CSCI 2400 – Models of Computation

Solution for Homework #5

1. Show language \( L = \{ a^n b^i c^k : k = jn \} \) is not context-free.

   \textit{Solution}

   Assume that \( L \) is a context free language.
   Let \( w = a^m b^n c^m \), \( w \in L \). By the Pumping Lemma, \( w \) can be decomposed as \( w = uvxyz \) with \( |vzx| \leq m \) and \( |vy| \geq 1 \) such that \( uv^i xy^i z \in L \), \( i \leq 0 \).

   \textit{case 1} \( \underbrace{a \ldots ab \ldots bc \ldots c}_{uvxy} z \)
   
   If \( i = 0 \), \( uv^0 xy^0 z = a^m - |vy| b^n c^m \not\in L \).

   \textit{case 2} \( \underbrace{a \ldots ab \ldots bc \ldots c}_{uv} \)
   
   If \( i = 0 \), \( uv^0 xy^0 z = a^m b^n c^m \not\in L \).

   \textit{case 3} \( \underbrace{a \ldots ab \ldots bc \ldots c}_{uv} \)
   
   If \( i = 0 \), \( uv^0 xy^0 z = a^m b^n c^m \not\in L \).

   \textit{case 4} \( \underbrace{a \ldots ab \ldots bc \ldots c}_{uv} \)
   
   If \( i = 0 \), \( uv^0 xy^0 z = a^m b^n c^m - |vy| \not\in L \).

   \textit{case 5} \( \underbrace{a \ldots ab \ldots bc \ldots c}_{uv} \)
   
   If \( i = 0 \), \( uv^0 xy^0 z = a^m b^n c^m - |vy| \not\in L \).

   \textit{case 6} \( v \) or \( y \) containing \( ab \) or \( bc \)
   
   If \( i > 1 \), \( uv^i xy^i z \) would be \( a \ldots ab \ldots ba \ldots ab \ldots b \ldots c \ldots c \) or \( a \ldots ab \ldots bc \ldots cb \ldots c \).

   This is contradictory to the assumption that language \( L \) is context free. Therefore \( L \) is not context free.

2. Show language \( L = \{ w \in \{a, b, c\}^* : n_a(w) + n_b(w) = 2n_c(w) \} \) is not context-free.

3. Show language \( L = \{ww^Ra^{|w|} : w \in \{a, b\}^* \} \) is not context-free.

   \textit{Solution}

   Assume that \( L \) is a context free language. Let \( w = a^m b^m \). Then \( ww^Ra^{|w|} = a^m b^m b^m a^m a^{2m} \in L \). By the Pumping Lemma, \( ww^Ra^{|w|} \) can be decomposed as \( ww^Ra^{|w|} = uvxyz \) with \( |vzx| \leq m \) and \( |vy| \geq 1 \) such that \( uv^i xy^i z \in L \), \( i \leq 0 \).

   \textit{case 1} \( \underbrace{a \ldots ab \ldots ba \ldots aa \ldots a}_{uvxy} z \)
   
   If \( i = 0 \), \( |w| = 2m - |vy| \) is less than \( |w^R| \). So \( uv^0 xy^0 z \not\in L \).

   \textit{case 2} \( \underbrace{a \ldots ab \ldots ba \ldots aa \ldots a}_{uv} \)
   
   If \( i = 0 \), \( |w^R| = 2m - |vy| \) is less than \( |w| \). So \( uv^0 xy^0 z \not\in L \).
case 3 \( u_{a...ab...bb...ba...aa...a} \)

If \( i = 0 \), \( |a^{|w|}| = 2m - |vx| \) is less than \( |w| \). So \( u^0 x y^0 z \notin L \).

case 4 \( u_{a...ab...bb...ba...aa...a} \)

If \( i = 0 \), \( |u w^R| = 4m - |vy| \) is less than \( 2 * |a^{|w|}| = 4m \). So \( u^0 x y^0 z \notin L \).

case 5 \( u_{a...ab...bb...ba...aa...a} \)

If \( i = 0 \), \( |w^R a^{|w|}| = 4m - |vy| \) is less than \( 2 * |w| = 4m \). So \( u^0 x y^0 z \notin L \).

This is contradictory to the assumption that language \( L \) is context free. Therefore \( L \) is not context free.

4. Construct a Turing machine that will accepts language \( L = \{a^n b^m : n \geq 1, n \neq m\} \).

   Solution

![Turing Machine Diagram]

Figure 1: Turing Machine that accepts \( L = \{a^n b^m : n \geq 1, n \neq m\} \)

5. Construct a Turing machine to compute the function

\[ f(w) = w^R \]

where \( w \in \{0,1\}^+ \).

   Solution
Figure 2: Turing Machine to compute $f(w) = w^R$