**Fuzzy Rules and Fuzzy Reasoning**

*(chapter 3)*

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(adapted from slides by R. Jang)

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**Outline**

- Extension principle
- Fuzzy relations
- Fuzzy IF-THEN rules
- Compositional rule of inference
- Fuzzy reasoning

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**Extension Principle**

Extends crisp domains of mathematical expressions to fuzzy domains

\( A \) is a fuzzy set on \( X \):

\[ A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n \]

The image of \( A \) under \( f() \) is a fuzzy set \( B \), i.e., \( B=f(A) \)

\[ B = \mu_B(x_1)/y_1 + \mu_B(x_2)/y_2 + \cdots + \mu_B(x_n)/y_n \]

where \( y_i = f(x_i) \), \( i = 1 \) to \( n \).

If \( f() \) is a many-to-one mapping, then

\[ \mu_B(y) = \max_{y_i=f(x)} \mu_A(x) \]

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**Extension Principle: Example**

\( A = 0.1/2 + 0.4/1 + 0.8/0.9 + 0.3/2 \)

\( f(x) = x^2-3 \)

\( B = 0.1/1 + 0.4/2 + 0.8/3 + 0.9/2 + 0.3/1 \)

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**Extension Principle, continuous vars.**

Let \( \mu_A(x) = bell(x;15,2,0.5) \)

and \( f(x) = \begin{cases} (x-1)^2 - 1 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases} \)

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Useful for:
- Automatic Control
- Expert Systems
- Pattern Recognition
- Time Series Prediction
- Data Classification
Fuzzy Relations

A fuzzy relation \( R \) is a 2D MF:
\[ R = \{(x, y), \mu_R(x, y)\} \forall (x, y) \in X \times Y \]

Examples:
- \( x \) depends on \( y \) and \( y \) are events
- If \( x \) is large, then \( y \) is small (\( x \) is an observed reading and \( y \) is a corresponding action)
- \( y \) is much greater than \( x \) (\( x \) and \( y \) are numbers)

If \( X = (3, 4, 5) \) and \( Y = (3, 4, 5, 6, 7) \)

Let
\[ R_1 = \text{"x is relevant to y"} \]
\[ R_2 = \text{"y is relevant to z"} \]

where
\[ X = \{a, \beta, \chi, \delta\} \]
\[ Y = \{1, 2, 3\} \]
\[ Z = \{a, b\} \]

Max-Star Composition

Max-product composition:
\[ \mu_{R_1 \wedge R_2}(x, z) = \max_y \{\mu_{R_1}(x, y) \cdot \mu_{R_2}(y, z)\} \]

In general, we have max-\( \ast \) composition:
\[ \mu_{R_1 \ast R_2}(x, z) = \max_y \{\mu_{R_1}(x, y) \ast \mu_{R_2}(y, z)\} \]

where \( \ast \) is a T-norm operator.

Even more generally, we have (S-norm)-(T-norm) compositions

Max-Min Composition

The max-min composition of two fuzzy relations \( R_1 \) (defined on \( X \) and \( Y \)) and \( R_2 \) (defined on \( Y \) and \( Z \)) is
\[ \mu_1(x, z) = \min_{y \in Y} \{\mu_1(x, y) \vee \mu_2(y, z)\} \]

Note: calculation very similar to matrix multiplication

Properties:
- Associativity: \( R \otimes (S \circ T) = (R \circ S) \otimes T \)
- Distributivity over union:
\[ R \otimes (S \cup T) = (R \circ S) \cup (R \circ T) \]
- Weak distributivity over intersection:
\[ R \otimes (S \cap T) \subseteq (R \circ S \cap T) \]
- Monotonicity:
\[ S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T) \]

Max-Min Composition: Example

Let
\[ R_1 = \text{"x is relevant to y"} \]
\[ R_2 = \text{"y is relevant to z"} \]

where
\[ X = \{0.1, 0.3, 0.5, 0.7\} \]
\[ Y = \{0.4, 0.2, 0.8, 0.9\} \]
\[ Z = \{0.6, 0.8, 0.3, 0.2\} \]

Calculate:
\[ X = \{0.1, 0.3, 0.5, 0.7\} \]
\[ Y = \{0.4, 0.2, 0.8, 0.9\} \]
\[ Z = \{0.6, 0.8, 0.3, 0.2\} \]

Linguistic Variables (knowledge rep.)

Precision vs. significance

A numeric variable can take on numerical values (Age is 65) or linguistic values (Age is old)

A linguistic value is characterized by the variable name (age) and a fuzzy set.

All linguistic values form a term set:
\[ T(\text{age}) = \text{young, not young, very young, ...} \]
\[ \text{middle aged, not middle aged, ...} \]
\[ \text{old, not old, very old, more or less old, ...} \]
\[ \text{not very young and not very old, ...} \]

The syntactic rule refers to how the linguistic values are generated.
The semantic rule defines the membership value of each linguistic variable.
Linguistic Values (Terms)

Operations on Linguistic Values

Fuzzy IF-THEN Rules

Two ways to interpret “If x is A then y is B”:
- A coupled with B: \((A \land B)\)
  \[ R = A \rightarrow B = A \times B = \{ \mu_a(x) \times \mu_b(y) | x, y \} \]
- A entails B: \((\neg A \lor B)\)
  - Material implication \(\neg A \lor B\)
  - Propositional calculus \((\neg A \land \neg B) \lor B\)
  - Extended propositional calculus \((\neg A \land B) \lor B\)
  - Generalization of modus ponens
  \[ \mu_x(x, y) = \max\{ \mu_a(x) \times \mu_b(y) | c \leq \mu_a(x) \land 0 \leq c \leq 1 \} \]
  - Note: these all reduce to \(A \lor B\) in two-valued logic

Fuzzy implication function:
- A coupled with B
  \[ \mu_{\rightarrow}(x, y) = \min(\mu_a(x), \mu_b(y)) \]
  \[ \mu_{\rightarrow}(x, y) = \max(\mu_a(x), \mu_b(y)) \]
  \[ \mu_{\rightarrow}(x, y) = \mu_a(x) \times \mu_b(y) \]
  \[ \mu_{\rightarrow}(x, y) = \mu_a(x) \lor \mu_b(y) \]
  - \(0 \leq \mu_a(x) \leq 1\)
  - \(0 \leq \mu_b(y) \leq 1\)
  - \(0 \leq \mu_{\rightarrow}(x, y) \leq 1\)

Effects of Contrast Intensification

Fuzzy IF-THEN Rules
Fuzzy IF-THEN Rules

A coupled with B
Let
\[ \mu_A(x) = \text{bell}(x; 4.3, 10) \]
\[ \mu_B(y) = \text{bell}(y; 4.3, 10) \]

min algebraic product bounded product drastic product

Zadeh's arithmetic rule Zadeh's max-min rule Boolean Fuzzy Implication Gougen's Fuzzy Impl.

Compositional Rule of Inference

Same idea as max-min composition

Derivation of \( y = b \) from \( x = a \) and \( y = f(x) \):

To find the resulting interval \( y = b \)
(which corresponds to \( x = a \))

- Construct a cylindrical extension of \( a \)
- Find intersection with curve
- Project intersection to \( y \)-axis

Recall: Cylindrical Extension

\[
\mu_{\text{ext}}(x, y) = \mu_A(x)
\]
Recall: 2D MF Projection

Two-dimensional MF Projection onto X and onto Y

\[ \mu_{B}(x, y) = \max_{y} \mu_{A}(x, y) \]

\[ \mu_{B}(x) = \max_{y} \mu_{A}(x, y) \]

\[ \mu_{B}(y) = \max_{x} \mu_{A}(x, y) \]

Compositional Rule of Inference

\( a \) is a fuzzy set and \( y = f(x) \) is a fuzzy relation:

\[ \mu_{B}(y) = \max_{x} \min_{a} \mu_{A}(x, y) \]

\[ \mu_{B}(x) = \min_{y} \max_{a} \mu_{A}(x, y) \]

Crisp Reasoning

Modus Ponens:

Fact: \( x \) is A

Rule: IF \( x \) is A THEN \( y \) is B

Conclusion: \( y \) is B

Putting it together: Comp. rule of Inf.

Cylindrical extension with base A

\[ \mu_{\text{cyl}}(x, y) = \mu_{A}(x) \]

Intersection of c(A) with F

\[ \mu_{\text{int}}(x, y) = \min \left[ \mu_{\text{cyl}}(x, y), \mu_{F}(y) \right] \]

Projection onto y-axis

\[ \mu_{Y}(y) = \max_{x} \min \left[ \mu_{X}(x), \mu_{F}(y) \right] \]

\[ = \min_{y} \max_{x} \mu_{X}(x) \cdot \mu_{F}(y) \]

\[ = \vee \left[ \mu_{X}(x) \cdot \mu_{F}(y) \right] \]

Representation: \( B = A \times F \)

Note relation to extension principle

Fuzzy Reasoning

Single rule with single antecedent

Fact: \( x \) is A

Rule: IF \( x \) is A then \( y \) is B

Conclusion: \( y \) is B

(Graphical Modus Ponens)

Graphic Representation:

FR - Single Rule, Single Antecedent

Graphical Representation:

- Find degree of match \( w \) between \( \mu_{A}(x) \) and \( \mu_{B}(x) \)

Intuitively: degree of belief for antecedent which gets propagated; result should not be greater than \( w \)
Fuzzy Reasoning: Single Antecedent

Let A, A', and B be fuzzy sets of X, X, and Y, respectively.
Assumption: the fuzzy implication A→B is expressed as a fuzzy relation R on XxY
The fuzzy set B induced by
fact: x is A' and
premise: IF x is A then y is B
or: B = A→B = A→(A→B)

Fuzzy Reasoning

Single rule with multiple antecedents
Facts: x is A' and y is B'
Rule: if x is A and y is B then z is C
Conclusion: z is C'
μ_C(z) = sup [μ_A(x) ∧ μ_B(y)](μ_C(z))
= sup [μ_A(x) ∧ μ_B(y)](μ_C(z))
= sup [μ_A(x) ∧ μ_B(y)](μ_C(z))
C = [A→(A→C)] ∧ [B→(B→C)]

FR - Single Rule, Multiple Antecedents

Graphical Representation
- w denotes degree of compatibility between A and A'
and w between B and B'
- w, w is degree of fulfillment of the rule

Fuzzy Reasoning

Multiple rules with multiple antecedent
Fact: x is A' and y is B'
Rule 1: if x is A and y is B then z is C_1
Rule 2: if x is A and y is B then z is C_2
Conclusion: z is C'
Graphic Representation: (next slide)