retaining the first four terms in (2.1.2) and (2.1.5) and adding the resulting expressions to get

\[
(U_{xx})^n_j = \frac{U^n_{j+1} - 2U^n_j + U^n_{j-1}}{\Delta x^2}.
\]  

(2.1.9a)

The discretization error of this approximation is

\[
\tau^n_j = -\frac{1}{12}(u_{xxxx})^n_{j+\xi}\Delta x^2, \quad -1 < \xi < 1,
\]

(2.1.9b)

where the \(\xi\) of (2.1.9) is a generic symbol and has no relation to the \(\xi\) used in (2.1.3) or (2.1.4b).

Centered differences only have a higher order of accuracy than forward or backward differences on uniform grids. To see this, consider three points \(x_{j-1}\), \(x_j\), and \(x_{j+1}\) of a nonuniform grid as shown in Figure 2.1.2. Let \(\Delta x_L \equiv x_j - x_{j-1}\) and \(\Delta x_R \equiv x_{j+1} - x_j\) and construct the Taylor’s series expansions

\[
\begin{align*}
\Delta x_L & \quad \Delta x_R \\
\hline
n & \quad n + 1 \\
\hline
j - 1 & \quad j & \quad j + 1
\end{align*}
\]

Figure 2.1.2: Two neighboring intervals of a nonuniform grid.

\[
u^n_{j+1} = u^n_j + \Delta x_R (u_x)^n_j + \frac{1}{2}\Delta x_R^2 (u_{xx})^n_j + O(\Delta x_R^3),
\]

(2.1.10a)

\[
u^n_{j-1} = u^n_j - \Delta x_L (u_x)^n_j + \frac{1}{2}\Delta x_L^2 (u_{xx})^n_j + O(\Delta x_L^3).
\]

(2.1.10b)

Divide (2.1.10a) by \(\Delta x_R\), divide (2.1.10b) by \(\Delta x_L\), and subtract the results to get

\[
(u_x)^n_j = \frac{1}{2}\left[\frac{u^n_{j+1} - u^n_j}{\Delta x_R} + \frac{u^n_j - u^n_{j-1}}{\Delta x_L}\right] - \frac{1}{4}(\Delta x_R - \Delta x_L)(u_{xx})^n_j + O(\Delta x^2).
\]  

(2.1.11a)
where $\Delta x \equiv \max(\Delta x_L, \Delta x_R)$. Let us also divide (2.1.10a) by $\Delta x_R$, divide (2.1.10b) by $\Delta x_L$, and add the results to get

$$
(u_{xx})_j^n = \frac{2}{\Delta x_R + \Delta x_L} \left[ \frac{u_{j+1}^n - u_j^n}{\Delta x_R} - \frac{u_j^n - u_{j-1}^n}{\Delta x_L} \right] - \frac{1}{6} (\Delta x_R - \Delta x_L)(u_{xxx})_j^n + O(\Delta x^2).
$$

(2.1.11b)

The approximations of $(u_x)_j^n$ and $(u_{xx})_j^n$ that are obtained by retaining the first terms of (2.1.11a) and (2.1.11b) are only accurate to $O(\Delta x)$. If $\Delta x_L = \Delta x_R$, the $u_{xx}$ terms in (2.1.11a) and the $u_{xxx}$ terms in (2.1.11b) cancel and the accuracy of both formulas is $O(\Delta x^2)$.

It will be convenient to have a shorthand operator notation for the finite difference operators in the same way that such notation is used for derivatives. The notation shown in Table 2.1.1 is relatively standard and will be used throughout these notes. For simplicity, we have suppressed the time dependence and only show spatial difference operators in Table 2.1.1. Thus, we have assumed that $u$ is a function of $x$ only with $u_j \equiv u(x)$. Temporal difference operators are defined in analogous fashion. Some examples follow.

<table>
<thead>
<tr>
<th>Operator</th>
<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>Forward Difference</td>
<td>$\Delta$</td>
<td>$\Delta u_j \equiv u_{j+1} - u_j$</td>
</tr>
<tr>
<td>Backward Difference</td>
<td>$\nabla$</td>
<td>$\nabla u_j \equiv u_j - u_{j-1}$</td>
</tr>
<tr>
<td>Central Difference</td>
<td>$\delta$</td>
<td>$\delta u_j \equiv u_{j+1/2} - u_{j-1/2}$</td>
</tr>
<tr>
<td>Average</td>
<td>$\mu$</td>
<td>$\mu u_j \equiv (u_{j+1/2} + u_{j-1/2})/2$</td>
</tr>
<tr>
<td>Shift</td>
<td>$E$</td>
<td>$Eu_j \equiv u_{j+1}$</td>
</tr>
<tr>
<td>Derivative</td>
<td>$D$</td>
<td>$Du_j \equiv (u_x)_j$</td>
</tr>
</tbody>
</table>

Table 2.1.1: Definition of finite difference operators.

Example 2.1.1. The centered difference formula (2.1.7a) can be expressed in terms of the central difference and averaging operators $\delta$ and $\mu$. Observe that

$$
\mu \delta u_j = \mu(u_{j+1/2} - u_{j-1/2}) = \frac{1}{2}(u_{j+1} - u_{j-1}).
$$

Thus,

$$
\frac{\mu \delta u_j}{\Delta x} = \frac{u_{j+1} - u_{j-1}}{2 \Delta x}.
$$
Example 2.1.2. An operator appearing to a positive integral power is iterated; thus,

\[ \delta^2 u_j = \delta (u_{j+1/2} - u_{j-1/2}) = u_{j+1} - 2u_j + u_{j-1}. \]

Thus, the centered second difference approximation (2.1.9a) of the second derivative can be written as

\[ (u_{xx})_j \approx \frac{\delta^2 u_j}{\Delta^2}. \]

Example 2.1.3. Let us expand \( u_{j+1} \) in a Taylor’s series about \( x_j \) to obtain

\[ u_{j+1} = u_j + \Delta x (u_x)_j + \frac{1}{2} \Delta x^2 (u_{xx})_j + \ldots, \]

or, using the derivative operator \( D \) defined in Table 2.1.1,

\[ u_{j+1} = [1 + \Delta x D + \frac{1}{2} \Delta x^2 D^2 + \ldots] u_j. \]

We may use the Taylor’s series expansion of the exponential function and the shift operator of Table 2.1.1 to write this in the short-hand form

\[ E u_j = u_{j+1} = e^{\Delta x D} u_j. \]

We can, thus, infer the identity between the shift, exponential, and derivative operators

\[ E = e^{\Delta x D}. \] (2.1.12)

Additional relationships can be obtained by noting that \( \Delta u_j = (E - 1) u_j \), which implies that \( \Delta = E - 1 \) or \( E = 1 + \Delta \). Treating the operators in (2.1.12) as algebraic quantities, we find

\[ \Delta x D = \ln E = \ln (1 + \Delta) = \Delta - \frac{1}{2} \Delta^2 + \frac{1}{3} \Delta^3 - \ldots, \] (2.1.13a)

where the series expansion of \( \ln (1 + x) \), \(|x| < 1\), has been used. A similar relation in terms of the backward difference operator can be constructed by noting that \( \nabla = 1 - E^{-1} \); thus,

\[ \Delta x D = \ln E = -\ln (1 - \nabla) = \nabla + \frac{1}{2} \nabla^2 + \frac{1}{3} \nabla^3 + \ldots, \] (2.1.13b)

These identities can be used to derive high-order finite difference approximations of the first derivative. For example, retaining the first two terms in (2.1.13a)

\[ \Delta x D u_j \approx [\Delta - \frac{1}{2} \Delta^2] u_j, \]
or
\[
\Delta x Du_j \approx [(u_{j+1} - u_j) - \frac{1}{2}(u_{j+2} - 2u_{j+1} + u_j)],
\]
or
\[
Du_j \approx \frac{-u_{j+2} + 4u_{j+1} - 3u_j}{2\Delta x}.
\]
This formula can be verified as an \(O(\Delta x^2)\) approximation of \((u_x)_j\).

The formal manipulations used in Examples 2.1.1 - 2.1.3 have to be verified as being rigorous. Estimates of local discretization errors must also be obtained. Nevertheless, using the formal operators of Table 2.1.1 provides a simple way of developing high-order finite difference approximations.

**Problems**

1. High-order centered difference approximations can be constructed by manipulating identities involving the central difference operator ([3], Chapter 1)
\[
\delta u_j \equiv u_{j+1/2} - u_{j-1/2}.
\]

1.1. Use Taylor’s series expansions of a function \(u(x)\) on a uniform mesh of spacing \(\Delta x\) to show that
\[
\begin{align*}
 u_{j+1/2} &= e^{\frac{\Delta x}{2}} Du_j, \\
 u_{j-1/2} &= e^{-\frac{\Delta x}{2}} Du_j
\end{align*}
\]
where \(Du_j \equiv u'(j\Delta x)\). Use the definition of the central difference operator to infer
\[
\delta = 2\sinh \frac{\Delta x}{2} D,
\]
or, inverting,
\[
\Delta x D = 2\sinh^{-1} \frac{\delta}{2} = \delta - \frac{1}{2^2 3!} \delta^3 + \frac{3^2}{2^4 5!} \delta^5 - \ldots .
\]
This relationship can be used to construct central difference approximations of \(u_x\).

1.2. Use the result of Question 1.1 to show that
\[
\Delta x^2 D^2 u_j = \Delta x^2 (u_{xx})_j = [\delta^2 - \frac{1}{12} \delta^4 + \frac{1}{90} \delta^6 + \ldots ]u_j.
\]
Truncating this relationship gives higher-order centered difference approximations of the second derivative.
2.2 Simple Difference Schemes for a Kinematic Wave Equation

We have developed more than enough finite difference formulas to begin solving some simple problems. Let us begin with the kinematic wave propagation problem

\[ u_t + a(u)u_x = 0, \quad -\infty < x < \infty, \quad t > 0, \]

\[ u(x, 0) = \phi(x), \quad -\infty < x < \infty, \]

where the wave speed \( a(u) \) is a real function of \( u \). We have neglected boundary conditions for our initial study, so, in order to have a finite spatial domain, we will require that \( \phi(x) \) either have compact support

\[ \phi(x) \equiv 0, \quad |x| > X, \]

or be periodic

\[ \phi(x + X) = \phi(x), \]

where \( X \) is a positive constant.

Perhaps the simplest strategy for solving (2.2.1) is to approximate both time and spatial derivatives by first-forward differences. As in Section 2.1, let us cover the half-plane \( t > 0 \) by a uniform rectangular space-time mesh having cells of size \( \Delta x \times \Delta t \) (Figure 2.1.1), evaluate (2.2.1a) at \((j\Delta x, n\Delta t)\), and use the forward difference approximations (2.1.3, 2.1.8) to obtain

\[ (u_t)_j^n + a(u^n)_j^n(u_x)_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta t} - \frac{\Delta t}{2} (u_t)_j^n + \alpha(u^n)_j^n \left( \frac{u_{j+1}^n - u_j^n}{\Delta x} - \frac{\Delta x}{2} (u_x)_j^n \right) = 0. \]

Neglecting the second-order derivative terms in the local discretization errors, we obtain

the finite difference equation

\[ \frac{U_{j+1}^n - U_j^n}{\Delta t} + a(U_j^n) \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0. \]

Solving for \( U_j^{n+1} \), we obtain

\[ U_j^{n+1} = (1 + \alpha_j^n) U_j^n - \alpha_j^n U_{j+1}^n, \]
Figure 2.2.1: Computational stencil of the forward time-forward space finite difference scheme (2.2.2b).

where

\[ \alpha_j^n = a(U_j^n) \frac{\Delta t}{\Delta x}. \]  

(2.2.3)

The parameter \( \alpha_j^n \) is called the Courant number.

The finite difference equation (2.2.2b) involves three points as indicated in the stencil of Figure 2.2.1. It is easy to solve using the prescribed initial data (2.2.1b). Knowing \( U_j^0 = \phi(j \Delta x) \) for all \( j \), we calculate \( U_j^1 \) for all \( j \) of interest. Then, knowing \( U_j^1 \) for all \( j \), we repeat the process to obtain \( U_j^2 \), etc.

Forward differences are appropriate for approximating time derivatives in this type of marching procedure, but it seems reasonable to also consider backward differences or centered differences for approximating the spatial derivatives in (2.2.1a). Thus, using (2.1.6, 2.1.8) in (2.2.1a) we obtain the forward time-backward space difference scheme

\[ U_j^{n+1} = (1 - \alpha_j^n)U_j^n + \alpha_j^n U_{j-1}^n. \]  

(2.2.4)

Using (2.1.7, 2.1.8) in (2.2.1a) yields the forward time-centered space difference scheme

\[ U_j^{n+1} = U_j^n - \frac{\alpha_j^n}{2}(U_{j+1}^n - U_{j-1}^n). \]  

(2.2.5)

These two schemes have the computational stencils shown in Figure 2.2.2. They are used in exactly the same way as the forward time-forward space scheme (2.2.2b). Each scheme has about the same computational complexity.

The question to ask at this juncture is whether or not there are any significant differences between the three schemes (2.2.2b), (2.2.4), and (2.2.5). We have not yet studied
their discretization errors; however, based on the analyses of Section 2.1, we might expect that solutions obtained by (2.2.5) have a higher order of spatial accuracy than those obtained by either (2.2.2b) or (2.2.4) (cf. (2.1.4b, 2.1.6b, 2.1.7b)). This would be enough to abandon schemes (2.2.2b) and (2.2.4), if it were the only difference between the methods. Let’s apply the methods to two simple examples.

**Example 2.2.1.** Consider (2.2.1) with $a = 1$ and

$$
\phi(x) = \begin{cases} 
  x, & \text{if } x \leq 0 \\
  0, & \text{if } x > 0
\end{cases}.
$$

The solution of this problem is easily obtained by the method of characteristics (Section 1.3) as

$$
u(x, t) = \phi(x - t),$$

which is a sloping ramp moving in the positive $x$ direction with unit speed.

Let us, rather arbitrarily, choose $\Delta x = 1/10$ and $\Delta t = 1/20$ and solve this problem for several spatial locations and a few time levels by the forward time-forward space and forward time-backward space finite difference schemes (2.2.2b) and (2.2.4), respectively. The Courant number $\alpha^n_j = 1/2$ in each case. The results are shown in Tables 2.2.1 and 2.2.2 and Figure 2.2.3.

The solution obtained by the forward time-forward space scheme (2.2.2b) bears little resemblance to the exact solution. It is oscillatory and growing as the time level increases. The solution obtained by the forward time-backward space difference scheme (2.2.4) appears to be a good approximation of the exact solution. We will be more precise about the accuracy in Section 2.3, for now, we seek an explanation for the catastrophic failure
Table 2.2.1: Solution of Example 2.2.1 using the forward time-forward space scheme (2.2.2b).

<table>
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<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
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Table 2.2.2: Solution of Example 2.2.1 using the forward time-backward space scheme (2.2.4).

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</tr>
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<td>5</td>
<td>-0.65</td>
<td>-0.55</td>
<td>-0.45</td>
<td>-0.35</td>
<td>-0.25</td>
<td>-0.153</td>
<td>-0.072</td>
<td>-0.022</td>
<td>-0.003</td>
</tr>
</tbody>
</table>

of the forward time-forward space scheme (2.2.2b). To rule out the possibility that the problems could be due to a lack of smoothness in the data and to further study the performance of these two schemes, let us study a second example.

**Example 2.2.2.** Consider (2.2.1) with \( a = 1 \) and

\[
\phi(x) = \sin x;
\]

hence, the exact solution is the traveling sinusoidal wave

\[
u(x, t) = \sin(x - t).
\]

We’ll solve this initial value problem on \( 0 \leq x \leq 2\pi \) using \( \Delta x = 2\pi/16 \) and \( \Delta t = 2\pi/32 \), so \( a_j^n = 1/2 \). Solutions of the forward time-forward space (2.2.2b) and forward time-backward space (2.2.4) schemes are shown in Figure 2.2.4. The forward space scheme appears to be working; however, its solution is increasing in amplitude. Oscillations are not present on this time scale but would appear had we computed for longer times. (The
Figure 2.2.3: Solutions of Example 2.2.1 obtained by the forward time-forward space scheme (2.2.2b) (top) and forward time-backward space scheme (2.2.4) (bottom). Each solution has a Courant number of 1/2.
oscillations take longer to develop with smoother initial data.) Once again, the backward
space scheme is producing a reasonable approximation of the exact solution.

Let us postpone treatment of the forward time-centered space scheme (2.2.5) and
seek to understand the difficulty with scheme (2.2.2b). Thus, consider (2.2.1) when \( a \)
is a positive constant. The solution of (2.2.1) at a point \((x, t)\) is determined by the initial
data at the point \((x - at, 0)\) (Figure 2.2.5).

**Definition 2.2.1.** The domain of dependence of a point \((x, t)\) for the initial value prob-
lem (2.2.1) is the set of all points at \( t = 0 \) that determine the solution at \((x, t)\).

**Remark 1.** Definition 2.2.1 is particular to (2.2.1). It will have to be amended to
account for inhomogeneous equations, vector systems, and initial-boundary value prob-
lems.

As noted, the domain of dependence of the point \((x, t)\) for the initial value problem
(2.2.1) is the single point \((x - at, 0)\). In a similar manner, let us use Definition 2.2.1
to identify domains of dependence of the finite difference schemes (2.2.2b) and (2.2.4).
The solution of the forward time-backward space scheme (2.2.4) at a point \((j \Delta x, n \Delta t)\)
is determined by the initial data on the interval \([ (j - n) \Delta x, j \Delta x] \) at \( t = 0 \) (Figure 2.2.5).
Thus, following Definition 2.2.1, we’ll call this interval the domain of dependence of the
point \((j \Delta x, n \Delta t)\) for the difference scheme (2.2.4). The domain of dependence of the
forward time-forward space scheme (2.2.2b), however, is the interval \([ j \Delta x, (j + n) \Delta x] \),
which cannot possibly be correct (Figure 2.2.5). This scheme does not use the correct
initial data to determine the solution at \((j \Delta x, n \Delta t)\). These simple arguments lead to
the famous Courant, Friedrichs, Lewy Theorem [2, 1] which we’ll state in the context of
(2.2.1).

**Theorem 2.2.1.** (Courant, Friedrichs, Lewy). A necessary condition for the conver-
gence of the solution of a finite-difference approximation to the solution of (2.2.1) for
arbitrary initial data is that the domain of dependence of the finite-difference approxima-
tion contain the domain of dependence of the partial differential equation (2.2.1).

**Remark 2.** We’ll give a formal definition of convergence in the next section; however,
informally, convergence implies that the solution of the difference scheme approaches the
Figure 2.2.4: Solutions of Example 2.2.2 obtained by the forward time-forward space scheme (2.2.2b) (top) and forward time-backward space scheme (2.2.4) (bottom). Each solution has a Courant number of 1/2.
solution of the partial differential equation as the space and time steps are reduced.

Remark 3. Theorem 2.2.1 implies that (2.2.2b) is useless when $a$ is positive and that (2.2.4) is useless when $a$ is negative. Thus, we should maintain $0 \leq a \leq \Delta x/\Delta t$ for (2.2.4) and $-\Delta x/\Delta t \leq a \leq 0$ for (2.2.2b). Stated in terms of the Courant number, we should maintain $0 \leq \alpha^n_j \leq 1$ for (2.2.4) and $-1 \leq \alpha^n_j \leq 0$ for (2.2.2b).

Remark 4. The Courant-Friedrichs-Levy Theorem is applicable under more general circumstances than stated here. It is, furthermore, usual to state it as a stability condition rather than one on the convergence of a difference scheme. We’ll consider other forms of the theorem in Chapter 6.

Proof. We use a straightforward contradiction argument. Since the difference scheme is required to converge for all initial conditions, consider space-time points where the two domains of dependence are disjoint and choose initial data that is nonzero on the domain of dependence of the partial differential equation and zero on the domain of dependence of the difference scheme. The solution of the difference scheme must be trivial at these points, whereas the solution of the partial differential equation at the same points will be nonzero. Refining the mesh so that the two domains of dependence remain disjoint does not alter this conclusion; hence, the solution of the difference equation cannot possibly
converge to that of the partial differential equation. Therefore, the domain of dependence of the difference scheme must contain that of the partial differential equation.

Problems

1. Suppose $a$ is a positive constant. The scheme (2.2.4) only depends on the Courant number $a^n_j$ which, for Examples 2.2.1 and 2.2.2, is the constant $a \Delta t / \Delta x$. Is there a particular choice of the Courant number on $(0, 1]$ that produces more accurate solutions than others? Answer the same question for (2.2.2b) when $a < 0$.

2. What restrictions, if any, should be placed on the Courant number $a^n_j$ for the forward time-centered space scheme (2.2.5) to satisfy the Courant, Friedrichs, Lewy theorem? We must, of course, study the behavior of (2.2.5); however, prior to doing this in Chapter 3, experiment by applying the method to the problems of Examples 2.2.1 and 2.2.2. Use the same mesh spacings as the earlier examples.

### 2.3 A Simple Difference Scheme for the Heat Equation

Let us determine whether similar or different phenomena occur when solving a simple initial-boundary value problem for the heat conduction equation. In particular, consider

\[ u_t = \sigma u_{xx}, \quad 0 < x < 1, \quad t > 0, \]  
\[ u(x, 0) = \phi(x), \quad 0 \leq x \leq 1, \]  
\[ u(0, t) = u(1, t) = 0, \]

where the diffusivity $\sigma$ is positive.

In order to construct finite-difference approximations of (2.3.1): (i) introduce a uniform grid of spacing $\Delta x = 1/J$, $J > 0$, and $\Delta t$ on the strip $(0, 1) \times (t > 0)$ (Figure 2.3.1); (ii) evaluate (2.3.1a) at the mesh point $(j \Delta x, n \Delta t)$; and (iii) replace the partial derivatives by forward time (2.1.8) and centered space (2.1.9) differences to obtain
2.2. A Heat Equation

Figure 2.3.1: Computational grid used for the finite difference solution of the heat conduction problem (2.3.1).

\[ \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\Delta t}{2} (u_{tt})^n = \sigma \left[ \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \frac{\Delta x^2}{12} (u_{xxxx})_{j+1}^{n+\theta} \right], \tag{2.3.2a} \]

where \(-1 < \xi < 1, 0 < \theta < 1\). (With second spatial derivatives in (2.3.1) there seems little point in using forward or backward spatial derivatives and we have not done so.)

Neglecting the discretization error terms, we get the finite difference scheme

\[ \frac{U_j^{n+1} - U_j^n}{\Delta t} = \sigma \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} \tag{2.3.2b} \]

or, solving for \(U_j^{n+1}\),

\[ U_j^{n+1} = rU_{j-1}^n + (1 - 2r)U_j^n + rU_{j+1}^n, \tag{2.3.3a} \]

where

\[ r = \sigma \frac{\Delta t}{\Delta x^2}. \tag{2.3.3b} \]

The initial and boundary conditions (2.3.1b, 2.3.1c) are

\[ U_j^0 = \phi(j\Delta x), \quad j = 0, 1, \ldots, J, \tag{2.3.3c} \]
Finite Difference Methods

\[ U_0^n = U_j^n = 0, \quad n > 0. \]  \hspace{1cm} (2.3.3d)

The computational stencil for (2.3.3a) is shown in Figure 2.3.2. The parameter \( r \) is analogous to the Courant number (2.2.3) for the kinematic wave equation (2.2.1).

The solution of (2.3.3a) is obtained in the same manner as the finite difference solutions of (2.2.2b) and (2.2.4) for the kinematic wave equation (2.2.1). Thus, using the initial data (2.3.3c), we calculate a solution \( U_j^1 \) at the interior mesh points, \( j = 1, 2, \ldots, J - 1 \), of time level 1 using (2.3.3a) with \( j \) ranging from 1 to \( j - 1 \). The boundary conditions (2.3.3d) with \( n = 1 \) determine \( U_0^1 \) and \( U_J^1 \). Knowing the discrete solution at time level 1, we proceed to determine it at time level 2, \( \text{etc.} \) in the same manner.

**Example 2.3.1.** Let us use the forward time-centered space scheme (2.3.3a) to obtain an approximate solution of (2.3.1) with \( \sigma = 1 \) and the initial data

\[ \phi(x) = \begin{cases} 
2x, & \text{if } 0 \leq x < 1/2 \\
2(1-x), & \text{if } 1/2 < x \leq 1
\end{cases} . \]

In Section 1.2, we saw that the exact Fourier series solution of this problem is

\[ u(x, t) = \sum_{k=1}^{\infty} \frac{8}{(k\pi)^2} e^{-k^2\pi^2 t} \sin k\pi x. \]

We'll solve this problem with \( \Delta x = 0.1 \) (\( J = 10 \)) and either \( \Delta t = 0.001 \) or 0.01. Again, the selection of these parameters is rather arbitrary. Using (2.3.3b) with \( \Delta t = 0.001 \) and \( \Delta x = 0.1 \) gives \( r = 0.1 \). Similarly, with \( \Delta t = 0.01 \) we find \( r = 1 \). The two solutions are
2.2. A Heat Equation

Table 2.3.1: Solution of Example 2.3.1 using the forward time-centered space scheme (2.3.3a) with \( r = 0.1 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( n )</th>
<th>( x = 0 )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( j = 0 )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>0.0</td>
<td>0</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>1.0</td>
<td>0.8</td>
</tr>
<tr>
<td>0.001</td>
<td>1</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>0.96</td>
<td>0.8</td>
</tr>
<tr>
<td>0.002</td>
<td>2</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.796</td>
<td>0.928</td>
<td>0.796</td>
</tr>
<tr>
<td>0.003</td>
<td>3</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.600</td>
<td>0.790</td>
<td>0.901</td>
<td>0.790</td>
</tr>
<tr>
<td>0.004</td>
<td>4</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.599</td>
<td>0.782</td>
<td>0.879</td>
<td>0.782</td>
</tr>
<tr>
<td>0.005</td>
<td>5</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.400</td>
<td>0.597</td>
<td>0.773</td>
<td>0.860</td>
</tr>
<tr>
<td>0.006</td>
<td>6</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.400</td>
<td>0.595</td>
<td>0.764</td>
<td>0.842</td>
</tr>
<tr>
<td>0.007</td>
<td>7</td>
<td>0.0</td>
<td>0.200</td>
<td>0.399</td>
<td>0.592</td>
<td>0.755</td>
<td>0.827</td>
<td>0.755</td>
</tr>
<tr>
<td>0.008</td>
<td>8</td>
<td>0.0</td>
<td>0.200</td>
<td>0.399</td>
<td>0.589</td>
<td>0.746</td>
<td>0.813</td>
<td>0.746</td>
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<tr>
<td>0.009</td>
<td>9</td>
<td>0.0</td>
<td>0.200</td>
<td>0.398</td>
<td>0.586</td>
<td>0.737</td>
<td>0.799</td>
<td>0.737</td>
</tr>
<tr>
<td>0.010</td>
<td>10</td>
<td>0.0</td>
<td>0.200</td>
<td>0.397</td>
<td>0.582</td>
<td>0.728</td>
<td>0.787</td>
<td>0.728</td>
</tr>
</tbody>
</table>

Table 2.3.2: Solution of Example 2.3.1 using the forward time-centered space scheme (2.3.3a) with \( r = 1 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( n )</th>
<th>( x = 0 )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( j = 0 )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>0.0</td>
<td>0</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>1.0</td>
<td>0.8</td>
</tr>
<tr>
<td>0.01</td>
<td>1</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>0.02</td>
<td>2</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.4</td>
<td>1.0</td>
<td>0.4</td>
</tr>
<tr>
<td>0.03</td>
<td>3</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.2</td>
<td>1.2</td>
<td>-0.2</td>
<td>1.2</td>
</tr>
<tr>
<td>0.04</td>
<td>4</td>
<td>0.0</td>
<td>0.2</td>
<td>0.0</td>
<td>1.4</td>
<td>-1.2</td>
<td>2.6</td>
<td>-1.2</td>
</tr>
</tbody>
</table>

Table 2.3.3: Errors at \( x = 0.3 \) (\( j = 3 \)) and \( x = 0.5 \) (\( j = 5 \)) for the solution of Example 2.3.1 using the forward time-centered space scheme (2.3.3a) with \( r = 0.1 \).

| \( t \) | \( n \) | \( x = 0.3(j = 3) \) | \( |u^n_j - U^n_3| \) | \( x = 0.5(j = 5) \) | \( |u^n_j - U^n_5| \) |
|--------|-------|----------------|----------------|----------------|----------------|
| 0.005  | 5     | 0.0008         | 0.023          |                |
| 0.01   | 10    | 0.004          | 0.016          |                |
| 0.1    | 100   | 0.011          | 0.012          |                |

shown in Tables 2.3.1 and 2.3.2 for a few time steps. Errors of the solution with \( r = 0.1 \) at \( x = 0.3 \) and 0.5 are presented in Table 2.3.3 for a few times. The solutions are also shown in Figure 2.3.3.

As shown in Tables 2.3.1 and 2.3.3, the solution of (2.3.3a) with \( r = 0.1 \) is producing a reasonable approximation of the exact solution. The larger errors at \( x = 0.5 \) for
small times are due to the discontinuity in the initial data there. When \( r = 1 \), the finite difference solution bears little resemblance to the exact solution. As with the forward time-forward space scheme (2.2.2b) for the kinematic wave equation (2.2.1), it is oscillatory and increasing in amplitude. Apparently, some restriction must be placed on \( r \); however, the explanation for this restriction is not as simple as it was with (2.2.1). We’ll discuss it in the next chapter.
Figure 2.3.3: Solutions of Example 2.3.1 obtained by the forward time-centered space scheme (2.3.3a) with \( r = 0.1 \) (top) and \( r = 1 \) (bottom).
Bibliography

