Computational Finance – Computing Risk

1 Introduction

Estimation of the risk in a portfolio is one of the most ill-defined, yet important tasks of a big financial institution. The challenge has been to come up with a meaningful risk measure which captures both the typical fluctuations in the portfolio as well as the “low probability” worst case loss. In general these are competing goals. One typical approach to capturing the typical fluctuations is through the volatility $\sigma$ in the value of the portfolio. For a normally distributed value, some multiple of the volatility, for example $5\sigma$ also captures the low probability tail risk. Such an approach estimates the worst case that one has to consider as $E[V] - 5\sigma$. Such rules of thumb are useful, but it is also useful to have some more principled metrics for risk computation and techniques for computing them.

Suppose that one has instruments with values $s_1, \ldots, s_n$ which comprise a portfolio $\Pi$ whose value is given by $V = \sum_{i=1}^{n} w_i s_i$ at some risk horizon $T$. Typically, the risk horizon is 1 year, and by measuring the risk, one considers the distribution of future values of the portfolio $V$ under the assumption that the portfolio is static, i.e., there is no portfolio management. The loss $L = -V$.

Let $F_V$ be the distribution function of the value,

$$F_V(v) = P[V \leq v].$$

The distribution function for the loss $L$, denoted by $F_L$, is related to $F_V$.

Exercise 1.1

Show that $F_L(\ell) = 1 - F_V(-\ell)$.

1.1 Risk Metrics

A risk metric captures the riskiness of the portfolio $\Pi$, which is a function of the loss distribution $F_L$. Corresponding to the loss distribution, assume that there is also a loss density $f_L$.

Volatility. The portfolio volatility is the standard deviation of the loss distribution,

$$\sigma = \sqrt{Var[L]},$$

$$= \sqrt{\int_{-\infty}^{\infty} d\ell \ f_L(\ell) \ell^2 - \left(\int_{-\infty}^{\infty} d\ell \ f_L(\ell)\ell\right)^2}.$$
1.1 Risk Metrics

For portfolios whose loss distribution is Gaussian or near-Gaussian, the volatility works well as a measure of the typical fluctuation as well as the maximum fluctuation. For typical portfolios, which exhibit a fat tailed loss distribution, the volatility is not a good measure of the maximum fluctuation (or risk) and it is better to resort to quantile-type measures.

**Loss Probability.** Given a loss threshold, $L^*$ one can compute the probability of a loss at least as large as $L^*$,

$$P[L \geq L^*] = 1 - F_L(L^*),$$

$$= F_V(-L^*).$$

**Expected Shortfall.** Given a loss threshold, $L^*$, the downside deviation is the expected loss conditioned on the loss being at least $L^*$,

$$\bar{\sigma}(L^*) = E[L|L \geq L^*],$$

$$= \frac{1}{1 - F_L(L^*)} \int_{L^*}^{\infty} d\ell f_L(\ell).$$

**The $\alpha$-Quantile Loss – The Value at Risk, $VaR(\alpha)$**. Given a probability $\alpha$, the quantile, one can compute the $\alpha$-quantile, which is the value $L^*$ for which the probability of a loss exceeding $L^*$ is $\alpha$. This is typically called the Value at Risk ($VaR$) at confidence $\alpha$. The Value at Risk $VaR(\alpha)$ is thus not defined explicitly but rather by a property that it should have, namely,

$$P[L \geq VaR(\alpha)] = \alpha.$$

In terms of the loss distribution function,

$$1 - F_L(VaR(\alpha)) = \alpha.$$

Inverting this equation to obtain $VaR(\alpha)$, we obtain

$$VaR(\alpha) = F_L^{-1}(1 - \alpha),$$

$$= -F_V^{-1}(\alpha).$$

Typical settings for $\alpha$ are 5%(0.05), 1%(0.01), 10bp (0.001), and 1bp (0.0001).

**The $\alpha$-Quantile Expected Shortfall.** While the $VaR(\alpha)$ captures the tail loss as parameterized by $\alpha$, it conveys no information about how large that loss can be. It simply says that with probability $\alpha$, the loss is at least $VaR(\alpha)$. One can obtain more information by computing the tail expectation by setting the threshold $L^* = VaR(\alpha)$ and then computing the expected shortfall with respect to this threshold.

$$\bar{\sigma}(\alpha) = E[L|L \geq VaR(\alpha)],$$

$$= \frac{1}{\alpha} \int_{VaR(\alpha)}^{\infty} d\ell f_L(\ell).$$

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In our opinion, $\bar{\sigma}(\alpha)$ is probably the most informative risk metric for an appropriately set $\alpha$. In words, it is the expected loss with probability $\alpha$. However, the most popular risk measure is the Value at Risk $VaR(\alpha)$, which is the risk measure we will focus on in our discussion.

1.2 Value at Risk for the Normal and Log-Normal

2 Disaggregation of Risk
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10,000 Monte Carlo Iterations, n=100

10,000 Monte Carlo Iterations, n=500

100,000 Monte Carlo Iterations, n=100

100,000 Monte Carlo Iterations, n=500

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