Foundations of Computer Science

Lecture 3

Making Precise Statements

Propositions
Compound Propositions and Truth Tables
Satisfiability
Predicates and Quantifiers
Last Time

1. Sets

2. Sequences


Examples of basic proofs.

- $x^2$ is even “is the same as” $x$ is even.
- In *any* group of 6 people there is an orgy of 3 mutual friends or a war of 3 mutual enemies.
- Well ordering principle (axiom)
- $\sqrt{2}$ is not rational.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. 
Today: Making Precise Statements

1. Making a precise statement: the proposition

2. Complicated precise statements: the compound proposition
   - Truth tables

3. Claims about many things
   - Predicates
   - Quantifiers
   - Proofs with quantifiers
2+2=4.
2 + 2 = 4. \quad T
Statements aka Propositions and Ambiguity

1. $2+2=4$. T
2. $2+2=5$. F
Statements aka Propositions and Ambiguity

1. $2+2=4.$  \hspace{1cm} T

2. $2+2=5.$  \hspace{1cm} F
Statements aka Propositions and Ambiguity

1. $2+2=4.$ \quad T

2. $2+2=5.$ \quad F

3. You may have cake OR ice-cream.
Statements aka Propositions and Ambiguity

1. $2 + 2 = 4$.  
   \[ T \]

2. $2 + 2 = 5$.  
   \[ F \]

3. You may have cake OR ice-cream.  
   (Can you have both?)
Statements aka Propositions and Ambiguity

1. $2+2=4$. \hspace{1cm} T

2. $2+2=5$. \hspace{1cm} F

3. You may have cake OR ice-cream. \hspace{1cm} (Can you have both?)

4. IF pigs can fly THEN you get an A.
1. $2+2=4$.  
   \[ T \]

2. $2+2=5$.  
   \[ F \]

3. You may have cake OR ice-cream.  
   (Can you have both?)

4. IF pigs can fly THEN you get an A.  
   (Pigs can’t fly. So, can you get an A?)
Statements aka Propositions and Ambiguity

1. $2 + 2 = 4.$ \quad T

2. $2 + 2 = 5.$ \quad F

3. You may have cake or ice-cream. (Can you have both?)

4. \textbf{IF} pigs can fly \textbf{THEN} you get an $A$. (Pigs can’t fly. So, can you get an $A$?)

5. \textbf{EVERY} American has an $A$ dream.
2+2=4.  
2+2=5.  
You may have cake OR ice-cream.  
IF pigs can fly THEN you get an A.  
EVERY American has A dream.  
5(a) There is a single dream that EVERY American shares.
Statements aka Propositions and Ambiguity

1. $2+2=4.$ \hspace{1cm} T

2. $2+2=5.$ \hspace{1cm} F

3. You may have cake OR ice-cream. \hspace{1cm} (Can you have both?)

4. IF pigs can fly THEN you get an A. \hspace{1cm} (Pigs can’t fly. So, can you get an A?)

5. EVERY American has A dream.
   5(a) There is a single dream that EVERY American shares.
   5(b) EVERY American has their own personal dream.
Statements aka Propositions and Ambiguity

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   \[ T \]

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   5(a) There is a single dream that EVERY American shares.

   5(b) EVERY American has their own personal dream.

Why is ambiguity bad? **Proof!**
Statements aka Propositions and Ambiguity

1. $2+2=4.$ \hspace{1cm} \text{T}

2. $2+2=5.$ \hspace{1cm} \text{F}

3. You may have cake \textbf{OR} ice-cream. \hspace{1cm} \text{(Can you have both?)}

4. \textbf{IF} pigs can fly \textbf{THEN} you get an A. \hspace{1cm} \text{(Pigs can’t fly. So, can you get an A?)}

5. \textbf{EVERY} American has \textbf{A} dream.
   5(a) There is a single dream that \textbf{EVERY} American shares.
   5(b) \textbf{EVERY} American has their own personal dream.

Why is ambiguity bad? \textbf{Proof!}

We asked questions of our friends to prove 5(b).

$A$ dreams of PB & J sandwiches;
$B$ dreams of garlic croutons;
$C$ dreams of PB & J sandwiches;
$D$ dreams of red Porsches;
$E$ dreams of PB & J sandwiches;
$F$ dreams of jelly beans.
Statements aka Propositions and Ambiguity

1. 2+2=4. \( T \)

2. 2+2=5. \( F \)

3. You may have cake OR ice-cream. (Can you have both?)

4. IF pigs can fly THEN you get an A. (Pigs can’t fly. So, can you get an A?)

5. EVERY American has A dream.
   5(a) There is a single dream that EVERY American shares.
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Why is ambiguity bad? Proof!

We asked questions of our friends to prove 5(b).

Pop Quiz How to prove 5(a)?

A dreams of PB & J sandwiches;
B dreams of garlic croutons;
C dreams of PB & J sandwiches;
D dreams of red Porsches;
E dreams of PB & J sandwiches;
F dreams of jelly beans.
Propositions are T or F

We use the letters $p, q, r, s, \ldots$ to represent propositions.

- $p$: Porky the pig can fly. F
- $q$: You got an $A$. T?
- $r$: Kilam is an American. T?
- $s$: $4^2$ is even. T

To get complex statements, combine basic propositions using logical connectors.
Compound Propositions

$p$: Porky the pig can fly. \hspace{1cm} F
$q$: You got an A. \hspace{1cm} T?
$r$: Kilam is an American. \hspace{1cm} T?
$s$: $4^2$ is even. \hspace{1cm} T

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## Compound Propositions

\[ p: \text{Porky the pig can fly.} \quad F \]
\[ q: \text{You got an A.} \quad T? \]
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\[ s: \text{4² is even.} \quad T \]

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$r$: Kilam is an American.  \( T \)?

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The negation $\neg p$ is $T$ when $p$ is $F$, and the negation $\neg p$ is $F$ when $p$ is $T$.

“Porky the pig can fly” is $F$
Negation (NOT), $\neg p$

The negation $\neg p$ is T when $p$ is F, and the negation $\neg p$ is F when $p$ is T.

“Porky the pig can fly” is F

So,

IT IS NOT THE CASE THAT (Porky the pig can fly) is T
Conjunction (AND), \( p \land q \)

Both \( p \) and \( q \) must be \( T \) for \( p \land q \) to be \( T \); otherwise \( p \land q \) is \( F \).
Conjunction (AND), $p \land q$

Both $p$ and $q$ must be T for $p \land q$ to be T; otherwise $p \land q$ is F.

“Porky the pig can fly” is F
Conjunction (AND), $p \land q$

Both $p$ and $q$ must be T for $p \land q$ to be T; otherwise $p \land q$ is F.

"Porky the pig can fly" is F

We don’t know whether “You got an A”.
Conjunction (AND), $p \land q$

Both $p$ and $q$ must be T for $p \land q$ to be T; otherwise $p \land q$ is F.

“Porky the pig can fly” is F

We don’t know whether “You got an A”.

It does not matter.
Conjunction (AND), $p \land q$

Both $p$ and $q$ must be $T$ for $p \land q$ to be $T$; otherwise $p \land q$ is $F$.

"Porky the pig can fly" is $F$

We don’t know whether "You got an A".

It does not matter.

$(\text{Porky the pig can fly}) \land (\text{You got an A})$ is $F$
Disjunction (OR), $p \lor q$

Both $p$ and $q$ must be F for $p \lor q$ to be F; otherwise $p \lor q$ is T.
Disjunction (OR), $p \lor q$

Both $p$ and $q$ must be $F$ for $p \lor q$ to be $F$; otherwise $p \lor q$ is $T$.

“Porky the pig can fly” is $F$. 
Disjunction (OR), $p \lor q$

Both $p$ and $q$ must be F for $p \lor q$ to be F; otherwise $p \lor q$ is T.

“Porky the pig can fly” is F

We don’t know whether “You got an A”.

Disjunction (OR), $p \lor q$

Both $p$ and $q$ must be F for $p \lor q$ to be F; otherwise $p \lor q$ is T.

“Porky the pig can fly” is F

We don’t know whether “You got an A”.

Now it matters.
Disjunction (OR), $p \lor q$

Both $p$ and $q$ must be F for $p \lor q$ to be F; otherwise $p \lor q$ is T.

“Porky the pig can fly” is F

We don’t know whether “You got an A”.

Now it matters.

$(\text{Porky the pig can fly}) \lor (\text{You got an A})$ is T or F

( Depends on whether you got an A.)
Disjunction (OR), $p \lor q$

Both $p$ and $q$ must be F for $p \lor q$ to be F; otherwise $p \lor q$ is T.

"Porky the pig can fly" is F

We don’t know whether “You got an A”.

Now it matters.

$$(\text{Porky the pig can fly}) \lor (\text{You got an A})$$ is T or F

( Depends on whether you got an A.)

**Exercise:** “You can have cake” OR “You can have ice-cream.”
Truth Tables

Implication (if... then...), $p \rightarrow q$
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## Truth Tables

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The truth table defines the “meaning” of these logical connectors.
Implication (IF... THEN...), $p \rightarrow q$

**IF** “Porky the pig can fly” **THEN** “You got an A.” (T/F?)
Implication (IF... THEN...), \( p \rightarrow q \)

**IF** “Porky the pig can fly”  **THEN** “You got an A.”

Suppose \( T \). Since pigs can’t fly, does it mean you can’t get an A?

(T/F?)
Implication (IF... THEN...), \( p \rightarrow q \)

**IF** “Porky the pig can fly” **THEN** “You got an A.”

Suppose \( T \). Since pigs can’t fly, does it mean you can’t get an A?

\( (T/F?) \)

**IF** “\( n^2 \) is even”, **THEN** “\( n \) is even.”

\( (T) \)
Implication (\textbf{IF} \ldots \textbf{THEN} \ldots), \, p \rightarrow q

\textbf{IF} \, \text{“Porky the pig can fly”} \, \textbf{THEN} \, \text{“You got an A.”} \quad (T/F?)

Suppose \( T \). Since pigs can’t fly, does it mean you can’t get an A?

\textbf{IF} \, \text{“}n^2\text{ is even”}, \, \textbf{THEN} \, \text{“}n\text{ is even.”} \quad (T)

Suppose \( n^2 \) is even. Can we conclude \( n \neq 5 \)?
Implication (IF... THEN... ), $p \rightarrow q$

IF “Porky the pig can fly” THEN “You got an A.”
Suppose $T$. Since pigs can’t fly, does it mean you can’t get an A?

IF “$n^2$ is even”, THEN “$n$ is even.”
Suppose $n^2$ is even. Can we conclude $n \neq 5$?

IF “it rained last night” THEN “the grass is wet.”

(T/F?)

(T)

(T)
Implication (IF... THEN...), $p \rightarrow q$

IF “Porky the pig can fly” THEN “You got an A.”
Suppose $T$. Since pigs can’t fly, does it mean you can’t get an A?

IF “$n^2$ is even”, THEN “$n$ is even.”
Suppose $n^2$ is even. Can we conclude $n \neq 5$?

IF “it rained last night” THEN “the grass is wet.”

$p$ : it rained last night
$q$ : the grass is wet

$p \rightarrow q$
Implication (IF . . . THEN . . .), $p \rightarrow q$

**IF** “Porky the pig can fly” **THEN** “You got an A.”  
Suppose T. Since pigs can’t fly, does it mean you can’t get an A? (T/F?)

**IF** “$n^2$ is even”, **THEN** “$n$ is even.”  
Suppose $n^2$ is even. Can we conclude $n \neq 5$? (T)

**IF** “it rained last night” **THEN** “the grass is wet.” (T)

$p :$ it rained last night  
$q :$ the grass is wet

$p \rightarrow q$

What does it mean for this common-sense implication to be true?
Implication (IF... THEN...), \( p \rightarrow q \)

IF “Porky the pig can fly” THEN “You got an A.” (T/F?)
Suppose \( T \). Since pigs can’t fly, does it mean you can’t get an A?

IF “\( n^2 \) is even”, THEN “\( n \) is even.” (T)
Suppose \( n^2 \) is even. Can we conclude \( n \neq 5 \)?

IF “it rained last night” THEN “the grass is wet.” (T)

\[
\begin{align*}
  p & : \text{it rained last night} \\
  q & : \text{the grass is wet}
\end{align*}
\]

\( p \rightarrow q \)

What does it mean for this common-sense implication to be true?
What can you conclude? Did it rain last night? Is the grass wet?
Adding New Information to a True Implication: \( p \) is T

IF “it rained last night” THEN “the grass is wet.”

\[ p : \text{it rained last night} \]
\[ q : \text{the grass is wet} \]

\[ p \rightarrow q \]

Weather report in morning paper: rain last night. ← new information
Adding New Information to a True Implication: \( p \) is \( T \)

**IF** “it rained last night” \( \text{THEN} \) “the grass is wet.”

\[ p : \text{it rained last night} \]
\[ q : \text{the grass is wet} \]

\[ p \rightarrow q \]

**Weather report in morning paper: rain last night.**

\[-\text{new information}\]

\[ (\text{it rained last night}) \text{ THEN } (\text{the grass is wet}) \]
\[ \text{It rained last night (from the weather report)} \]
\[ \text{Is the grass wet?} \quad \text{YES!} \]
Adding New Information to a True Implication: $p$ is $T$

IF “it rained last night” THEN “the grass is wet.”

\[ p : \text{it rained last night} \]
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Weather report in morning paper: rain last night.

\[
\begin{array}{c|c|c}
\text{IF (it rained last night) THEN (the grass is wet)} & \text{T} & p \rightarrow q & \text{T} \\
\text{It rained last night (from the weather report)} & \text{T} & p & \text{T} \\
\text{Is the grass wet?} & \text{YES!} & q & \text{T} \\
\end{array}
\]

For a true implication $p \rightarrow q$, when $p$ is $T$, you can conclude $q$ is $T$. 
Adding New Information to a True Implication: \( q \) is True

IF “it rained last night” THEN “the grass is wet.”

\[
p : \text{ it rained last night} \\
q : \text{ the grass is wet}
\]

\[p \rightarrow q\]

While picking up the morning paper, you see the grass is wet. \( \leftarrow \text{ new information} \)
Adding New Information to a True Implication: \(q\) is T

IF “it rained last night” THEN “the grass is wet.”

\[ p : \text{it rained last night} \]
\[ q : \text{the grass is wet} \]
\[ p \rightarrow q \]

While picking up the morning paper, you see the grass is wet. ← new information

\[
\begin{array}{c}
\text{IF (it rained last night) THEN (the grass is wet)} \quad \text{T} \\
\text{The grass is wet (from walking outside)} \quad \text{T} \\
\text{Did it rain last night?} \\
\end{array}
\]
Adding New Information to a True Implication: $q$ is T

**IF “it rained last night” THEN “the grass is wet.”**

\[ p : \text{it rained last night} \]
\[ q : \text{the grass is wet} \]

\[ p \rightarrow q \]

**While picking up the morning paper, you see the grass is wet. ← new information**

\[
\begin{array}{ccc}
\text{IF (it rained last night) THEN (the grass is wet)} & T & p \rightarrow q \ T \\
\text{The grass is wet (from walking outside)} & T & q \ T \\
\text{Did it rain last night?} & \smile & .\ :. \ p \ T \ or \ F
\end{array}
\]

**For a true implication $p \rightarrow q$, when $q$ is T, you cannot conclude $p$ is T.**
Adding New Information to a True Implication: $p$ is F

IF “it rained last night” THEN “the grass is wet.”

\[ p : \text{it rained last night} \]
\[ q : \text{the grass is wet} \]
\[ p \rightarrow q \]

Weather report in morning paper: no rain last night. ← new information
Adding New Information to a True Implication: \( p \) is \( F \)

IF “it rained last night” THEN “the grass is wet.”

\[ p : \text{it rained last night} \]
\[ q : \text{the grass is wet} \]

\[ p \rightarrow q \]

Weather report in morning paper: no rain last night. ← new information

\[
\begin{array}{ccc}
\text{IF (it rained last night) THEN (the grass is wet)} & T & p \rightarrow q & T \\
\text{It rained last night (from the weather report)} & F & p & F \\
\text{Is the grass wet?} & ? & \therefore q & T \text{ or } F \\
\end{array}
\]

For a true implication \( p \rightarrow q \), when \( p \) is \( F \), you cannot conclude \( q \) is \( F \).
You are a scientist collecting data to verify that the implication is valid (true).
You are a scientist collecting data to *verify* that the implication is valid (true).

One night it rained. In the morning the grass was dry. ← new information
Falsifying an Implication

- You are a scientist collecting data to *verify* that the implication is valid (true).
- **One night it rained. In the morning the grass was dry.** ← new information
- What do you think about the implication now?
Falsifying an Implication

- You are a scientist collecting data to verify that the implication is valid (true).
- One night it rained. In the morning the grass was dry. ← new information
- What do you think about the implication now?

This is a falsifying scenario.

\[ p \rightarrow q \text{ is } F \text{ only when } p \text{ is } T \text{ and } q \text{ is } F. \text{ In all other cases } p \rightarrow q \text{ is } T. \]
Falsifying an Implication

- You are a scientist collecting data to *verify* that the implication is valid (true).
- **One night it rained. In the morning the grass was dry.** ← new information
- What do you think about the implication now?

This is a *falsifying scenario*.

\[
\text{IF (it rains) THEN (the grass is wet)} \quad \leftarrow \text{not T}
\]

\[
p \rightarrow q \text{ is F only when } p \text{ is T and } q \text{ is F. In all other cases } p \rightarrow q \text{ is T.}
\]

\[
\begin{array}{c}
\text{IF (Porky the pig can fly) THEN (You got an A)}\\
\hline
F\\
\hline
\end{array}
\quad \text{can be T or F (phew)}
\]
Implication is *Extremely Important*, $p \rightarrow q$

All these are $p \rightarrow q$ ($p =$ “it rained last night” and $q =$ “the grass is wet”):

If it rained last night then the grass is wet.  

IF $p$ THEN $q$
Implication is *Extremely* Important, $p \rightarrow q$

All these are $p \rightarrow q$ ($p =$ “it rained last night” and $q =$ “the grass is wet”):

If it rained last night then the grass is wet.        IF $p$ THEN $q$
It rained last night implies the grass is wet.      $p$ IMPLIES $q$
Implication is *Extremely Important, $p \rightarrow q$*

All these are $p \rightarrow q$ ($p =$ “it rained last night” and $q =$ “the grass is wet”):

- If it rained last night then the grass is wet.  
- It rained last night implies the grass is wet.  
- It rained last night only if the grass is wet.

$\text{IF } p \text{ THEN } q$  
$p \text{ IMPLIES } q$  
$p \text{ ONLY IF } q$
Implication is \textit{Extremely Important}, $p \rightarrow q$

All these are $p \rightarrow q$ ($p =$ “it rained last night” and $q =$ “the grass is wet”):

If it rained last night then the grass is wet. \hspace{1cm} \text{IF } p \text{ THEN } q

It rained last night implies the grass is wet. \hspace{1cm} p \text{ IMPLIES } q

It rained last night only if the grass is wet. \hspace{1cm} p \text{ ONLY IF } q

The grass is wet if it rained last night. \hspace{1cm} q \text{ IF } p
Implication is *Extremely Important*, $p \rightarrow q$

All these are $p \rightarrow q$ ($p =$ “it rained last night” and $q =$ “the grass is wet”):

If it rained last night then the grass is wet. \hspace{3cm} \text{IF} \ p \ \text{THEN} \ q

It rained last night implies the grass is wet. \hspace{3cm} \text{p IMPLIES} \ q

It rained last night only if the grass is wet. \hspace{3cm} \text{p ONLY IF} \ q

The grass is wet if it rained last night. \hspace{3cm} q \text{ IF} \ p

The grass is wet whenever it rains. \hspace{3cm} q \text{ WHENEVER} \ p

**Truth Tables:**

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$p \land q$</th>
<th>$p \lor q$</th>
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Example: \( \text{IF} \ (\text{you are hungry OR you are thirsty}) \ \text{THEN} \ \text{you visit the cafeteria} \)

\((p \lor q) \rightarrow r\) where
\( p: \text{you are hungry} \)
\( q: \text{you are thirsty} \)
\( r: \text{you visit the cafeteria} \)

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Example: If (you are hungry OR you are thirsty) THEN you visit the cafeteria

\[(p \lor q) \rightarrow r\]

where

\[p : \text{you are hungry}\]
\[q : \text{you are thirsty}\]
\[r : \text{you visit the cafeteria}\]

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Example: IF (you are hungry OR you are thirsty) THEN you visit the cafeteria

\[(p \lor q) \rightarrow r\]

where

\(p\) : you are hungry
\(q\) : you are thirsty
\(r\) : you visit the cafeteria

\[
\begin{array}{cccccc}
 p & q & r & (p \lor q) & (p \lor q) \rightarrow r \\
1. & F & F & F & F & T \\
2. & F & F & T & F & T \\
3. & F & T & F & T & F \\
4. & F & T & T & T & T \\
5. & T & F & F & T & F \\
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8. & T & T & T & T & T \\
\end{array}
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\[(p \lor q) \rightarrow r\]

where

- \(p\) : you are hungry
- \(q\) : you are thirsty
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\[(p \lor q) \rightarrow r\]
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\[(p \lor q) \rightarrow r\]

where

\[p : \text{you are hungry}\]
\[q : \text{you are thirsty}\]
\[r : \text{you visit the cafeteria}\]

- You are thirsty: \(q\) is \(T\).

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\[(p \lor q) \rightarrow r\]

where

\[p : \text{you are hungry}\]
\[q : \text{you are thirsty}\]
\[r : \text{you visit the cafeteria}\]

- **You are thirsty**: \(q\) is T. In both cases \(r\) is T.

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Example: IF (you are hungry OR you are thirsty) THEN you visit the cafeteria

\[(p \vee q) \rightarrow r\]

where

\[p : you\ are\ hungry\]
\[q : you\ are\ thirsty\]
\[r : you\ visit\ the\ cafeteria\]

- You are thirsty: \(q\) is \(T\). In both cases \(r\) is \(T\).
- You did visit the cafeteria: \(r\) is \(T\).

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Example: If (you are hungry OR you are thirsty) THEN you visit the cafeteria

\[(p \lor q) \rightarrow r\]

where

- \(p\): you are hungry
- \(q\): you are thirsty
- \(r\): you visit the cafeteria

You are thirsty: \(q\) is \(T\). In both cases \(r\) is \(T\).
(you visit the cafeteria)

You did visit the cafeteria: \(r\) is \(T\).
Are you hungry? We don’t know.
Are you thirsty? We don’t know.
(You accompanied your hungry friend (row 2).)

\[\begin{array}{ccc|c}
 p & q & r & (p \lor q) \rightarrow r \\
1. & F & F & F & T \\
2. & F & F & T & T \\
3. & F & T & F & F \\
4. & F & T & T & T \\
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6. & T & F & T & T \\
7. & T & T & F & F \\
8. & T & T & T & T \\
\end{array}\]
Example: \textbf{IF} (you are hungry \textbf{OR} you are thirsty) \textbf{THEN} you visit the cafeteria

\[(p \lor q) \rightarrow r\] where

\[p : \text{you are hungry}\]
\[q : \text{you are thirsty}\]
\[r : \text{you visit the cafeteria}\]

- \textbf{You are thirsty: }\(q\) is \text{T}. In both cases \(r\) is \text{T}. (you visit the cafeteria)
- \textbf{You did visit the cafeteria: }\(r\) is \text{T}. Are you hungry? We don’t know. Are you thirsty? We don’t know. (You accompanied your hungry friend (row 2).)
- \textbf{You did not visit the cafeteria: }\(r\) is \text{F}.

\[
\begin{array}{ccc|c}
 p & q & r & (p \lor q) \rightarrow r \\
 1. & F & F & F & T \\
 2. & F & F & T & T \\
 3. & F & T & F & F \\
 4. & F & T & T & T \\
 5. & T & F & F & F \\
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 7. & T & T & F & F \\
 8. & T & T & T & T \\
\end{array}
\]
Example: IF (you are hungry OR you are thirsty) THEN you visit the cafeteria

\[(p \lor q) \rightarrow r\]

where
\[p : \text{you are hungry}\]
\[q : \text{you are thirsty}\]
\[r : \text{you visit the cafeteria}\]

- **You are thirsty:** \(q\) is T. In both cases \(r\) is T. (you visit the cafeteria)

- **You did visit the cafeteria:** \(r\) is T.
  Are you hungry? We don’t know.
  Are you thirsty? We don’t know.
  (You accompanied your hungry friend (row 2).)

- **You did not visit the cafeteria:** \(r\) is F.
  \(p\) and \(q\) are both F.
  (You are neither hungry nor thirsty.)

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### Equivalent Compound Statements

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- rains $\rightarrow$ wet grass
- dry grass $\rightarrow$ no rain
- no rain $\lor$ wet grass
- wet grass $\rightarrow$ rain

$$
p \rightarrow q \equiv \neg q \rightarrow \neg p \equiv \neg p \lor q
$$
Equivalent Compound Statements

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| rains $\rightarrow$ wet grass | dry grass $\rightarrow$ no rain | no rain $\lor$ wet grass | wet grass $\rightarrow$ rain |

$p \rightarrow q \equiv \neg q \rightarrow \neg p \equiv \neg p \lor q$

Order is very important: $p \rightarrow q$ and $q \rightarrow p$ do not mean the same thing.

IF it’s raining, THEN I have an umbrella vs. IF I have an umbrella, THEN it’s raining
**Equivalent Compound Statements**

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
<th>$\neg q \rightarrow \neg p$</th>
<th>$\neg p \lor q$</th>
<th>$q \rightarrow p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
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</table>

- rains → wet grass
- dry grass → no rain
- no rain ∨ wet grass
- wet grass → rain

$p \rightarrow q \equiv \neg q \rightarrow \neg p \equiv \neg p \lor q$

Order is very important: $p \rightarrow q$ and $q \rightarrow p$ **do not** mean the same thing.

**IF** it’s raining, **THEN** I have an umbrella **vs.** **IF** I have an umbrella, **THEN** it’s raining

**Pop Quiz 3.5.** Compound propositions are used for program control flow, especially **IF**... **THEN**...

if($x > 0 \ || \ (y > 1 \ & \ & \ x < y)$)

Execute some instructions.

if($x > 0 \ || \ y > 1$)

Execute some instructions.

Use truth-tables to show that both do the same thing. Which do you prefer and why?
Proving an Implication: Reasoning Without Facts

IF \( n^2 \) is even THEN \( n \) is even.

\[
p : n^2 \text{ is even} \\
q : n \text{ is even} \\
p \rightarrow q
\]

<p>| | | |</p>
<table>
<thead>
<tr>
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What is \( n \)? How to prove?
Proving an Implication: Reasoning Without Facts

IF \((n^2 \text{ is even})\) THEN \((n \text{ is even})\).

<table>
<thead>
<tr>
<th>(p)</th>
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<th>(p \rightarrow q)</th>
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<tbody>
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</table>

What is \(n\)? How to prove?

We must show that the highlighted row cannot occur.
Proving an Implication: Reasoning Without Facts

If \( n^2 \) is even then \( n \) is even.

\[
\begin{array}{c|c|c}
 p & q & p \rightarrow q \\
 F & F & T \\
 F & T & T \\
 T & F & F \\
 T & T & T \\
\end{array}
\]

What is \( n \)? How to prove?

We must show that the highlighted row cannot occur.

In this row, \( q \) is \( F \): \( n = 2k + 1 \).
Proving an Implication: Reasoning Without Facts

IF \((n^2 \text{ is even})\) THEN \((n \text{ is even})\).

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What is \(n\)? How to prove?

We must show that the highlighted row \textit{cannot} occur.

In this row, \(q\) is \(F\): \(n = 2k + 1\).

\[n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1\]
Proving an Implication: Reasoning Without Facts

IF \( n^2 \) is even THEN \( n \) is even.

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What is \( n \)? How to prove?

We must show that the highlighted row cannot occur.

In this row, \( q \) is F: \( n = 2k + 1 \).

\[
\begin{align*}
n^2 &= (2k + 1)^2 \\
&= 2(2k^2 + 2k) + 1
\end{align*}
\]

\( p \) cannot be T.
Proving an Implication: Reasoning Without Facts

IF \((n^2 \text{ is even})\) THEN \((n \text{ is even})\).

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What is \(n\)? How to prove?

We must show that the highlighted row cannot occur.

In this row, \(q\) is F: \(n = 2k + 1\).

\[
n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1
\]

\(p\) cannot be T. This row cannot happen: \(p \rightarrow q\) is always T.

\(\blacksquare\)
Quantifiers

EVERY American has A dream.

Kilam has some gray hair.
Quantifiers

EVERY American has A dream.

Kilam has some gray hair.
Everyone has some gray hair.
Quantifiers

\[ \text{EVERY American has A dream.} \]

Kilam has some gray hair.
Everyone has some gray hair.
Any map can be colored with 4 colors with adjacent countries having different colors.
**EVERY** American has *a* dream.

Kilam has *some* gray hair.

*Everyone* has *some* gray hair.

*Any* map can be colored with 4 colors with adjacent countries having different colors.

*Every* even integer $n > 2$ is the sum of 2 primes (*Goldbach, 1742*).
Kilam has some gray hair.
Everyone has some gray hair.
Any map can be colored with 4 colors with adjacent countries having different colors.
Every even integer $n > 2$ is the sum of 2 primes (Goldbach, 1742).
Someone broke this faucet.
Kilam has some gray hair.
Everyone has some gray hair.
Any map can be colored with 4 colors with adjacent countries having different colors.
Every even integer $n > 2$ is the sum of 2 primes (Goldbach, 1742).
Someone broke this faucet.
There exists a creature with blue eyes and blonde hair.
Quantifiers

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There exists a creature with blue eyes and blonde hair.
All cars have four wheels.
Quantifiers

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There exists a creature with blue eyes and blonde hair.
All cars have four wheels.

These statements are more complex because of quantifiers:

\( \text{EVERY}; \ A; \ \text{SOME}; \ \text{ANY}; \ \text{ALL}; \ \text{THERE EXISTS}. \)
Quantifiers

**EVERY** American has **A** dream.

Kilam has **some** gray hair.

Everyone has **some** gray hair.

Any map can be colored with 4 colors with adjacent countries having different colors.

Every even integer \( n > 2 \) is the sum of 2 primes (*Goldbach, 1742*).

Someone broke this faucet.

There exists a creature with blue eyes and blonde hair.

All cars have four wheels.

These statements are more complex because of *quantifiers*:

EVERY; A; SOME; ANY; ALL; THERE EXISTS.

Compare:

My Ford Escort has four wheels;

ALL cars have four wheels.
ALL cars have four wheels
Predicates Are Like Functions

ALL cars have four wheels

Define \textit{predicate} \( P(c) \) and its \textit{domain}
Predicates Are Like Functions

**ALL** cars have four wheels

Define *predicate* $P(c)$ and its *domain*

$$C = \{c \mid c \text{ is a car}\} \leftarrow \text{set of cars}$$
ALL cars have four wheels

Define predicate $P(c)$ and its domain

\[
C = \{c | c \text{ is a car}\} \quad \leftarrow \text{set of cars}
\]
\[
P(c) = \text{"car } c \text{ has four wheels"}
\]
ALL cars have four wheels

Define *predicate* $P(c)$ and its *domain*

$$C = \{c|c \text{ is a car}\} \quad \leftarrow \text{set of cars}$$

$$P(c) = \text{“car } c \text{ has four wheels”}$$

“for all $c$ in $C$, the statement $P(c)$ is true.”

$$\forall c \in C : P(c).$$

($\forall$ means “for all”)

(\text{Creator: Malik Magdon-Ismail}
ALL cars have four wheels

Define *predicate* $P(c)$ and its *domain*

\[ C = \{ c | c \text{ is a car} \} \quad \leftarrow \text{set of cars} \]

\[ P(c) = \text{“car } c \text{ has four wheels”} \]

“for all $c$ in $C$, the statement $P(c)$ is true.”

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ALL cars have four wheels

Define *predicate* \( P(c) \) and its *domain*

\[
C = \{ c | c \text{ is a car} \} \quad \leftarrow \text{set of cars}
\]

\[
P(c) = \text{“car } c \text{ has four wheels”}
\]

“for all \( c \) in \( C \), the statement \( P(c) \) is true.”

\[
\forall c \in C : P(c).
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(\( \forall \) means “for all”)

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<td>( f(x) = x^2 )</td>
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ALL cars have four wheels

Define *predicate* $P(c)$ and its *domain*

$C = \{c | c \text{ is a car}\}$ ← set of cars

$P(c) = \text{“car } c \text{ has four wheels”}$

“for all $c$ in $C$, the statement $P(c)$ is true.”

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<tr>
<td></td>
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<td>parameter $x \in \mathbb{R}$</td>
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Creator: Malik Magdon-Ismail
Predicates Are Like Functions

ALL cars have four wheels

Define *predicate* \( P(c) \) and its *domain* 

\[
C = \{ c | \text{c is a car} \} \quad \leftarrow \text{set of cars}
\]

\[
P(c) = \text{“car c has four wheels”}
\]

“for all \( c \) in \( C \), the statement \( P(c) \) is true.”

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<tr>
<td>Output</td>
<td>parameter ( c \in C )</td>
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</tr>
<tr>
<td></td>
<td>\textbf{statement} ( P(c) )</td>
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Predicates Are Like Functions

ALL cars have four wheels

Define predicate $P(c)$ and its domain

$$C = \{c|c \text{ is a car}\} \leftarrow \text{set of cars}$$
$$P(c) = \text{“car } c \text{ has four wheels”}$$

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<td><strong>statement</strong> $P(c)$</td>
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</tr>
<tr>
<td>Example</td>
<td>$P(\text{Jen’s VW}) = \text{“car ‘Jen’s VW’ has four wheels”}$</td>
<td>$f(5) = 25$</td>
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Predicates Are Like Functions

ALL cars have four wheels

Define *predicate* $P(c)$ and its *domain*

$$C = \{c | c \text{ is a car} \}$$

$$P(c) = \text{“car } c \text{ has four wheels”}$$

“for all $c$ in $C$, the statement $P(c)$ is true.”

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($\forall$ means “for all”)

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<td>$f(5) = 25$</td>
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<tr>
<td>$\forall c \in C : P(c)$</td>
<td>$\forall x \in \mathbb{R}, f(x) \geq 0$</td>
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Predicates Are Like Functions

**ALL cars have four wheels**

Define *predicate* $P(c)$ and its *domain*

$$C = \{c|c \text{ is a car}\} \quad \leftarrow \text{set of cars}$$

$$P(c) = \text{“car } c \text{ has four wheels”}$$

“for all $c$ in $C$, the statement $P(c)$ is true.”

$$\forall c \in C : P(c).$$

($\forall$ means “for all”)

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<td><strong>Meaning</strong></td>
<td>For all $c \in C$, the statement $P(c)$ is $\top$.</td>
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</table>
There \textbf{EXISTS} a Creature with Blue eyes and Blonde Hair

Define \textit{predicate} \(Q(a)\) and its \textit{domain}

\[ A = \{a | a \text{ is a creature}\} \quad \leftarrow \text{set of creatures} \]
There exists a Creature with Blue eyes and Blonde Hair

Define predicate $Q(a)$ and its domain

$$A = \{a | a \text{ is a creature}\} \quad \leftarrow \text{set of creatures}$$

$$Q(a) = \text{“}a \text{ has blue eyes and blonde hair”}$$
There EXISTS a Creature with Blue eyes and Blonde Hair

Define predicate $Q(a)$ and its domain

$$A = \{a | a \text{ is a creature}\} \quad \leftarrow \text{set of creatures}$$

$$Q(a) = \text{“a has blue eyes and blonde hair”}$$

“there exists $a$ in $A$ for which the statement $Q(a)$ is true.”

$$\exists a \in A : Q(a).$$

($\exists$ means “there exists”)

(creator: Malik Magdon-Ismail)
There EXISTS a Creature with Blue eyes and Blonde Hair

Define predicate $Q(a)$ and its domain

$$A = \{a | a \text{ is a creature}\} \quad \leftrightarrow \quad \text{set of creatures}$$

$$Q(a) = \text{“a has blue eyes and blonde hair”}$$

“there exists $a$ in $A$ for which the statement $Q(a)$ is true.”

$$\exists a \in A : Q(a).$$

($\exists$ means “there exists”)

$$G(a) = \text{“a has blue eyes”}$$
There EXISTS a Creature with Blue eyes and Blonde Hair

Define predicate $Q(a)$ and its domain

$$A = \{a | a \text{ is a creature}\} \quad \leftarrow \text{set of creatures}$$

$$Q(a) = \text{“a has blue eyes and blonde hair”}$$

“there exists $a$ in $A$ for which the statement $Q(a)$ is true.”

$$\exists a \in A : Q(a).$$

($\exists$ means “there exists”)

$$G(a) = \text{“a has blue eyes”}$$

$$H(a) = \text{“a has blonde hair”}$$
There EXISTS a Creature with Blue eyes and Blonde Hair

Define predicate $Q(a)$ and its domain

$$A = \{a | a \text{ is a creature}\} \quad \leftarrow \text{set of creatures}$$

$$Q(a) = \text{“} a \text{ has blue eyes and blonde hair}\text{”}$$

“there exists $a$ in $A$ for which the statement $Q(a)$ is true.”

$$\exists a \in A : Q(a).$$

($\exists$ means “there exists”)

$G(a) = \text{“} a \text{ has blue eyes}\text{”}$

$H(a) = \text{“} a \text{ has blonde hair}\text{”}$

$$\exists a \in A : (G(a) \land H(a))$$

(compound predicate)
There EXISTS a Creature with Blue eyes and Blonde Hair

Define predicate $Q(a)$ and its domain

\[
A = \{a | a \text{ is a creature}\} \quad \leftarrow \text{set of creatures}
\]

$Q(a) = “a \text{ has blue eyes and blonde hair”}$

“there exists $a$ in $A$ for which the statement $Q(a)$ is true.”

$$\exists a \in A : Q(a).$$

($\exists$ means “there exists”)

\[
G(a) = “a \text{ has blue eyes”}
\]

\[
H(a) = “a \text{ has blonde hair”}
\]

$$\exists a \in A : (G(a) \land H(a))$$

\underline{compound predicate}

(When the domain is understood, we don’t need to keep repeating it. We write $\exists a : Q(a)$, or $\exists a : (G(a) \land H(a)).$)
IT IS NOT THE CASE THAT (There is creature with blue eyes and blonde hair)
Negating Quantifiers

IT IS NOT THE CASE THAT(There is creature with blue eyes and blonde hair)

Same as: “All creatures don’t have blue eyes and blonde hair”

\[ \neg \left( \exists a \in A : Q(a) \right) \quad \equiv \quad \forall a \in A : \neg Q(a) \]
IT IS NOT THE CASE THAT (There is creature with blue eyes and blonde hair)

Same as: “All creatures don’t have blue eyes and blonde hair”

\[ \neg \left( \exists a \in A : Q(a) \right) \equiv \forall a \in A : \neg Q(a) \]

IT IS NOT THE CASE THAT (All cars have four wheels)
Negating Quantifiers

IT IS NOT THE CASE THAT (There is creature with blue eyes and blonde hair)

Same as: “All creatures don’t have blue eyes and blonde hair”

\[-\left( \exists a \in A : Q(a) \right) \equiv \forall a \in A : \neg Q(a)\]

IT IS NOT THE CASE THAT (All cars have four wheels)

Same as: “There is a car which does not have four wheels”

\[-\left( \forall c \in C : P(c) \right) \equiv \exists c \in C : \neg P(c)\]
Negating Quantifiers

IT IS NOT THE CASE THAT (There is creature with blue eyes and blonde hair)

Same as: “All creatures don’t have blue eyes and blonde hair”

\[ \neg \left( \exists a \in A : Q(a) \right) \equiv \forall a \in A : \neg Q(a) \]

IT IS NOT THE CASE THAT (All cars have four wheels)

Same as: “There is a car which does not have four wheels”

\[ \neg \left( \forall c \in C : P(c) \right) \equiv \exists c \in C : \neg P(c) \]

When you take the negation inside the quantifier and negate the predicate, you must switch quantifiers: \( \forall \rightarrow \exists, \exists \rightarrow \forall \)
Define domains and a predicate.

\[ A = \{ a \mid a \text{ is an American}\}; \]
\[ D = \{ d \mid d \text{ is a dream}\}. \]
Every American Has a Dream

Define domains and a predicate.

\[ A = \{ a \mid a \text{ is an American}\}; \]
\[ D = \{ d \mid d \text{ is a dream}\}. \]

\[ P(a, d) = \text{“American } a \text{ has dream } d.” \]
Every American Has a Dream

Define domains and a predicate.

\[
A = \{ a \mid \text{a is an American}\}; \\
D = \{ d \mid \text{d is a dream}\}.
\]

\[
P(a, d) = \text{“American a has dream d.”}
\]

- There is some special dream \( d \), and every American \( a \) has that dream.
Define domains and a predicate.

\[ A = \{ a \mid a \text{ is an American} \}; \]
\[ D = \{ d \mid d \text{ is a dream} \}. \]

\[ P(a, d) = \text{“American } a \text{ has dream } d.” \]

There is some special dream \( d \), and every American \( a \) has that dream.

\[ \exists d \in D : (\forall a \in A : P(a, d)). \]
Every American Has a Dream

Define domains and a predicate.

\[ A = \{ a \mid a \text{ is an American}\}; \]
\[ D = \{ d \mid d \text{ is a dream}\}. \]

\[ P(a, d) = \text{“American\ } a \text{ has dream}\ d.” \]

- There is some special dream \( d \), and every American \( a \) has that dream.

\[ \exists d \in D : (\forall a \in A : P(a, d)). \]

- For every American \( a \), they have their own private dream \( d \).
Every American Has a Dream

Define domains and a predicate.

\[ A = \{a \mid a \text{ is an American}\}; \]
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When quantifiers are mixed, the order in which they appear is important for the meaning. Order generally cannot be switched.
Proofs with Quantifiers

Claim 1. \( \forall n > 2 : \text{IF } n \text{ is even, THEN } n \text{ is a sum of two primes.} \) (Goldbach, 1742)

Claim 2. \( \exists (a, b, c) \in \mathbb{N}^3 : a^2 + b^2 = c^2. \) \( ((a, b, c) \in \mathbb{N}^3 \text{ means triples of natural numbers})\)

Claim 3. \( \neg \exists (a, b, c) \in \mathbb{N}^3 : a^3 + b^3 = c^3. \)

Claim 4. \( \forall (a, b, c) \in \mathbb{N}^3 : a^3 + b^3 \neq c^3. \)

Think about what it would take to prove these claims.