Foundations of Computer Science
Lecture 6

Strong Induction

Strengthening the Induction Hypothesis
Strong Induction
Many Flavors of Induction
Proving “for all”:

- $P(n) : 4^n - 1$ is divisible by 3.  \( \forall n : P(n) \) ?
- $P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$.  \( \forall n : P(n) \) ?
- $P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n + 1)(2n + 1)$.  \( \forall n : P(n) \) ?

Induction.

Induction and Well-Ordering.
Today: Twists on Induction

1. Solving Harder Problems with Induction
   \[ \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \]

2. Strengthening the Induction Hypothesis
   - \( n^2 < 2^n \)
   - \( L \)-tiling.

3. Many Flavors of Induction
   - Leaping Induction
     - Postage; \( n^3 < 2^n \)
   - Strong Induction
     - Fundamental Theorem of Arithmetic
     - Games of Strategy
A Hard Problem: \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n \)

Proof. \( P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \).

1. [Base case] \( P(1) \) claims that \( 1 \leq 2 \times \sqrt{1} \), which is clearly T.
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$

Proof. $P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

1: [Base case] $P(1)$ claims that $1 \leq 2 \times \sqrt{1}$, which is clearly T.

2: [Induction step] Show $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$ (direct proof)

Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

Show $P(n + 1)$ is T: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n + 1}$.
A Hard Problem: \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n \)

Proof. \( P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \).

1. **[Base case]** \( P(1) \) claims that \( 1 \leq 2 \times \sqrt{1} \), which is clearly T.

2. **[Induction step]** Show \( P(n) \rightarrow P(n + 1) \) for all \( n \geq 1 \) (direct proof)

   Assume (induction hypothesis) \( P(n) \) is T: \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \).

   Show \( P(n + 1) \) is T: \( \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n + 1} \).

\[
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}
\]
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$

Proof. $P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

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2: [Induction step] Show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$ (direct proof)

Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

Show $P(n+1)$ is T: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$.

\[
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}
\]
A Hard Problem: \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n \)

**Proof.** \( P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \).

1: **[Base case]** \( P(1) \) claims that \( 1 \leq 2 \times \sqrt{1} \), which is clearly T.

2: **[Induction step]** Show \( P(n) \rightarrow P(n+1) \) for all \( n \geq 1 \) (direct proof)
   
   Assume (induction hypothesis) \( P(n) \) is T:\n   
   \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \).
   
   Show \( P(n+1) \) is T:\n   
   \( \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1} \).

\[
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \\
\leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}
\]

**Lemma.** \( 2\sqrt{n+1}/\sqrt{n+1} \leq 2\sqrt{n+1} \)

**Proof.** By contradiction.

\[
2\sqrt{n+1}/\sqrt{n+1} > 2\sqrt{n+1} \\
\rightarrow 2\sqrt{n+1} + 1 > 2(n+1) \\
\rightarrow 4n(n+1) > (2n+1)^2 \\
\rightarrow 0 > 1 \text{ FISHY!}
\]
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$

**Proof.** $P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

1: **[Base case]** $P(1)$ claims that $1 \leq 2 \times \sqrt{1}$, which is clearly T.

2: **[Induction step]** Show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$ (direct proof)

   Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

   Show $P(n+1)$ is T: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$.

   $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}$

   $\leq \text{IH} + \frac{1}{\sqrt{n+1}}$

   $\leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$

   By the lemma $2\sqrt{n+1}/\sqrt{n+1} \leq 2\sqrt{n+1}$

   $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$

   So, $P(n+1)$ is T.

3: By induction, $P(n)$ is T $\forall n \geq 1$. 

**Lemma.** $2\sqrt{n+1}/\sqrt{n+1} \leq 2\sqrt{n+1}$

**Proof.** By contradiction.

$2\sqrt{n+1}/\sqrt{n+1} > 2\sqrt{n+1}$

$\rightarrow 2\sqrt{n+1} + 1 > 2(n+1)$

$\rightarrow 4n(n+1) > (2n+1)^2$

$\rightarrow 0 > 1$ **FISHY!**
$n^2 \leq 2^n$ for $n \geq 4$. 
\[ n^2 \leq 2^n \text{ for } n \geq 4. \]

**Induction Step.** Must use \( n^2 \leq 2^n \) to show \((n + 1)^2 \leq 2^{n+1}\).

\[ (n + 1)^2 = n^2 + 2n + 1 \]
Proving Stronger Claims

\[ n^2 \leq 2^n \quad \text{for } n \geq 4. \]

**Induction Step.** Must use \( n^2 \leq 2^n \) to show \((n + 1)^2 \leq 2^{n+1}\).

\[
(n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1
\]
$n^2 \leq 2^n$ for $n \geq 4$.

**Induction Step.** Must use $n^2 \leq 2^n$ to show $(n + 1)^2 \leq 2^{n+1}$.

$$(n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n + 1 \leq 2^n + 2^n = 2^{n+1}$$

What to do with the $2n + 1$?

Would be fine if $2n + 1 \leq 2^n$. 
Proving Stronger Claims

\[ n^2 \leq 2^n \quad \text{for } n \geq 4. \]

**Induction Step.** Must use \( n^2 \leq 2^n \) to show \((n + 1)^2 \leq 2^{n+1}\).

\[
(n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 \leq 2^n + 2^n = 2^{n+1}
\]

What to do with the \(2n + 1\)?

Would be fine if \(2n + 1 \leq 2^n\).

With induction, it can be easier to prove a stronger claim.
Strengthen the Claim: $Q(n)$ Implies $P(n)$

$Q(n): (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n.$

\[
Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots
\]
Strengthen the Claim: $Q(n)$ Implies $P(n)$

\[ Q(n) : (i) \; n^2 \leq 2^n \quad \text{AND} \quad (ii) \; 2n + 1 \leq 2^n. \]

\[ Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots \]

**Proof.** $Q(n) : (i) \; n^2 \leq 2^n \quad \text{AND} \quad (ii) \; 2n + 1 \leq 2^n.$

1: **[Base case]** $Q(4)$ claims $(i) \; 4^2 \leq 2^4 \quad \text{AND} \; (ii) \; 2 \times 4 + 1 \leq 2^4.$ Both clearly T.
Strengthen the Claim: $Q(n)$ Implies $P(n)$

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$Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots$

Proof. $Q(n): (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n.$

1. [Base case] $Q(4)$ claims (i) $4^2 \leq 2^4$ AND (ii) $2 \times 4 + 1 \leq 2^4.$ Both clearly T.

2. [Induction step] Show $Q(n) \rightarrow Q(n + 1)$ for $n \geq 4$ (direct proof).
Strengthen the Claim: $Q(n)$ Implies $P(n)$

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**Proof.** $Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n.$

1: **[Base case]** $Q(4)$ claims (i) $4^2 \leq 2^4$ AND (ii) $2 \times 4 + 1 \leq 2^4$. Both clearly T.

2: **[Induction step]** Show $Q(n) \rightarrow Q(n + 1)$ for $n \geq 4$ (direct proof).
   Assume (induction hypothesis) $Q(n)$ is T: (i) $n^2 \leq 2^n$ AND (ii) $2n + 1 \leq 2^n$.
   Show $Q(n + 1)$ is T: (i) $(n + 1)^2 \leq 2^{n+1}$ AND (ii) $2(n + 1) + 1 \leq 2^{n+1}$. 
Strengthen the Claim: \( Q(n) \) Implies \( P(n) \)

\[
Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n.
\]

\[
Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots
\]

**Proof.** \( Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n. \)

1: **[Base case]** \( Q(4) \) claims \( (i) \ 4^2 \leq 2^4 \) AND \( (ii) \ 2 \times 4 + 1 \leq 2^4. \) Both clearly T.

2: **[Induction step]** Show \( Q(n) \rightarrow Q(n+1) \) for \( n \geq 4 \) (direct proof).

Assume (induction hypothesis) \( Q(n) \) is T: \( (i) \ n^2 \leq 2^n \) AND \( (ii) \ 2n + 1 \leq 2^n. \)

Show \( Q(n+1) \) is T: \( (i) \ (n + 1)^2 \leq 2^{n+1} \) AND \( (ii) \ 2(n + 1) + 1 \leq 2^{n+1}. \)

\[
(i) \quad (n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \checkmark
\]

(because from the induction hypothesis \( n^2 \leq 2^n \) and \( 2n + 1 \leq 2^n \))
Strengthen the Claim: $Q(n)$ Implies $P(n)$

$Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n.$

$Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots$

**Proof.** $Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n.$

1: **[Base case]** $Q(4)$ claims (i) $4^2 \leq 2^4$ AND (ii) $2 \times 4 + 1 \leq 2^4$. Both clearly T.

2: **[Induction step]** Show $Q(n) \rightarrow Q(n+1)$ for $n \geq 4$ (direct proof).

Assume (induction hypothesis) $Q(n)$ is T: (i) $n^2 \leq 2^n$ AND (ii) $2n + 1 \leq 2^n$.

Show $Q(n+1)$ is T: (i) $(n+1)^2 \leq 2^{n+1}$ AND (ii) $2(n+1) + 1 \leq 2^{n+1}$.

\[
(i) \quad (n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \quad \checkmark \\
\quad \text{(because from the induction hypothesis } n^2 \leq 2^n \text{ and } 2n + 1 \leq 2^n)\\
(ii) \quad 2(n + 1) + 1 = 2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \quad \checkmark \\
\quad \text{(because } 2 \leq 2^n \text{ and from the induction hypothesis } 2n + 1 \leq 2^n)\]
Strengthen the Claim: \( Q(n) \) Implies \( P(n) \)

\[
Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n.
\]

\[
[Q(4)] ightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots
\]

Proof. \( Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n. \)

1: [Base case] \( Q(4) \) claims \( (i) \ 4^2 \leq 2^4 \ \text{AND} \ (ii) \ 2 \times 4 + 1 \leq 2^4. \quad \text{Both clearly T.} \)

2: [Induction step] Show \( Q(n) \rightarrow Q(n+1) \) for \( n \geq 4 \) (direct proof).

Assume (induction hypothesis) \( Q(n) \) is T: \( (i) \ n^2 \leq 2^n \ \text{AND} \ (ii) \ 2n + 1 \leq 2^n. \)

Show \( Q(n+1) \) is T: \( (i) \ (n+1)^2 \leq 2^{n+1} \ \text{AND} \ (ii) \ 2(n+1) + 1 \leq 2^{n+1}. \)

\[
(i) \quad (n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \quad \checkmark
\]

(because from the induction hypothesis \( n^2 \leq 2^n \ \text{and} \ 2n + 1 \leq 2^n)\)

\[
(ii) \quad 2(n + 1) + 1 = 2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \quad \checkmark
\]

(because \( 2 \leq 2^n \) and from the induction hypothesis \( 2n + 1 \leq 2^n \))

So, \( Q(n+1) \) is T.

3: By induction, \( Q(n) \) is T \( \forall n \geq 4. \)
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\square$ – tiles?
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only □ – tiles?

TINKER!
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only 🟢 – tiles?

TINKER!
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only \[ \Box \] – tiles?

TINKER!
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\blacksquare$ – tiles?

TINKER!
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\square$ tiles?

**TINKER!**
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only \[ \square \] tiles?

TINKER!
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\square$ – tiles?

TINKER!

$P(n)$ : The $2^n \times 2^n$ grid minus a center-square can be $L$-tiled.
Suppose $P(n)$ is T. What about $P(n + 1)$?
Suppose $P(n)$ is true. What about $P(n + 1)$?

The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.
Suppose $P(n)$ is true. What about $P(n+1)$?

The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.
Suppose $P(n)$ is true. What about $P(n + 1)$?

The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.

**Problem.** Corner squares are missing. $P(n)$ can be used only if center-square is missing.
Suppose $P(n)$ is true. What about $P(n + 1)$?

The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.

**Problem.** Corner squares are missing. $P(n)$ can be used only if center-square is missing.

**Solution.** Strengthen claim to also include patios missing corner-squares.

$Q(n)$: (i) The $2^n \times 2^n$ grid missing a **center-square** can be $L$-tiled; AND (ii) The $2^n \times 2^n$ grid missing a **corner-square** can be $L$-tiled.
Assume $Q(n)$: (i) The $2^n \times 2^n$ grid missing a center-square can be $L$-tiled; AND (ii) The $2^n \times 2^n$ grid missing a corner-square can be $L$-tiled.

Induction step: Must prove two things for $Q(n+1)$, namely (i) and (ii).
Assume \( Q(n) : \)

(i) The \( 2^n \times 2^n \) grid missing a **center-square** can be \( L \)-tiled; AND

(ii) The \( 2^n \times 2^n \) grid missing a **corner-square** can be \( L \)-tiled.

Induction step: Must prove two things for \( Q(n+1) \), namely (i) *and* (ii).

(i) Center square missing.

![Diagram](image.png)

use \( Q(n) \) with corner squares.
Assume $Q(n)$: (i) The $2^n \times 2^n$ grid missing a **center-square** can be $L$-tiled; AND (ii) The $2^n \times 2^n$ grid missing a **corner-square** can be $L$-tiled.

Induction step: Must prove two things for $Q(n+1)$, namely (i) **and** (ii).

(i) Center square missing.

![Diagram of a $2^n \times 2^n$ grid with a center square missing, labeled with $Q(n)$ with corner squares.]

(ii) Corner square missing.

![Diagram of a $2^n \times 2^n$ grid with a corner square missing, labeled with $Q(n)$ with corner squares.]

use $Q(n)$ with corner squares.
Assume \( Q(n) \) : (i) The \( 2^n \times 2^n \) grid missing a **center-square** can be \( L \)-tiled; AND (ii) The \( 2^n \times 2^n \) grid missing a **corner-square** can be \( L \)-tiled.

Induction step: Must prove two things for \( Q(n + 1) \), namely (i) *and* (ii).

(i) Center square missing.

![Diagram of a \( 2^n \times 2^n \) grid with a center square missing, where the grid is used \( Q(n) \) with corner squares.]

(ii) Corner square missing.

![Diagram of a \( 2^n \times 2^n \) grid with a corner square missing, where the grid is used \( Q(n) \) with corner squares.]

Your task: Add base cases and complete the formal proof.

**Exercise 6.4.** What if the missing square is some random square? Strengthen further.
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  

(Exercise 6.2)
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for} \ n \geq 10. \quad \text{(Exercise 6.2)} \]

Suppose \( P(n) \) is T. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8
\]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  
(Exercise 6.2)

Suppose \( P(n) \) is true. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8 \\
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \\
(n \geq 10 \rightarrow 6 < n; \ 12 < n^2; \ 8 < n^3)
\]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  

(Exercise 6.2)

Suppose \( P(n) \) is \( \top \). Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2} \)?

\[(n + 2)^3 = n^3 + 6n^2 + 12n + 8\]

\[< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \quad (n \geq 10 \rightarrow 6 < n; 12 < n^2; 8 < n^3)\]

\[= 4n^3\]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  

(Exercise 6.2)

Suppose \( P(n) \) is true. Consider \( P(n+2) : (n+2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8
\]

\[
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \quad (n \geq 10 \rightarrow 6 < n; \ 12 < n^2; \ 8 < n^3)
\]

\[
= 4n^3 < 4 \cdot 2^n = 2^{n+2} \quad (P(n) \text{ gives } n^3 < 2^n)
\]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  
(Exercise 6.2)

Suppose \( P(n) \) is true. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8 \\
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \\
= 4n^3 < 4 \cdot 2^n = 2^{n+2}
\]

\( P(n) \rightarrow P(n + 2). \)
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  

(Exercise 6.2)

Suppose \( P(n) \) is T. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8
\]
\[
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3
\]
\[
= 4n^3 < 4 \cdot 2^n = 2^{n+2}
\]

\( P(n) \rightarrow P(n + 2). \)

Base case. \( P(10) : 10^3 < 2^{10} \)

[Diagram showing the progression from \( P(10) \) to \( P(21) \) and beyond]
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  
(Exercise 6.2)

Suppose \( P(n) \) is T. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2?} \)

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8
\]

\[
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \quad \text{(} n \geq 10 \rightarrow 6 < n; \ 12 < n^2; \ 8 < n^3) \]

\[
= 4n^3 < 4 \cdot 2^n = 2^{n+2} \quad \text{(} P(n) \text{ gives } n^3 < 2^n) \]

\[ P(n) \rightarrow P(n + 2). \]

Base cases. \( P(10) : 10^3 < 2^{10} \checkmark \quad \text{and} \quad P(11) : 11^3 < 2^{11} \checkmark \)
**Leaping Induction**

**Induction.** One base case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots \]
**Leaping Induction**

*Induction.* One base case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots \]

*Leaping Induction.* More than one base case.

\[ P(1) \quad \boxed{P(2)} \quad P(3) \quad P(4) \quad P(5) \quad P(6) \quad P(7) \quad P(8) \quad P(9) \quad P(10) \quad P(11) \quad P(12) \quad \cdots \]
Leaping Induction

**Induction.** One base case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots \]

**Leaping Induction.** More than one base case.

\[
\begin{array}{ccccccccccc}
\end{array}
\]

**Example.** Postage greater than 5¢ can be made using 3¢ and 4¢ stamps.

<table>
<thead>
<tr>
<th>3¢</th>
<th>4¢</th>
<th>5¢</th>
<th>6¢</th>
<th>7¢</th>
<th>8¢</th>
<th>9¢</th>
<th>10¢</th>
<th>11¢</th>
<th>12¢</th>
<th>⋯</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>–</td>
<td><strong>3,3</strong></td>
<td><strong>3,4</strong></td>
<td><strong>4,4</strong></td>
<td><strong>3,3,3</strong></td>
<td><strong>3,3,4</strong></td>
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<td>⋯</td>
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**Leaping Induction**

**Induction.** One base case.

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**Leaping Induction.** More than one base case.

![Diagram showing multiple base cases leading to all other cases]

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<td>–</td>
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\[ P(n) : \text{Postage of } n \text{ cents can be made using only 3¢ and 4¢ stamps.} \]
**Leaping Induction**

**Induction.** One base case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots \]

**Leaping Induction.** More than one base case.

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\[ P(n) : \text{Postage of } n \text{ cents can be made using only 3¢ and 4¢ stamps.} \]

\[ P(n) \rightarrow P(n + 3) \quad \text{(add a 3¢ stamp to } n) \]
Leaping Induction

**Induction.** One base case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots \]

**Leaping Induction.** More than one base case.

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\[ P(n) \rightarrow P(n + 3) \quad \text{(add a 3¢ stamp to } n) \]

**Base cases:** 6¢, 7¢, 8¢.

**Practice.** Exercise 6.6
Fundamental Theorem of Arithmetic

\[ 2015 = 5 \times 13 \times 31. \]
Fundamental Theorem of Arithmetic

\[ 2015 = 5 \times 13 \times 31. \]

Theorem. (The Primes \( \mathcal{P} = \{2, 3, 5, 7, 11, \ldots \} \) are the atoms for numbers.)

Suppose \( n \geq 2 \). Then,

1. \( n \) can be written as a product of factors all of which are prime.
2. The representation of \( n \) as a product of primes is unique (up to reordering).

\[ P(n) : n \text{ is a product of primes.} \]
2015 = 5 \times 13 \times 31.

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What’s the first thing we do?
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What’s the first thing we do? TINKER!

$2016 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7$. 
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2. The representation of $n$ as a product of primes is unique (up to reordering).

What’s the first thing we do? **TINKER!**

$2015 = 5 \times 13 \times 31$.

$2016 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7$.

Wow! No similarity between the factors of 2015 and those of 2016.
Fundamental Theorem of Arithmetic

2015 = 5 × 13 × 31.

Theorem. (The Primes $\mathcal{P} = \{2, 3, 5, 7, 11, \ldots\}$ are the atoms for numbers.)

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- $n$ can be written as a product of factors all of which are prime.
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**How will $P(n)$ help us to prove $P(n+1)$?**
Fundamental Theorem of Arithmetic

2015 = 5 × 13 × 31.

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\( P(n) : n \) is a product of primes.

What’s the first thing we do? **TINKER!**

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**How will** \( P(n) \) **help us to prove** \( P(n + 1) \)?
Much “Stronger” Induction Claim

Do smaller values of \( n \) help with 2016? Yes!

\[
2016 = 32 \times 63
\]

\[P(32) \land P(63) \rightarrow P(2016)\]  

(like leaping induction)
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Q(n) : 2, 3, \ldots, n \text{ are all products of primes.}
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(like leaping induction)

Much Stronger Claim:

$Q(n) : 2, 3, \ldots, n$ are all products of primes.

$P(n) : n$ is a product of primes.

(Qualify)

$$Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n).$$
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**Much Stronger Claim:**

$$Q(n) : 2, 3, \ldots, n \text{ are all products of primes.}$$

$$P(n) : n \text{ is a product of primes.}$$

(Compare)

$$Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n).$$

**Surprise!** The much stronger claim is *much* easier to prove. Also, $Q(n) \rightarrow P(n)$.
Fundamental Theorem of Arithmetic: Proof of Part (i)

\[ P(n) : n \text{ is a product of primes.} \]

\[ Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n). \]

Proof. (By Induction that \( Q(n) \) is \( \top \) for \( n \geq 2 \).)
**Fundamental Theorem of Arithmetic: Proof of Part (i)**

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**Proof.** (By Induction that \( Q(n) \) is \( \top \) for \( n \geq 2 \).)

1. **[Base case]** \( Q(1) \) claims that 2 is a product of primes, which is clearly \( \top \).
Fundamental Theorem of Arithmetic: Proof of Part (i)

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Proof. (By Induction that \( Q(n) \) is \( T \) for \( n \geq 2 \).)

1: **[Base case]** \( Q(1) \) claims that 2 is a product of primes, which is clearly \( T \).

2: **[Induction step]** Show \( Q(n) \to Q(n + 1) \) for \( n \geq 2 \) (direct proof).
Fundamental Theorem of Arithmetic: Proof of Part (i)

\[ P(n) : n \text{ is a product of primes.} \]

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\[ \text{Assume } Q(n) \text{ is } T: \text{ each of } 2, 3, \ldots, n \text{ are a product of primes.} \]

\[ \text{Show } Q(n+1) \text{ is } T: \text{ each of } 2, 3, \ldots, n, n+1 \text{ is a product of primes.} \]
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   Assume \( Q(n) \) is \( T \): each of 2, 3, \ldots, \( n \) are a product of primes.
   Show \( Q(n + 1) \) is \( T \): each of 2, 3, \ldots, \( n \), \( n + 1 \) is a product of primes.
   Since we assumed \( Q(n) \), we already have that 2, 3, \ldots, \( n \) are products of primes.
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   Since we assumed \( Q(n) \), we already have that 2, 3, \ldots, \( n \) are products of primes. **To prove \( Q(n + 1) \), we only need to prove \( n + 1 \) is a product of primes.**
Fundamental Theorem of Arithmetic: Proof of Part (i)

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   - \( n + 1 \) is prime. Done (nothing to prove).
Fundamental Theorem of Arithmetic: Proof of Part (i)

\[ P(n) : n \text{ is a product of primes.} \]

\[ Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n). \]

Proof. (By Induction that \( Q(n) \) is true for \( n \geq 2 \).)

1: **[Base case]** \( Q(1) \) claims that 2 is a product of primes, which is clearly true.

2: **[Induction step]** Show \( Q(n) \to Q(n + 1) \) for \( n \geq 2 \) (direct proof).

Assume \( Q(n) \) is true: each of 2, 3, \ldots, \( n \) are a product of primes.

Show \( Q(n + 1) \) is true: each of 2, 3, \ldots, \( n, n + 1 \) is a product of primes.

Since we assumed \( Q(n) \), we already have that 2, 3, \ldots, \( n \) are products of primes.

To prove \( Q(n + 1) \), we only need to prove \( n + 1 \) is a product of primes.

- \( n + 1 \) is prime. Done (nothing to prove).
- \( n + 1 \) is not prime, \( n + 1 = k\ell \), where \( 2 \leq k, \ell \leq n \).
proof. (By Induction that $Q(n)$ is $\text{T}$ for $n \geq 2$.)

1: [Base case] $Q(1)$ claims that 2 is a product of primes, which is clearly $\text{T}$.

2: [Induction step] Show $Q(n) \rightarrow Q(n + 1)$ for $n \geq 2$ (direct proof).

Assume $Q(n)$ is $\text{T}$: each of 2, 3, ..., $n$ are a product of primes.
Show $Q(n + 1)$ is $\text{T}$: each of 2, 3, ..., $n$, $n + 1$ is a product of primes.

Since we assumed $Q(n)$, we already have that 2, 3, ..., $n$ are products of primes.
**To prove $Q(n + 1)$, we only need to prove $n + 1$ is a product of primes.**

- $n + 1$ is prime. Done (nothing to prove).

- $n + 1$ is not prime, $n + 1 = k \ell$, where $2 \leq k, \ell \leq n$.

  $P(k) \rightarrow k$ is a product of primes.
  $P(\ell) \rightarrow \ell$ is a product of primes.
Fundamental Theorem of Arithmetic: Proof of Part (i)

\( P(n) : n \) is a product of primes.

\[ Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n). \]

**Proof.** (By Induction that \( Q(n) \) is \( T \) for \( n \geq 2 \).)

1: **[Base case]** \( Q(1) \) claims that 2 is a product of primes, which is clearly \( T \).

2: **[Induction step]** Show \( Q(n) \rightarrow Q(n + 1) \) for \( n \geq 2 \) (direct proof).

Assume \( Q(n) \) is \( T \): each of 2, 3, \ldots, \( n \) are a product of primes.

Show \( Q(n + 1) \) is \( T \): each of 2, 3, \ldots, \( n, n + 1 \) is a product of primes.

Since we assumed \( Q(n) \), we already have that 2, 3, \ldots, \( n \) are products of primes. **To prove \( Q(n + 1) \), we only need to prove \( n + 1 \) is a product of primes.**

- \( n + 1 \) is prime. Done (nothing to prove).
- \( n + 1 \) is not prime, \( n + 1 = k \ell \), where \( 2 \leq k, \ell \leq n \).
  - \( P(k) \rightarrow k \) is a product of primes.
  - \( P(\ell) \rightarrow \ell \) is a product of primes.
  
  \( n + 1 = k \ell \) is a product of primes and \( Q(n + 1) \) is \( T \).
Fundamental Theorem of Arithmetic: Proof of Part (i)

\[ P(n) : n \text{ is a product of primes.} \]

\[ Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n). \]

**Proof.** (By Induction that \( Q(n) \) is T for \( n \geq 2 \).)

1: **[Base case]** \( Q(1) \) claims that 2 is a product of primes, which is clearly T.

2: **[Induction step]** Show \( Q(n) \rightarrow Q(n + 1) \) for \( n \geq 2 \) (direct proof).

Assume \( Q(n) \) is T: each of 2, 3, \ldots, \( n \) are a product of primes.

Show \( Q(n + 1) \) is T: each of 2, 3, \ldots, \( n, n + 1 \) is a product of primes.

Since we assumed \( Q(n) \), we already have that 2, 3, \ldots, \( n \) are products of primes. **To prove \( Q(n + 1) \), we only need to prove \( n + 1 \) is a product of primes.**

- \( n + 1 \) is prime. Done (nothing to prove).
- \( n + 1 \) is not prime, \( n + 1 = k\ell \), where \( 2 \leq k, \ell \leq n \).
  
  \[ P(k) \rightarrow k \text{ is a product of primes.} \]
  \[ P(\ell) \rightarrow \ell \text{ is a product of primes.} \]

  \( n + 1 = k\ell \) is a product of primes and \( Q(n + 1) \) is T.

3: By induction, \( Q(n) \) is T \( \forall n \geq 2 \).
**Strong Induction.** To prove $P(n) \forall n \geq 1$ by strong induction, you use induction to prove the *stronger* claim:

$$Q(n) : \text{each of } P(1), P(2), \ldots, P(n) \text{ are T.}$$
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Ordinary Induction

Base Case  Prove $P(1)$
**Strong Induction.** To prove $P(n) \forall n \geq 1$ by strong induction, you use induction to prove the *stronger* claim:

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### Ordinary Induction

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$$Q(n) : \text{each of } P(1), P(2), \ldots, P(n) \text{ are } T.$$
Every $n \geq 1$ Has a Binary Expansion

$P(n) :$ Every $n \geq 1$ is a sum of distinct powers of two (its binary expansion).

$$22 = 2^1 + 2^2 + 2^4.$$  \hspace{1cm} (22_{\text{binary}} = 10110.)
Every $n \geq 1$ Has a Binary Expansion

$P(n)$: Every $n \geq 1$ is a sum of distinct powers of two (its binary expansion).

$$22 = 2^1 + 2^2 + 2^4.$$ (22_{\text{binary}} = 1\ 0\ 1\ 1\ 0.)

**Base Case:** $P(1)$ is true: $1 = 2^0$
Every $n \geq 1$ Has a Binary Expansion

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\[
22 = 2^1 + 2^2 + 2^4. \quad (22_{\text{binary}} = 1011010)
\]

**Base Case:** $P(1)$ is T: $1 = 2^0$

**Strong Induction:** Assume $P(1) \land P(2) \land \cdots \land P(n)$ and prove $P(n + 1)$. 
Every $n \geq 1$ Has a Binary Expansion

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**Strong Induction:** Assume $P(1) \land P(2) \land \ldots \land P(n)$ and prove $P(n + 1)$.

If $n$ is even, then $n + 1 = 2^0 + \text{binary expansion of } n$,

\[ \text{e.g. } 23 = 2^0 + 2^1 + 2^2 + 2^4 \]
Every $n \geq 1$ Has a Binary Expansion

$P(n)$: Every $n \geq 1$ is a sum of distinct powers of two (its binary expansion).

$$22 = 2^1 + 2^2 + 2^4. \quad \text{(22)}_{\text{binary}} = 1 \ 0 \ 1 \ 1 \ 0 .$$

**Base Case:** $P(1)$ is T: $1 = 2^0$

**Strong Induction:** Assume $P(1) \land P(2) \land \cdots \land P(n)$ and prove $P(n + 1)$.

If $n$ is even, then $n + 1 = 2^0 + \text{binary expansion of } n$,

\[ \text{e.g. } 23 = 2^0 + 2^1 + 2^2 + 2^4 \]

If $n$ is odd, then multiply each term in the expansion of $\frac{1}{2}(n + 1)$ by 2 to get $n + 1$.

\[ \text{e.g. } 24 = 2 \times (2^2 + 2^3) = 2^3 + 2^4 \]

**Exercise.** Give the formal proof by strong induction.
The Many Applications of Induction

Tournament rankings, greedy or recursive algorithms, games of strategy, ....
The Many Applications of Induction

Tournament rankings, greedy or recursive algorithms, \textit{games of strategy}, \ldots.

Equal Pile Nim (old English/German: to steal or pilfer)
The Many Applications of Induction

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**Equal Pile Nim** (old English/German: to steal or pilfer)

![Diagram of a game of Equal Pile Nim](image)
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Tournament rankings, greedy or recursive algorithms, **games of strategy**, ...
The Many Applications of Induction

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Equal Pile Nim (old English/German: to steal or pilfer)

```
5  4  3  2  1  |  5  4  3  2  1  |  5  4  3  2  1  |  5  4  3  2  1  |  5  4  3  2  1
---|---|---|---|---
1  2  3  4  5  |  | | | | 
player 1 | player 1 | player 1 | player 2 | 
| player 2 | player 2 | 
```

-
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Equal Pile Nim (old English/German: to steal or pilfer)

Player 1  

Player 2  

Player 1  

Player 2  

Player 1 wins
The Many Applications of Induction

Tournament rankings, greedy or recursive algorithms, **games of strategy**, ....

**Equal Pile Nim (old English/German: to steal or pilfer)**

\[ \begin{align*}
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\end{align*} \]

player 1 \hspace{1cm} player 2 \hspace{1cm} player 1 \hspace{1cm} player 2

\[ P(n) : \text{Player 2 can win the game that starts with } n \text{ pennies in each row.} \]
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Equalization strategy:
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Player 2 can win the game that starts with \( n \) pennies in each row.

**Equalization strategy:**

Player 2 can always return the game to *smaller* equal piles.
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**Equal Pile Nim** (old English/German: to steal or pilfer)

- **P(n)**: Player 2 can win the game that starts with \( n \) pennies in each row.

**Equalization strategy:**

Player 2 can always return the game to *smaller* equal piles.
If Player 2 wins the smaller game, Player 2 wins the larger game. That’s strong induction!

**Exercise.** Give the full formal proof by strong induction.

**Challenge.** What about more than 2 piles. What about unequal piles. (Problem 6.20).
Uniqueness of binary representation as a sum of distinct powers of 2:

Problem 6.27

General Nim:

Problem 6.39
Checklist When Approaching an Induction Problem.

- Are you trying to prove a “For all . . .” claim?
Please, Please, Please! Become Good at Induction!

Checklist When Approaching an Induction Problem.

- Are you trying to prove a “For all . . .” claim?
- Identify the claim $P(n)$, especially the parameter $n$. Here is an example.
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  Prove: geometric mean $\leq$ arithmetic mean.
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- Identify the claim $P(n)$, especially the parameter $n$. Here is an example.
  Prove: geometric mean $\leq$ arithmetic mean. What is $P(n)$?
Checklist When Approaching an Induction Problem.

- Are you trying to prove a “For all . . .” claim?

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- Identify the claim $P(n)$, especially the parameter $n$. Here is an example.
  - **Prove:** geometric mean $\leq$ arithmetic mean. What is $P(n)$? What is $n$?
  - $P(n)$: geometric mean $\leq$ arithmetic mean for every set of $n$ positive numbers.
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  $P(n)$ : geometric mean $\leq$ arithmetic mean for every set of $n$ positive numbers.
  
  Identifying the right claim is important.

You may fail because you try to prove too much. Your $P(n + 1)$ is too heavy a burden. You may fail because you try to prove too little. Your $P(n)$ is too weak a support. You must balance the strength of your claim so that the support is just enough for the burden. — G. Polya (paraphrased).
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- Tinker. Does the claim hold for small $n$ ($n = 1, 2, 3, \ldots$)? These become base cases.
Please, Please, Please! Become Good at Induction!

Checklist When Approaching an Induction Problem.

- Are you trying to prove a “For all ...” claim?
- Identify the claim \( P(n) \), especially the parameter \( n \). Here is an example.

  Prove: geometric mean \( \leq \) arithmetic mean. What is \( P(n) \)? What is \( n \)?
  \( P(n) : \) geometric mean \( \leq \) arithmetic mean for every set of \( n \) positive numbers.

  *Identifying the right claim is important.*
  You may fail because you try to prove too much. Your \( P(n+1) \) is too heavy a burden. You may fail because you try to prove too little. Your \( P(n) \) is too weak a support. You must balance the strength of your claim so that the support is just enough for the burden. — G. Polya (paraphrased).

- Tinker. Does the claim hold for small \( n \) (\( n = 1, 2, 3, \ldots \))? These become base cases.

- Tinker. Can you see why (say) \( P(5) \) follows from \( P(1), P(2), P(3), P(4) \)?
  This is the crux of induction; to build up from smaller \( n \) to a larger \( n \).
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Please, Please, Please! Become Good at Induction!

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- Write out the skeleton of the proof to see exactly what you need to prove.
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- Determine the type of induction: try strong induction first.

- Write out the skeleton of the proof to see exactly what you need to prove.

- Determine and prove the base cases.
Checklist When Approaching an Induction Problem.

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- Prove $P(n+1)$ in the induction step. You *must* use the induction hypothesis.