Problem 1 Determine a formula for the minimal size of a maximal matching in a cycle $C_n$ of length $n \geq 3$. Prove the formula.

The lengths of the maximal matchings in the cycles $C_6, C_7, C_8, C_9$ are 2, 3, 3, and 3, respectively.

Let $C_n = (x_0, x_1, \ldots, x_{n-1})$ be a cycle of length $n$. For any matching $M$ in $C_n$, if $|M| < n/3$, then $M$ contains at least one edge $(x_i, x_{i+1})$ such that the next two vertices, $x_{i+2}$ and $x_{i+3}$, are weak (not saturated by $M$). Thus, $M$ is not maximal. This implies that for every maximal matching $M$, its size $|M| \geq n/3$. Since, $|M|$ is an integer,

$$|M| \geq \left\lceil \frac{n}{3} \right\rceil.$$ 

The answer is $\left\lceil \frac{n}{3} \right\rceil$, since it is easy to construct a matching of this length in $C_n$. \qed

Problem 2 Let $\alpha(G)$ denote the maximal number of vertices such that no two of them are adjacent to each other. Let graph $G$ have $n$ vertices and let $d$ be the maximal vertex degree in $G$. Prove that

$$\alpha(G) \geq \frac{n}{d + 1}.$$
Solution. Let $S$ be a set of vertices such that no two are adjacent to each other. Then for every $x \in S$, all its neighbors are in $V - S$. Since the maximum vertex degree is $d$, the number of vertices in $V - S$ that are adjacent to vertices in $S$ doesn’t exceed $d|S|$. Together with $S$ itself, the number of vertices in $S$ and its neighborhood $\leq |S|(d+1)$. If $|S|(d+1) < n$, then there would be a vertex $x$ in $V - S$ which is not adjacent to any vertex in $S$. Thus $S$ wouldn’t be a maximal size set of vertices no two adjacent to each other. Therefore, $\alpha(G) \times (d + 1) \geq n$. \hfill \blacksquare

Problem 3 Prove: every tree has at most one perfect matching.

Proof. If a tree $T$ had two distinct perfect matchings $M_1$ and $M_2$, then their symmetric difference would be a collection of cycles, whose edges alternate between $M_1$ and $M_2$ (done in class). But no tree has cycles. \hfill \blacksquare