Chapter 7

Multidimensional Hyperbolic Problems

7.1 Split and Unsplit Difference Methods

Our study of multidimensional parabolic problems in Chapter 5 has laid most of the groundwork for our present task of creating difference approximations for multidimensional hyperbolic problems. By now we know that

- definitions of consistency, convergence, and stability for one-dimensional problems carry over to multiple dimensions;

- stability analyses by von Neumann’s method are equivalent to those in one dimension but algebraically more complex; and

- implicit schemes are difficult to implement without splitting the operator or alternating directions.

Geometrical complexities will, again, be postponed until Chapter 8, so we’ll consider two-dimensional conservation laws of the form

\[ u_t + f(u)_x + g(u)_y = 0 \]  \hspace{1cm} (7.1.1a)

where \( u, f, \) and \( g \) are \( m \)-vectors. The convective form of this system is

\[ u_t + Au_x + Bu_y = 0 \]  \hspace{1cm} (7.1.1b)
where

\[ A(u) = f_\mathbf{u}(u), \quad B(u) = g_\mathbf{u}(u). \]  

Multidimensional finite difference schemes can be simple extensions of one-dimensional methods. Thus, for example, a Lax-Friedrichs approximation of (7.1.1a) would be

\[
U_{jk}^{n+1} = \frac{1}{4} (U_{j+1,k}^{n} + U_{j-1,k}^{n} + U_{j,k+1}^{n} + U_{j,k-1}^{n})
- \frac{\Delta t}{2\Delta x} (f_{j+1,k}^{n} - f_{j-1,k}^{n}) - \frac{\Delta t}{2\Delta y} (g_{j,k+1}^{n} - g_{j,k-1}^{n}).
\]  

As in Chapter 5, a uniform mesh of spacing \(\Delta x \times \Delta y\) has been introduced with \(U_{jk}^{n}\) being the finite difference approximation of \(u(j\Delta x, k\Delta y, n\Delta t)\).

Extending the Richtmyer two-step method (6.4.1) to two dimensions is a bit more difficult. The predictor step is the Lax-Friedrichs method

\[
U_{lm}^{n+1} = \frac{1}{4} (U_{l+1,m}^{n} + U_{l-1,m}^{n} + U_{l,m+1}^{n} + U_{l,m-1}^{n}) - \frac{\Delta t}{2\Delta x} (f_{l+1,m}^{n} - f_{l-1,m}^{n})
- \frac{\Delta t}{2\Delta y} (g_{l,m+1}^{n} - g_{l,m-1}^{n}),
\]  

applied to the four points surrounding \((j, k)\) (Figure 7.1.1). The corrector step is the leap frog method

\[
U_{jk}^{n+2} = U_{jk}^{n} - \frac{\Delta t}{\Delta x} (f_{j+1,k}^{n+1} - f_{j-1,k}^{n+1}) - \frac{\Delta t}{\Delta y} (g_{j,k+1}^{n+1} - g_{j,k-1}^{n+1}).
\]

The computational stencil of (7.1.3) is shown in Figure 7.1.1. The mesh spacing \(\Delta t\), \(\Delta x\), and \(\Delta y\) has been doubled relative to the one-dimensional scheme (6.4.1). This was done to simplify the writing of the scheme. As shown on the bottom of Figure 7.1.1, the Richtmyer two-step scheme can be regarded as a nine-point difference formula on a staggered grid. This is possible because the physical flux vector and the divergence operator are rotationally invariant. Halving the mesh spacing to get a scheme that is more in line with our usual notation is much simpler with this staggered grid interpretation.

The Courant, Friedrichs, Lewy Theorem is still available to restrict the domain of dependence of a difference scheme to contain that of the partial differential equation. For example, the solution of the model initial value problem

\[
u_t + au_x + bu_y = 0, \quad -\infty < x, y < \infty, \quad t > 0,
\]  

(7.1.4a)
Figure 7.1.1: Computational stencil of the Richtmyer two-step method (7.1.3). Predicted solutions are shown with filled circles and corrected solutions are shown with blue circles and corrected solutions are shown in red. The Richtmyer two-step scheme can be regarded as a nine-point formula on a staggered grid (bottom).
\[ u(x, y, 0) = \phi(x, y), \quad -\infty < x < \infty, \quad (7.1.4b) \]

is

\[ u(x, y, t) = \phi(x - at, y - bt). \quad (7.1.4c) \]

Thus, the domain of dependence of a point \((x_0, y_0, t_0)\) is the single point \((x_0 - at_0, y_0 - bt_0)\).

Let us consider a five- or nine-point explicit difference scheme of the form

\[ U_{jk}^{n+1} = \sum_{l=j-1}^{j+1} \sum_{m=k-1}^{k+1} C_{lm} U_{lm}^n \]

The domain of dependence of the mesh point \((j, k, n + 1)\) for the nine-point scheme is the rectangle

\[ D = \{(x, y) | x_{j-1} \leq x \leq x_{j+1}, \ y_{k-1} \leq y \leq y_{k+1}\}. \quad (7.1.5a) \]

A five-point scheme would have \(C_{j+1,k+1} = 0\). Its domain of dependence would be less clear; however, for simplicity, we’ll define it as

\[ D = \{(x, y) | \frac{|x - x_j|}{\Delta x} + \frac{|y - y_k|}{\Delta y} \leq 1\}. \quad (7.1.5b) \]

The domain of dependence of the point \((j, k, n + 1)\) for the partial differential equation is the single point \((x_j - a \Delta t, y_k - b \Delta t)\). The various domains of dependence are shown in Figure 7.1.2. Thus, the Courant, Friedrichs, Lewy condition is

\[ \max\left(\frac{|a| \Delta t}{\Delta x}, \frac{|b| \Delta t}{\Delta y}\right) \leq 1. \quad (7.1.6a) \]

for the nine-point scheme and

\[ \frac{|a| \Delta t}{\Delta x} + \frac{|b| \Delta t}{\Delta y} \leq 1 \quad (7.1.6b) \]

for the five-point scheme.

The Lax-Friedrichs scheme applied to (7.1.4a) yields

\[ U_{jk}^{n+1} = \frac{1}{4}(1 - 2\alpha)U_{j+1,k}^n + \frac{1}{4}(1 + 2\alpha)U_{j-1,k}^n + \frac{1}{4}(1 - 2\beta)U_{j,k+1}^n + \frac{1}{4}(1 + 2\beta)U_{j,k-1}^n \]

where

\[ \alpha = \frac{a \Delta t}{\Delta x}, \quad \beta = \frac{b \Delta t}{\Delta y}. \]
Using the Maximum Principle, we find a sufficient condition for stability in the maximum norm as
\[
\max(|\alpha|, |\beta|) \leq \frac{1}{2},
\]
which is more restrictive than in one dimension. Using the von Neumann method (cf. Problem 1 at the end of this section), we may show that the Lax-Friedrichs scheme is stable in $L^2$ when
\[
\alpha^2 + \beta^2 \leq \frac{1}{2}.
\]

**Example 7.1.1.** With the more stringent stability restrictions, methods that use operator splitting become attractive. Let us consider splitting MacCormack’s predictor-corrector scheme (6.4.2) for the conservation laws (7.1.2). In the $x$-direction, we neglect $g$, predict using forward time-backward space differencing, and correct using backward time-forward space differencing for one half time step to obtain
\[
\hat{U}_{jk}^{n+1} = U_{jk}^n - \frac{\Delta t}{\Delta x} (f_{jk}^m - f_{j-1,k}^m),
\]
\[
U_{jk}^{n+1} = \frac{U_{jk}^n + \hat{U}_{jk}^{n+1}}{2} - \frac{\Delta t}{2\Delta x} (\hat{f}_{j+1,k}^{n+1} - \hat{f}_{jk}^{n+1}).
\]
where \( \hat{f}_{jk}^{n+1} \equiv f(\hat{U}_{jk}^{n+1}) \). The solution obtained at the conclusion of sweeping the mesh by rows is denoted as \( \hat{U}_{jk}^{m+1} \). The sweep in the \( y \)-direction follows the same pattern with \( f \) neglected and \( g \) included and the solution \( \hat{U}_{jk}^{m+1} \) used as initial data. Thus,

\[
\hat{U}_{jk}^{n+1} = \hat{U}_{jk}^{n+1} - \frac{\Delta t}{\Delta y} (g_{jk}^{n+1} - g_{jk-1}^{n+1}),
\]

(7.1.9)

\[
U_{jk}^{n+1} = \frac{U_{jk}^{n+1} + \hat{U}_{jk}^{n+1}}{2} - \frac{\Delta t}{2\Delta y} (g_{j,k+1}^{n+1} - g_{jk}^{n+1}).
\]

(7.1.10)

This split scheme merely needs to satisfy the one-dimensional stability conditions when the boundary conditions between the \( x \) and \( y \) sweeps are implemented with care (cf. Section 5.2).

**Problems**

1. Analyze the stability of the Lax-Friedrichs scheme (7.1.2) for the kinematic wave equation (7.1.4) using von Neumann’s method.