Foundations of Computer Science
Lecture 6

Strong Induction
Strengthening the Induction Hypothesis
Strong Induction
Many Flavors of Induction
Proving “for all”:

- $P(n) : 4^n - 1$ is divisible by 3. $\forall n : P(n)$?
- $P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$. $\forall n : P(n)$?
- $P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n + 1)(2n + 1)$. $\forall n : P(n)$?

Induction.

Induction and Well-Ordering.
Today: Twists on Induction

1. Solving Harder Problems with Induction
   - \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \)

2. Strengthening the Induction Hypothesis
   - \( n^2 < 2^n \)

3. Many Flavors of Induction
   - Leaping Induction
     - Postage; \( n^3 < 2^n \)
   - Strong Induction
     - Fundamental Theorem of Arithmetic
     - Games of Strategy
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$

Proof. $P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

1: [Base case] $P(1)$ claims that $1 \leq 2 \times \sqrt{1}$, which is clearly T.
A Hard Problem: \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n \)

**Proof.** \( P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \).

1. **[Base case]** \( P(1) \) claims that \( 1 \leq 2 \times \sqrt{1} \), which is clearly true.

2. **[Induction step]** Show \( P(n) \rightarrow P(n + 1) \) for all \( n \geq 1 \) (direct proof).

   Assume (induction hypothesis) \( P(n) \) is true: \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \).

   Show \( P(n + 1) \) is true: \( \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n + 1} \).
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$

Proof. $P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

1: [Base case] $P(1)$ claims that $1 \leq 2 \times \sqrt{1}$, which is clearly $T$.

2: [Induction step] Show $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$ (direct proof)
   Assume (induction hypothesis) $P(n)$ is $T$: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.
   Show $P(n + 1)$ is $T$: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$.

\[
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}
\]
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$

**Proof.** $P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

1: **[Base case]** $P(1)$ claims that $1 \leq 2 \times \sqrt{1}$, which is clearly T.

2: **[Induction step]** Show $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$ (direct proof)
   Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.
   Show $P(n + 1)$ is T: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n + 1}$.

\[
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n + 1}} \leq 2\sqrt{n + 1}
\]

**IH**
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$

Proof. $P(n): \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

1: [Base case] $P(1)$ claims that $1 \leq 2 \times \sqrt{1}$, which is clearly $T$.

2: [Induction step] Show $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$ (direct proof)

Assume (induction hypothesis) $P(n)$ is $T$: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

Show $P(n + 1)$ is $T$: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n + 1}$.

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n + 1}}$$

$$\leq 2\sqrt{n} + \frac{1}{\sqrt{n + 1}}$$

**Lemma.** $2\sqrt{n + 1}/\sqrt{n + 1} \leq 2\sqrt{n + 1}$

Proof. By contradiction.

$2\sqrt{n + 1}/\sqrt{n + 1} > 2\sqrt{n + 1}$
$
\rightarrow 2\sqrt{n(n + 1)} + 1 > 2(n + 1)$
$
\rightarrow 4n(n + 1) > (2n + 1)^2$
$
\rightarrow 0 > 1$ **FISHY!**
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$

Proof. $P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

1: **[Base case]** $P(1)$ claims that $1 \leq 2 \times \sqrt{1}$, which is clearly T.

2: **[Induction step]** Show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$ (direct proof)

   Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

   Show $P(n+1)$ is T: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$.

   \[
   \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}
   \]

   \[
   \leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \quad \text{(IH)}
   \]

   \[
   \leq 2\sqrt{n+1}
   \]

   (lemma)

   So, $P(n+1)$ is T.

3: By induction, $P(n)$ is T $\forall n \geq 1$. 

**Lemma.** $2\sqrt{n+1}/\sqrt{n+1} \leq 2\sqrt{n+1}$

**Proof.** By contradiction. 

\[
2\sqrt{n+1}/\sqrt{n+1} > 2\sqrt{n+1}
\]

\[
\rightarrow 2\sqrt{n(n+1)} + 1 > 2(n+1)
\]

\[
\rightarrow 4n(n+1) > (2n+1)^2
\]

\[
\rightarrow 0 > 1 \quad \text{FISHY!}
\]
Proving Stronger Claims

\[ n^2 \leq 2^n \quad \text{for } n \geq 4. \]
$n^2 \leq 2^n$ for $n \geq 4$.

**Induction Step.** Must use $n^2 \leq 2^n$ to show $(n + 1)^2 \leq 2^{n+1}$.

$$(n + 1)^2 = n^2 + 2n + 1$$
Proving Stronger Claims

\[ n^2 \leq 2^n \quad \text{for } n \geq 4. \]

**Induction Step.** Must use \( n^2 \leq 2^n \) to show \((n + 1)^2 \leq 2^{n+1} \).

\[
(n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1
\]
$n^2 \leq 2^n$ for $n \geq 4$.

**Induction Step.** Must use $n^2 \leq 2^n$ to show $(n + 1)^2 \leq 2^{n+1}$.

$$(n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 \leq 2^n + 2^n = 2^{n+1}$$

What to do with the $2n + 1$?

Would be fine if $2n + 1 \leq 2^n$. 

Creator: Malik Magdon-Ismail
Proving Stronger Claims

\[ n^2 \leq 2^n \quad \text{for } n \geq 4. \]

**Induction Step.** Must use \( n^2 \leq 2^n \) to show \( (n + 1)^2 \leq 2^{n+1} \).

\[
(n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 \leq 2^n + 2^n = 2^{n+1}
\]

What to do with the \( 2n + 1 \)?

Would be fine if \( 2n + 1 \leq 2^n \).

With induction, it can be easier to prove a stronger claim.
Strengthen the Claim: $Q(n)$ Implies $P(n)$

$Q(n)$: (i) $n^2 \leq 2^n$; AND, (ii) $2n + 1 \leq 2^n$.

$Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots$
Strengthen the Claim: \( Q(n) \) Implies \( P(n) \)

\[
Q(n) : (i) \ n^2 \leq 2^n; \ \text{AND,} \ (ii) \ 2n + 1 \leq 2^n.
\]

\[
\boxed{Q(4)} \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots
\]

Proof. \( Q(n) \) : (i) \( n^2 \leq 2^n \); AND, (ii) \( 2n + 1 \leq 2^n \).

1. **[Base case]** \( Q(4) \) claims (i) \( 4^2 \leq 2^4 \); AND, (ii) \( 2 \times 4 + 1 \leq 2^4 \). Both are clearly T.
Strengthen the Claim: $Q(n)$ Implies $P(n)$

$Q(n) : (i) \ n^2 \leq 2^n$; AND, $(ii) \ 2n + 1 \leq 2^n$. 

$Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \ldots$

**Proof.** $Q(n) : (i) \ n^2 \leq 2^n$; AND, $(ii) \ 2n + 1 \leq 2^n$.

1: **[Base case]** $Q(4)$ claims $(i) \ 4^2 \leq 2^4$; AND, $(ii) \ 2 \times 4 + 1 \leq 2^4$. Both are clearly T.

2: **[Induction step]** Show $Q(n) \rightarrow Q(n + 1)$ for $n \geq 4$ (direct proof).
   Assume (induction hypothesis) $Q(n)$ is T: $(i) \ n^2 \leq 2^n$; AND, $(ii) \ 2n + 1 \leq 2^n$.
   Show $Q(n + 1)$ is T: $(i) \ (n + 1)^2 \leq 2^{n+1}$; AND, $(ii) \ 2(n + 1) + 1 \leq 2^{n+1}$. 

Strengthen the Claim: $Q(n)$ Implies $P(n)$

$Q(n) : (i) \ n^2 \leq 2^n; \ \text{AND,} \ (ii) \ 2n + 1 \leq 2^n.$

$Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots$

Proof. $Q(n) : (i) \ n^2 \leq 2^n; \ \text{AND,} \ (ii) \ 2n + 1 \leq 2^n.$

1: [Base case] $Q(4)$ claims $(i) \ 4^2 \leq 2^4; \ \text{AND,} \ (ii) \ 2 \times 4 + 1 \leq 2^4.$ Both are clearly T.

2: [Induction step] Show $Q(n) \rightarrow Q(n+1)$ for $n \geq 4$ (direct proof).
   Assume (induction hypothesis) $Q(n)$ is T: $(i) \ n^2 \leq 2^n; \ \text{AND,} \ (ii) \ 2n + 1 \leq 2^n.$
   Show $Q(n+1)$ is T: $(i) \ (n+1)^2 \leq 2^{n+1}; \ \text{AND,} \ (ii) \ 2(n+1) + 1 \leq 2^{n+1}.$

$\ (i) \ \quad (n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1}\ \checkmark$

(because from the induction hypothesis $n^2 \leq 2^n$ \textbf{and} $2n + 1 \leq 2^n$)
Strengthen the Claim: $Q(n)$ Implies $P(n)$

$Q(n) : (i) \ n^2 \leq 2^n; \ \text{AND,} \ (ii) \ 2n + 1 \leq 2^n.$

$[Q(4)] \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots$

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1: [Base case] $Q(4)$ claims $(i) \ 4^2 \leq 2^4; \ \text{AND,} \ (ii) \ 2 \times 4 + 1 \leq 2^4.$ Both are clearly T.

2: [Induction step] Show $Q(n) \rightarrow Q(n + 1)$ for $n \geq 4$ (direct proof).

Assume (induction hypothesis) $Q(n)$ is T: $(i) \ n^2 \leq 2^n; \ \text{AND,} \ (ii) \ 2n + 1 \leq 2^n.$

Show $Q(n + 1)$ is T: $(i) \ (n + 1)^2 \leq 2^{n+1}; \ \text{AND,} \ (ii) \ 2(n + 1) + 1 \leq 2^{n+1}.$

$(i) \ (n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \ \checkmark$

(because from the induction hypothesis $n^2 \leq 2^n \ \text{and} \ 2n + 1 \leq 2^n$)

$(ii) \ 2(n + 1) + 1 = 2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \ \checkmark$

(because $2 \leq 2^n$ and from the induction hypothesis $2n + 1 \leq 2^n$)
Strengthen the Claim: $Q(n)$ Implies $P(n)$

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Proof. $Q(n): (i) \ n^2 \leq 2^n; \ \text{AND}, \ (ii) \ 2n+1 \leq 2^n.$

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Assume (induction hypothesis) $Q(n)$ is T: (i) $n^2 \leq 2^n; \ \text{AND}, \ (ii) \ 2n+1 \leq 2^n.$

Show $Q(n+1)$ is T: (i) $(n+1)^2 \leq 2^{n+1}; \ \text{AND}, \ (ii) \ 2(n+1)+1 \leq 2^{n+1}.$

$(i) \quad (n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \ \checkmark$

(because from the induction hypothesis $n^2 \leq 2^n \text{ and } 2n+1 \leq 2^n$)

$(ii) \quad 2(n+1) + 1 = 2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \ \checkmark$

(because $2 \leq 2^n$ and from the induction hypothesis $2n+1 \leq 2^n$)

So, $Q(n+1)$ is T.

3: By induction, $Q(n)$ is T $\forall n \geq 4.$
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\Box$ – tiles?
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only □ - tiles?

TINKER!
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only □ – tiles?

TINKER!
Can you tile a \(2^n \times 2^n\) patio missing a center square. You have only \(\square\) – tiles?

**TINKER!**
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\blacklozenge$ – tiles?

TINKER!
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\blacksquare$ – tiles?

TINKER!
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\square$ – tiles?

TINKER!
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\square$ - tiles?

**TINKER!**

\[ P(n) : \text{The } 2^n \times 2^n \text{ grid minus a center-square can be } L\text{-tiled.} \]
L-Tile Land: Induction Idea

Suppose $P(n)$ is T. What about $P(n + 1)$?
L-Tile Land: Induction Idea

Suppose $P(n)$ is true. What about $P(n + 1)$?

The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.
Suppose $P(n)$ is true. What about $P(n + 1)$?

The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.
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The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.

**Problem.** Corner squares are missing. $P(n)$ can be used only if center-square is missing.
Suppose $P(n)$ is true. What about $P(n + 1)$?

The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.

**Problem.** Corner squares are missing. $P(n)$ can be used only if center-square is missing.

**Solution.** Strengthen claim to also include patios missing corner-squares.

$Q(n)$:

(i) The $2^n \times 2^n$ grid missing a **center-square** can be $L$-tiled; AND
(ii) The $2^n \times 2^n$ grid missing a **corner-square** can be $L$-tiled.
Assume $Q(n)$: (i) The $2^n \times 2^n$ grid missing a **center-square** can be $L$-tiled; AND (ii) The $2^n \times 2^n$ grid missing a **corner-square** can be $L$-tiled.

Induction step: Must prove two things for $Q(n + 1)$, namely (i) *and* (ii).
Assume $Q(n)$:  

(i) The $2^n \times 2^n$ grid missing a **center-square** can be $L$-tiled; AND  

(ii) The $2^n \times 2^n$ grid missing a **corner-square** can be $L$-tiled.

Induction step: Must prove two things for $Q(n + 1)$, namely (i) *and* (ii).

(i) Center square missing.

![Diagram of a $2^n \times 2^n$ grid with a center square missing, labeled with $n$ and using $Q(n)$ with corner squares.]
Assume \( Q(n) : \) (i) The \( 2^n \times 2^n \) grid missing a **center-square** can be \( L \)-tiled; \textbf{AND} (ii) The \( 2^n \times 2^n \) grid missing a **corner-square** can be \( L \)-tiled.

Induction step: Must prove two things for \( Q(n+1) \), namely (i) \textit{and} (ii).

(i) Center square missing.

(ii) Corner square missing.

use \( Q(n) \) with corner squares.
Assume $Q(n)$: (i) The $2^n \times 2^n$ grid missing a center-square can be $L$-tiled; and (ii) The $2^n \times 2^n$ grid missing a corner-square can be $L$-tiled.

Induction step: Must prove two things for $Q(n + 1)$, namely (i) and (ii).

(i) Center square missing.

(ii) Corner square missing.

Your task: Add base cases and complete the formal proof.

Exercise 6.4
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  

(Exercise 6.2)
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  

(Exercise 6.2)

Suppose \( P(n) \) is true. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2} \)?

\[ (n + 2)^3 = n^3 + 6n^2 + 12n + 8 \]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  

(Exercise 6.2)

Suppose \( P(n) \) is true. Consider \( P(n+2) : (n+2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8
\]

\[
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \quad \text{(} n \geq 10 \rightarrow 6 < n; \ 12 < n^2; \ 8 < n^3 \text{)}
\]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \quad \text{(Exercise 6.2)} \]

Suppose \( P(n) \) is \( \text{T} \). Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8 \\
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \\
= 4n^3
\]

\( n \geq 10 \rightarrow 6 < n; \ 12 < n^2; \ 8 < n^3 \)
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \quad \text{(Exercise 6.2)} \]

Suppose \( P(n) \) is true. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2}? \)

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8
\]

\[
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \quad \text{(} n \geq 10 \rightarrow 6 < n; \ 12 < n^2; \ 8 < n^3 \text{)}
\]

\[
= 4n^3 < 4 \cdot 2^n = 2^{n+2} \quad \text{(} P(n) \text{ gives } n^3 < 2^n \text{)}
\]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  

(Exercise 6.2)

Suppose \( P(n) \) is true. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2?} \)

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8 \\
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \\
= 4n^3 < 4 \cdot 2^n = 2^{n+2} \\
\]

\[ P(n) \rightarrow P(n + 2). \]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \quad \text{(Exercise 6.2)} \]

Suppose \( P(n) \) is true. Consider \( P(n+2) : (n+2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8 \\
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \\
= 4n^3 < 4 \cdot 2^n = 2^{n+2} \\
\]

\[ P(n) \rightarrow P(n + 2). \]

Base case. \( P(10) : 10^3 < 2^{10} \checkmark \)

\[
\begin{align*}
P(10) & \quad P(11) \quad P(12) \quad P(13) \quad P(14) \quad P(15) \quad P(16) \quad P(17) \quad P(18) \quad P(19) \quad P(20) \quad P(21) \quad \cdots
\end{align*}
\]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \quad \text{(Exercise 6.2)} \]

Suppose \( P(n) \) is T. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8 < n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \quad (n \geq 10 \rightarrow 6 < n; 12 < n^2; 8 < n^3) \\
= 4n^3 < 4 \cdot 2^n = 2^{n+2} \quad (P(n) \text{ gives } n^3 < 2^n) \\
\]

\[ P(n) \rightarrow P(n + 2). \]

Base cases. \( P(10) : 10^3 < 2^{10} \checkmark \quad \text{and} \quad P(11) : 11^3 < 2^{11} \checkmark \)
Leaping Induction

**Induction.** One base case.

\[ P(1) \to P(2) \to P(3) \to P(4) \to P(5) \to \ldots \]
**Leaping Induction**

**Induction.** One base case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots \]

**Leaping Induction.** More than one base case.

\[ P(1) \rightarrow P(3) \rightarrow P(5) \rightarrow P(7) \rightarrow P(9) \rightarrow P(11) \rightarrow \cdots \]
Leaping Induction

**Induction.** One base case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots \]

Leaping Induction. More than one base case.

Example. Postage greater than 5¢ can be made using 3¢ and 4¢ stamps.

<table>
<thead>
<tr>
<th>3¢</th>
<th>4¢</th>
<th>5¢</th>
<th>6¢</th>
<th>7¢</th>
<th>8¢</th>
<th>9¢</th>
<th>10¢</th>
<th>11¢</th>
<th>12¢</th>
<th>\cdots</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>–</td>
<td>3,3</td>
<td>3,4</td>
<td>4,4</td>
<td>3,3,3</td>
<td>3,3,4</td>
<td>3,4,4</td>
<td>4,4,4</td>
<td>\cdots</td>
</tr>
</tbody>
</table>
Leaping Induction

**Induction.** One base case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots \]

**Leaping Induction.** More than one base case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow P(6) \rightarrow P(7) \rightarrow P(8) \rightarrow P(9) \rightarrow P(10) \rightarrow P(11) \rightarrow P(12) \rightarrow \cdots \]

**Example.** Postage greater than 5¢ can be made using 3¢ and 4¢ stamps.

<table>
<thead>
<tr>
<th>3¢</th>
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<th>6¢</th>
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**Base cases:** 6¢, 7¢, 8¢.

**Practice.** Exercise 6.6
Fundamental Theorem of Arithmetic

\[ 2015 = 5 \times 13 \times 31. \]
Theorem. (The Primes \( P = \{2, 3, 5, 7, 11, \ldots \} \) are the atoms for numbers.)

Suppose \( n \geq 2 \). Then,

(i) \( n \) can be written as a product of factors all of which are prime.

(ii) The representation of \( n \) as a product of primes is unique (up to reordering).

\[ P(n) : n \text{ is a product of primes.} \]
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$2016 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7$. 
Fundamental Theorem of Arithmetic

2015 = 5 × 13 × 31.

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Wow! No similarity between the factors of 2015 and those of 2016.
Fundamental Theorem of Arithmetic

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$2016 = 32 \times 63$

$P(32) \land P(63) \rightarrow P(2016)$

(like leaping induction)
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**Surprise!** The much stronger claim is much easier to prove. Also, $Q(n) \to P(n)$. 
Fundamental Theorem of Arithmetic: Proof of Part (i)

\[ P(n) : \text{n is a product of primes}. \]
\[ Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n). \]

Proof. (By Induction that \( Q(n) \) is \( \top \) for \( n \geq 2 \).)
$P(n) : n$ is a product of primes.

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**Proof.**  (By Induction that $Q(n)$ is $\top$ for $n \geq 2$.)

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**Proof.** (By Induction that \( Q(n) \) is T for \( n \geq 2 \).)

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2. **[Induction step]** Show \( Q(n) \rightarrow Q(n + 1) \) for \( n \geq 2 \) (direct proof).
   - Assume \( Q(n) \) is T: each of 2, 3, \ldots, \( n \) are a product of primes.
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   By \( Q(n) \), 2, 3, \ldots, \( n \) are already products of primes.

   **To prove \( Q(n + 1) \), we only need to prove \( n + 1 \) is a product of primes.**
   - \( n + 1 \) is prime. Done (nothing to prove).
   - \( n + 1 \) is not prime, \( n + 1 = k \ell \), where \( 2 \leq k, \ell \leq n \).
Fundamental Theorem of Arithmetic: Proof of Part (i)

\( P(n) : n \) is a product of primes.

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   Show \( Q(n + 1) \) is \( \top \): each of 2, 3, \ldots, \( n \), \( n + 1 \) is a product of primes.

By \( Q(n) \), 2, 3, \ldots, \( n \) are already products of primes.

**To prove** \( Q(n + 1) \), **we only need to prove** \( n + 1 \) **is a product of primes.**

- \( n + 1 \) is prime. Done (nothing to prove).
- \( n + 1 \) is not prime, \( n + 1 = k \ell \), where \( 2 \leq k, \ell \leq n \).
  \[ P(k) \rightarrow k \text{ is a product of primes.} \]
  \[ P(\ell) \rightarrow \ell \text{ is a product of primes.} \]
  \( n + 1 = k \ell \) is a product of primes and \( Q(n + 1) \) is \( \top \).
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   Assume \( Q(n) \) is T: each of 2, 3, \ldots, \( n \) are a product of primes.
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   By \( Q(n) \), 2, 3, \ldots, \( n \) are already products of primes.
   **To prove \( Q(n + 1) \), we only need to prove \( n + 1 \) is a product of primes.**
   - \( n + 1 \) is prime. Done (nothing to prove).
   - \( n + 1 \) is not prime, \( n + 1 = k\ell \), where \( 2 \leq k, \ell \leq n \).
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     \[ P(\ell) \rightarrow \ell \] is a product of primes.
     \( n + 1 = k\ell \) is a product of primes and \( Q(n + 1) \) is T.

3. By induction, \( Q(n) \) is T \( \forall n \geq 2 \). ■
**Strong Induction.** To prove $P(n) \forall n \geq 1$ by strong induction, you use induction to prove the *stronger* claim:

$$Q(n) : \text{each of } P(1), P(2), \ldots, P(n) \text{ are } \text{T}.$$
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**Ordinary Induction**

| Base Case | Prove $P(1)$ |
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Ordinary Induction

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Strong induction is always easier.
Every \( n \geq 1 \) Has a Binary Expansion

\[ P(n) : \text{Every } n \geq 1 \text{ is a sum of distinct powers of two (its binary expansion).} \]

\[ 22 = 2^1 + 2^2 + 2^4. \]

\((22_{\text{binary}} = 1 0 1 1 0.\))
Every $n \geq 1$ Has a Binary Expansion

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(22_{binary} = 1 \ 0 \ 1 \ 1 \ 0_2.)

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Every $n \geq 1$ Has a Binary Expansion

$P(n)$: Every $n \geq 1$ is a sum of distinct powers of two (its binary expansion).

$$22 = 2^1 + 2^2 + 2^4.$$  \hspace{1cm} (22_{\text{binary}} = 10110.)

**Base Case:** $P(1)$ is T: $1 = 2^0$

**Strong Induction:** Assume $P(1) \land P(2) \land \cdots \land P(n)$ and prove $P(n + 1)$.

If $n$ is even, then $n + 1 = 2^0 + \text{binary expansion of } n,$

\begin{equation*}
\text{e.g. } 23 = 2^0 + 2^1 + 2^2 + 2^4 \tag{22}_{\text{binary}}
\end{equation*}
Every $n \geq 1$ Has a Binary Expansion

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\[ 22 = 2^1 + 2^2 + 2^4. \quad (22_{\text{binary}} = 1 \ 0 \ 1 \ 1 \ 0.) \]

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If $n$ is even, then $n + 1 = 2^0 + \text{binary expansion of } n$,

\[
\text{e.g. } 23 = 2^0 + \underline{2^1 + 2^2 + 2^4}_{22}.
\]

If $n$ is odd, then multiply each term in the expansion of $\frac{1}{2}(n + 1)$ by 2 to get $n + 1$.

\[
\text{e.g. } 24 = 2 \times (\underline{2^2 + 2^3}_{12}) = 2^3 + 2^4.
\]
Every $n \geq 1$ Has a Binary Expansion

$P(n)$: Every $n \geq 1$ is a sum of distinct powers of two (its binary expansion).

$$22 = 2^1 + 2^2 + 2^4.$$  \hspace{1cm} (22_{\text{binary}} = 1 \ 0 \ 1 \ 1 \ 0.)

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e.g. $24 = 2 \times (2^2 + 2^3) = 2^3 + 2^4$

**Exercise.** Give the formal proof by strong induction.
The Many Applications of Induction

Tournament rankings, greedy or recursive algorithms, games of strategy, ....
The Many Applications of Induction

Tournament rankings, greedy or recursive algorithms, games of strategy, . . . .

Equal Pile Nim  (old English/German: to steal or pilfer)
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**Equal Pile Nim** (old English/German: to steal or pilfer)

---

6 dots → 3 dots

Player 1
The Many Applications of Induction

Tournament rankings, greedy or recursive algorithms, **games of strategy**, ....

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```
  ●●●●●    player 1    ●●●●●    player 2    ●●    player 1    ●
  ●●●●●    player 1
```
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```

```

player 1  player 2  player 1  player 2  ...

-
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Equal Pile Nim (old English/German: to steal or pilfer)

Creator: Malik Magdon-Ismail
The Many Applications of Induction

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**Equal Pile Nim** (old English/German: to steal or pilfer)

\[
P(n) : \text{Player 2 can win the game that starts with } n \text{ pennies in each row.}
\]
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Equalization strategy:
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![Game Diagram]

\[ P(n) : \text{Player 2 can win the game that starts with } n \text{ pennies in each row.} \]

Equalization strategy:

![Equalization Strategy Diagram]

Player 2 can always return the game to *smaller* equal piles.
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**Equal Pile Nim** (old English/German: to steal or pilfer)

\[ P(n) : \text{Player 2 can win the game that starts with } n \text{ pennies in each row.} \]

Equalization strategy:

Player 2 can always return the game to **smaller** equal piles.
If Player 2 wins the smaller game, Player 2 wins the larger game. That’s strong induction!
The Many Applications of Induction

Tournament rankings, greedy or recursive algorithms, games of strategy, . . . .

Equal Pile Nim (old English/German: to steal or pilfer)

Player 2 can win the game that starts with \( n \) pennies in each row.

Equalization strategy:

Player 2 can always return the game to smaller equal piles. If Player 2 wins the smaller game, Player 2 wins the larger game. That’s strong induction!

Exercise. Give the full formal proof by strong induction.

Challenge. What about more than 2 piles. What about unequal piles. (Problem 6.20).
Please, Please, Please! Become Good at Induction!

Checklist When Approaching an Induction Problem.

✓ Are you trying to prove a “For all . . .” claim?
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✓ Are you trying to prove a “For all . . . ” claim?

✓ Identify the claim $P(n)$; in particular the parameter $n$. Here is an example.
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  **Prove**: geometric mean $\leq$ arithmetic mean.
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- Are you trying to prove a “For all . . .” claim?
- Identify the claim \( P(n) \); in particular the parameter \( n \). Here is an example.
  - *Prove*: geometric mean \( \leq \) arithmetic mean. What is \( P(n) \)? What is \( n \)?
  - \( P(n) : \) geometric mean \( \leq \) arithmetic mean for every set of \( n \) positive numbers.
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✓ Are you trying to prove a “For all ...” claim?

✓ Identify the claim $P(n)$; in particular the parameter $n$. Here is an example.

Prove: geometric mean $\leq$ arithmetic mean. What is $P(n)$? What is $n$?

$P(n)$: geometric mean $\leq$ arithmetic mean for every set of $n$ positive numbers.

Identifying the right claim is important.
You may fail because you try to prove too much. Your $P(n + 1)$ is too heavy a burden. You may fail because you try to prove too little. Your $P(n)$ is too weak a support. You must balance the strength of your claim so that the support is just enough for the burden. — G. Polya (paraphrased).
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✓ Tinker. Does the claim hold for small $n$ ($n = 1, 2, 3, \ldots$)? These become base cases.
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✓ Tinker. Can you see why (say) $P(5)$ follows from $P(1), P(2), P(3), P(4)$?

This is the crux of induction; to build up from smaller $n$ to a larger $n$. 
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✓ Determine the type of induction: try strong induction first.

✓ Write out the skeleton of the proof to see exactly what you need to prove.

✓ Determine and prove the base cases.
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✓ Determine the type of induction: try strong induction first.

✓ Write out the skeleton of the proof to see exactly what you need to prove.

✓ Determine and prove the base cases.

✓ Prove $P(n+1)$ in the induction step. You must use the induction hypothesis.