Callibration of Multi-Factor Vasicek Interest Rate Models

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GOAL

• We have a set of instruments (eg interest rate swaps, caps, swaptions) and market prices – Historical Data.

• Assume a model: **Vasicek-k factor** model, which gives the interest rate dynamics.

• To the extent that the model describes the market dynamics, obtain the fair value of the instruments today.
MODEL: VASICEK K-Factor DYNAMICS

- \( dx_i(t) = k_i(m_i - x_i(t)) + \sigma_i dW_i(t) \)
  \( i = 1, \ldots, K \)

Parameters (Y): \( k_i, m_i, \sigma_i \) and \( \rho_{ij}. \) \( \left[ \frac{K(K+5)}{2} \right] \)

States (x): \( x_i. \) \( [K] \)

- Interest rate dynamics are given by

  \[ r(t) = \sum_{i=1}^{K} x_i(t) \]

- For any instrument, \( I, \) one can compute

  \[ \text{Price}(I|Y, x) \]
• Dynamics given by Vasicek model.

• $\varepsilon$ is a noise “shock”, assumed to have some distribution, à la maximum likelihood.

• Noise term represents (tradeable) mispricing that is expected to “revert” back to zero.

Determine $Y, x_0$. 
COMMONLY USED APPROACH

• Use *plausible* values for long term parameters.

• *Calibrate* each day’s states to that day’s prices.

• Is this curve fitting?
  – Fast
  – Does the callibration have any “economic” meaning?
  – Use small models, which may not realistically describe the dynamics.
  – As a result, can’t “really” trade on the model-market discrepancies.
HIDDEN MARKOV MODEL

FIRST DAY

- Naturally posed as a HMM with infinite dimensional state space and known non-linearity.

- Unknowns: $Y$, $x_i$: LOTS of them.

- Optimize $\text{Prob}[\text{Unknowns}|\text{Data}]$.

- Hard: Many variables, and Pricing function is complex.
• Break down into two easier problems.

  1. Fix $x_i$’s, and optimize $Y$.

  2. Fix $Y$, and optimize $x_i$’s.

  3. Iterate.

• Can solve in a sequential manner (convergence?).

• Finite iterations: order is important, especially at end points.
FITS TO SWAP DATA

EQUILIBRIUM SWAP RATES (PAR RATES) FIT

Day Number = 1                              MATURITY

% RATE

EQUILIBRIUM SWAP RATES (PAR RATES) FIT

Day Number = 280                              MATURITY

% RATE

But, ..., suppose that we want the volatility term structure as well.
HIGHER ORDER STRUCTURE

Histogram of First Factor

Histogram of Second Factor

Histogram of Third Factor

First and Second Factors

First and Third Factors

Second and Third Factors

Severe mismatch between model and historical!
WHAT HAPPENED

5th order polynomial fit assuming gaussian noise

Histogram of sample errors

Actual Fit
CORRECTING THE ERROR

1. Change model: assume different noise model, start over (eg. fat tails).

2. Control the way we search the space: regularize.
   - Callibrate
   - Make sure that model assumptions are satisfied.

Ensure that Final Model is **CONSISTENT** with **initial** assumptions
CONSISTENCY HINTS

1. **Consistency** of states with parameters.

2. **Plausibility** of the parameters.

3. **Validity** of the driver as a Wiener Process.

How does one get these?

- Probabilistic approach: Expand $\text{Prob}[(\cdot)|\text{Data}]$ in terms of conditionals. How to get these probabilities?

- Add error terms that enforce these “initial” assumptions with corresponding regularization parameters. How to get the regularization parameters?
FITS TO SWAP DATA – Part II

EQUILIBRIUM SWAP RATES (PAR RATES) FIT

Day Number = 1

% RATE

MATURITY

EQUILIBRIUM SWAP RATES (PAR RATES) FIT

Day Number = 280

% RATE

MATURITY
HIGHER ORDER STRUCTURE

Histogram of First Factor

First and Second Factors

Histogram of Second Factor

First and Third Factors

Histogram of Third Factor

Second and Third Factors
VOLATILITY TERM STRUCTURE

HISTORICAL (blue) AND MODEL−BASED (red) ANNUAL VOLATILITY

MODEL−BASED CORRELATION MATRIX

HISTORICAL CORRELATION MATRIX
STATISTICS FOR MISPRICINGS

- Errors are distributed according to our assumptions?

- Errors are mis-pricings?
OVERVIEW

Historical Data

Vasicek k–Factor Model
Parameters [Y]

CALLIBRATION
Consistency Hint
Plausibility Hint
Validity Hint

Todays Fair Value

TRADES

MONEY!!