Pricing The American Put Using a New Class of Tight Lower Bounds

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BACKGROUND
THE BOUNDS
USING THE BOUNDS
EXPERIMENTS
The American Put Option

- Continuous exercise (when to exercise?)
- What price to exercise at?
Survey

– **Data driven methods**, [Barone-Adesi87, Johnson83, Keber01, MacMillan86].

– **Discrete optimal exercise**, [Bunch92, Geske85, Huang96].

– **Bounds**, [Chen02, Levy85, Lo87].

– **Convergent numerical methods**,  
  *Simulation*, [Breen91, Broadie96, Laprise01, Longstaff01, Tsitsiklis01]  
  *Linear Programming*, [Dempster00]

For accurate pricing we need:

**Continuous and Optimal exercise.**
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Lower Bound

\[ \frac{dS}{S} = (r - \delta)dt + \sigma dW. \]

Exercise Strategy: \( e(t) \).

Lower Bound:

\[ P(S, K, T, r, \delta, \sigma) \geq E_P[Cash\ Flows|e(t)] \]

for any \( e(t) \).
Piecewise Exponential Exercise

\[ e(t) = e^{\beta_3 + \alpha_3 (t - \tau_2)} \]

\[ e^{\beta_2 + \alpha_2 (t - \tau_1)} \]

\[ e^{\beta_1 + \alpha_1 (t - \tau_0)} \]

\[ 0 = \tau_0 < \tau_1 < \tau_2 < T \]

\[ L_M(S, K, T|\alpha, \beta, \tau) = \sum_i a_i N_{p_i}(x_i, \Sigma_i), \quad p_i \leq M. \]

**M–Point Bound** (3M – 1 parameters):

**Special Cases:**

- Constant (0-Point Bound): \( \alpha = 0, \beta \).
- Exponential (1-Point Bound): \( \alpha, \beta \).
- Two Piece (2-Point Bound): \( \begin{bmatrix} \beta_1 \\ \alpha_1 \end{bmatrix}, \begin{bmatrix} \beta_2 \\ \alpha_2 \end{bmatrix}, \tau \).
Recursive Derivation of $L_M$

Analytic – can be automated!
벽돌  

认可背景  
认可边界条件  

⇒ 使用边界条件的实验
A Tighter Bound

- \( L_M(S, K, T|\alpha, \beta, \tau) \) is a **Lower Bound** \( \forall \{\alpha, \beta, \tau\} \).
- Maximize w.r.t. \( \{\alpha, \beta, \tau\} \) to get a tighter bound.

**Constrained Optimization Problem**

\[ M \to \infty, \text{ should have} \]

\[ P(S, K, T) = \sup_{\alpha, \beta, \tau} L_M(S, K, T|\alpha, \beta, \tau). \]
Practical Tradeoff

\[ M \uparrow \iff \text{more computation:} \]

- Some \( M \)-dim multivariate Normal integrals.
- \( O(12 \times 2^M) \) terms.

Small \( M \) are fast \((M = 1, 2)\):

- Bivariate Normal integrals.
- Small number of terms.
☑ BACKGROUND
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⇌ EXPERIMENTS
Simulations

Tested 0, 1, 2–Point bounds.

Canned optimization (MATLAB)

Ensemble of $\geq 130,000$ “randomly generated puts”

$$\{S, K, T, r, \delta, \sigma, P(S, K, T, r, \delta, \sigma)\}$$

$P(S, K, T, r, \delta, \sigma) \leftarrow 15000$ step binomial tree.
$S = 25$
$K = 20$
$T = 2.5$
$r = 0.06$
$\delta = 0.03$
$\sigma = 0.2$

\[ \rightarrow \quad P(S, K, T, r, \delta, \sigma) = 0.6321 \]

\[
\begin{array}{ll}
L_0 = 0.6258 & \text{time } \sim 0.004 \text{sec} \\
L_1 = 0.6309 & \text{time } \sim 0.008 \text{sec} \\
L_2 = 0.6317 & \text{time } \sim 0.142 \text{sec}
\end{array}
\]
Extensive Simulations

Comparing binomial trees to $M$-point Bounds.

Pricing with Bounds vs. Binomial Trees

2-point bound:
Accuracy: $10^{-3.71}$, time $\sim 0.15$ sec.

Binomial tree:
Comparable speed: $\sim \times 7$ less accurate.
Comparable accuracy: $\sim \times 25$ slower.
Histogram of Relative Errors

Histogram of Relative Errors for 2-point Bound

(a) Lower bounds.

(b) 1500 step binomial tree.
Thank You

- **Family** of Analytic bounds.

- **Optimize** to get tighter bounds.

- **Ongoing:**
  - Convergence?
  - Data driven discrepancy prediction.
  - Sensitivities.

Data available at:

www.cs.rpi.edu/~magdon