Foundations of Computer Science
Lecture 15

Probability
Computing Probabilities
Probability and Sets: Probability Space
Uniform Probability Spaces
Infinite Probability Spaces

The probable is what usually happens – Aristotle
To count complex objects, construct a sequence of “instructions” that can be used to construct the object uniquely. The number of possible sequences of instructions equals the number of possible complex objects.

**Counting**
- Sequences with and without repetition.
- Subsets with and without repetition.
- Sequences with specified numbers of each type of object: anagrams.

**Inclusion-Exclusion** (advanced technique).

**Pigeonhole principle** (simple but IMPORTANT technique).
Today: Probability

   - Outcome tree.
   - Event of interest.
   - Examples with dice.

2. Probability and sets.
   - The probability space.

3. Uniform probability spaces.

4. Infinite probability spaces.
The Chance of Rain Tomorrow is 40%

What does the title mean? Either it will rain tomorrow or it won’t.
The Chance of Rain Tomorrow is 40%

What does the title mean? Either it will rain tomorrow or it won’t.

The chances are 50% that a fair coin-flip will be H.
The Chance of Rain Tomorrow is 40%

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Flip 100 times. Approximately 50 will be H ← frequentist view.
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- You toss a fair coin 3 times. How many heads will you get?

- You keep tossing a fair coin until you get a head. How many tosses will you make?
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1 You toss a fair coin 3 times. How many heads will you get?

2 You keep tossing a fair coin until you get a head. How many tosses will you make?

There’s no answer. The outcome is uncertain. Probability is appropriate for such settings.
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Birth of Mathematical Probability.

Antoine Gombaud,: Should I bet even money on at least one ‘double-6’ in 24 rolls of two dice?
Chevalier de Méré: What about at least one 6 in 4 rolls of one die?
Blaise Pascal: Interesting question. Let’s bring Pierre de Fermat into the conversation.

…a correspondence is ignited between these two mathematical giants
You are analyzing an “experiment” whose outcome is uncertain.
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**Outcomes.** Identify all possible outcomes using a tree of outcome sequences.
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**Outcomes.** Identify all possible outcomes using a tree of outcome sequences.

```
   H
  /   \
/     \
H   T   H   T
```

Coin 1

Coin 2
You are analyzing an “experiment” whose outcome is uncertain.

**Outcomes.** Identify all possible outcomes using a tree of outcome sequences.
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**Outcomes.** Identify all possible outcomes using a tree of outcome sequences.

**Edge probabilities.** If one of \( k \) edges (options) from a vertex is chosen randomly then each edge has edge-probability \( \frac{1}{k} \).
You are analyzing an “experiment” whose outcome is uncertain.

Outcomes. Identify all possible outcomes using a tree of outcome sequences.

Edge probabilities. If one of $k$ edges (options) from a vertex is chosen randomly then each edge has edge-probability $\frac{1}{k}$.

Outcome-probability. Multiply edge-probabilities to get outcome-probabilities.
Toss two coins: you win if the coins match (HH or TT)

**Question:** When do you win?  

**Event:** Subset of outcomes where you win.
Toss two coins: you win if the coins match (HH or TT)

**Question:** When do you win?  

**Event:** *Subset* of outcomes where you win.

**Event of interest.** Subset of the outcomes where you win.

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The Outcome-Tree Method →

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Creator: Malik Magdon-Ismail

Probability: 6 / 14
Event of Interest

Toss two coins: you win if the coins match (HH or TT)

**Question:** When do you win?  

**Event:** Subset of outcomes where you win.

---

1. **Event of interest.** Subset of the outcomes where you win.

2. **Event-probability.** Sum of its outcome-probabilities.

   \[
   \text{event-probability} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
   \]

Probability that you win is \(\frac{1}{2}\), written as \(\mathbb{P}[\text{"YouWin"}] = \frac{1}{2}\).

Go and do this experiment at home. Toss two coins 1000 times and see how often you win.
The Outcome-Tree Method

Become familiar with this 6-step process for analyzing a probabilistic experiment.

1. **You are analyzing an experiment whose outcome is uncertain.**
2. **Outcomes.** Identify *all possible* outcomes, the tree of *outcome sequences*.
3. **Edge-Probability.** Each edge in the outcome-tree gets a probability.
4. **Outcome-Probability.** Multiply edge-probabilities to get outcome-probabilities.
5. **Event of Interest \( \mathcal{E} \).** Determine the subset of the outcomes you care about.
6. **Event-Probability.** The sum of outcome-probabilities in the subset you care about.

\[
\mathbb{P}[\mathcal{E}] = \sum_{\text{outcomes } \omega \in \mathcal{E}} P(\omega).
\]

\[\mathbb{P}[\mathcal{E}] \sim \text{frequency an outcome you want occurs over many repeated experiments.}\]

**Pop Quiz.** Roll two dice. Compute \(\mathbb{P}[\text{first roll is less than the second}]\).
1: Contestant at door 1.
2: Prize placed behind *random* door.
Let’s Make a Deal: The Monty Hall Problem

1: Contestant at door 1.
2: Prize placed behind *random* door.
3: Monty opens *empty* door (*randomly* if there’s an option).
1. Contestant at door 1.
2. Prize placed behind random door.
3. Monty opens empty door \((randomly\) if there’s an option).

- Outcome-tree and edge-probabilities.
Let’s Make a Deal: The Monty Hall Problem

1: Contestant at door 1.
2: Prize placed behind random door.
3: Monty opens empty door (randomly if there’s an option).

- Outcome-tree and edge-probabilities.
1: Contestant at door 1.
2: Prize placed behind \textit{random} door.
3: Monty opens \textit{empty} door (\textit{randomly} if there’s an option).

- Outcome-tree and edge-probabilities.
Let’s Make a Deal: The Monty Hall Problem

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- Outcome-tree and edge-probabilities.
- Outcome-probabilities.

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<tr>
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<th>Outcome</th>
<th>Probability</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>(1, 2)</td>
<td>$P(1, 2) = \frac{1}{6}$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(1, 3)</td>
<td>$P(1, 3) = \frac{1}{6}$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>(2, 3)</td>
<td>$P(2, 3) = \frac{1}{3}$</td>
</tr>
<tr>
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Let’s Make a Deal: The Monty Hall Problem

1: Contestant at door 1.
2: Prize placed behind *random* door.
3: Monty opens empty door (*randomly* if there’s an option).

- Outcome-tree and edge-probabilities.
- Outcome-probabilities.
- Event of interest: “WinBySwitching”.

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Let’s Make a Deal: The Monty Hall Problem

1: Contestant at door 1.
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3: Monty opens empty door (randomly if there’s an option).

- Outcome-tree and edge-probabilities.
- Outcome-probabilities.
- Event of interest: “WinBySwitching”.
- Event probability.

\[
\begin{array}{c|c|c}
\text{Prize} & \text{Host} & \text{Outcome} & \text{Probability} \\
\hline
1 & 2 & (1, 2) & P(1, 2) = \frac{1}{6} \\
2 & 3 & (1, 3) & P(1, 3) = \frac{1}{6} \\
1 & 3 & (2, 3) & P(2, 3) = \frac{1}{3} \\
3 & 2 & (3, 2) & P(3, 2) = \frac{1}{3} \\
\end{array}
\]

\[
\frac{1}{3} + \frac{1}{3} = \frac{2}{3} = \mathbb{P}[\text{“WinBySwitching”}]
\]
Non-Transitive Dice

\[ A: \begin{cases} \{1, 2, 3\} \\ \{4, 5, 6\} \end{cases} \quad B: \begin{cases} \{1, 2, 4\} \\ \{3, 5, 6\} \end{cases} \quad C: \begin{cases} \{1, 2, 5\} \\ \{3, 4, 6\} \end{cases} \]

Your friend picks a die; you pick a die.
e.g. friend picks die \( B \); you pick \( A \).
Non-Transitive Dice

A: \{\text{\ding{143}, \text{\ding{143}, \text{\ding{143}}, \text{\ding{143}}}}\}  \quad B: \{\text{\ding{143}, \text{\ding{143}, \text{\ding{143}}, \text{\ding{143}}}}\}  \quad C: \{\text{\ding{143}, \text{\ding{143}, \text{\ding{143}}, \text{\ding{143}}}}\}

Your friend picks a die; you pick a die.
e.g. friend picks die $B$; you pick $A$.

What is the probability that $A$ beats $B$?
Non-Transitive Dice

Your friend picks a die; you pick a die.
e.g. friend picks die $B$; you pick $A$.

What is the probability that $A$ beats $B$?

- Outcome-tree and outcome-probabilities.
Non-Transitive Dice

A: \begin{align*}
\{ & 1, 3, 5, 6 \\ & 2, 3, 4, 5 \} \\
\end{align*}

B: \begin{align*}
\{ & 1, 3, 5, 6 \\ & 2, 3, 4, 5 \} \\
\end{align*}

C: \begin{align*}
\{ & 1, 3, 5, 6 \\ & 2, 3, 4, 5 \} \\
\end{align*}

Your friend picks a die; you pick a die. e.g. friend picks die B; you pick A.

**What is the probability that A beats B?**

- Outcome-tree and outcome-probabilities.
Non-Transitive Dice

A: \{\text{\ding{192}, \ding{193}, \ding{194}, \ding{195}, \ding{196}}\}  
B: \{\text{\ding{192}, \ding{193}, \ding{194}, \ding{195}, \ding{196}}\}  
C: \{\text{\ding{192}, \ding{193}, \ding{194}, \ding{195}, \ding{196}}\}

Your friend picks a die; you pick a die.  
e.g. friend picks die B; you pick A.

What is the probability that A beats B?

- Outcome-tree and outcome-probabilities.
- Uniform probabilities.
Non-Transitive Dice

A: \{\text{\ding{192} \ding{193} \ding{194} \ding{195} \ding{196}}\}  
  \text{\ding{192} \ding{193} \ding{194} \ding{195} \ding{196}} 

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Your friend picks a die; you pick a die. 
e.g. friend picks die B; you pick A.

What is the probability that A beats B?

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- Uniform probabilities.
- Even of interest: outcomes where you win.
Non-Transitive Dice

Your friend picks a die; you pick a die.
e.g. friend picks die $B$; you pick $A$.

What is the probability that $A$ beats $B$?

- Outcome-tree and outcome-probabilities.
- Uniform probabilities.
- Even of interest: outcomes where you win.
- Number of outcomes where you win: 5.
- Probability you win, $\mathbb{P}[A \text{ beats } B] = \frac{5}{9}$. 

Die $A$  Die $B$  Probability

$\begin{array}{ll}
A: & \left\{ \begin{array}{ll}
\CDots & 0.333
\end{array} \right. \\
B: & \left\{ \begin{array}{ll}
\CDots & 0.333
\end{array} \right. \\
C: & \left\{ \begin{array}{ll}
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\end{array}$
Non-Transitive Dice

Your friend picks a die; you pick a die. e.g. friend picks die $B$; you pick $A$.

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- Outcome-tree and outcome-probabilities.
- Uniform probabilities.
- Even of interest: outcomes where you win.
- Number of outcomes where you win: 5.
- Probability you win, $\mathbb{P}[A$ beats $B] = \frac{5}{9}$.

Conclusion: Die $A$ beats Die $B$.

**Sample Space** $\Omega = \{\omega_1, \omega_2, \ldots\}$, set of *possible* outcomes.

**Probability Function** $P(\cdot)$. Non-negative function $P(\omega)$, normalized to 1:

$$0 \leq P(\omega) \leq 1 \quad \text{and} \quad \sum_{\omega \in \Omega} P(\omega) = 1.$$

Die $A$ versus $B$

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>$P(\omega)$</th>
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<tbody>
<tr>
<td>{11, 12, 13, 21, 22, 23, 31, 32, 33}</td>
<td>$\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$</td>
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<tr>
<td>${\text{\ding{58}, \ding{59}, \ding{60}, \ding{61}, \ding{62}, \ding{63}, \ding{64}, \ding{65}, \ding{66}}}$</td>
<td>$\frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9}$</td>
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Events $\mathcal{E} \subseteq \Omega$ are subsets. Event probability $\mathbb{P}[\mathcal{E}]$ is the sum of outcome-probabilities.

"A > B" $\mathcal{E}_1 = \{\text{\ding{58}, \ding{59}, \ding{60}, \ding{61}, \ding{62}}\}$

"Sum > 8" $\mathcal{E}_2 = \{\text{\ding{63}, \ding{64}, \ding{65}, \ding{66}, \ding{67}}\}$

"B < 9" $\mathcal{E}_3 = \{\text{\ding{68}, \ding{69}, \ding{70}, \ding{71}, \ding{72}}\}$
Sample Space $\Omega = \{\omega_1, \omega_2, \ldots\}$, set of possible outcomes.

Probability Function $P(\cdot)$. Non-negative function $P(\omega)$, normalized to 1:

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Die $A$ versus $B$

$\Omega = \{\text{11, 12, 21, 22, 31, 32, 41, 42, 51, 52, 61, 62, 71, 72}\}$

$P(\omega) = \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}$

Events $\mathcal{E} \subseteq \Omega$ are subsets. Event probability $\mathbb{P}[\mathcal{E}]$ is the sum of outcome-probabilities.

"$A > B$" $\quad \mathcal{E}_1 = \{\text{12, 22, 32, 42, 52}\}$

"Sum $> 8$" $\quad \mathcal{E}_2 = \{\text{13, 23, 33, 43, 53}\}$

"$B < 9$" $\quad \mathcal{E}_3 = \{\text{11, 12, 21, 22, 31, 32, 41, 42}\}$

Combining events using logical connectors corresponds to set operations:
Sample Space $\Omega = \{\omega_1, \omega_2, \ldots\}$, set of possible outcomes.

Probability Function $P(\cdot)$. Non-negative function $P(\omega)$, normalized to 1:

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Die $A$ versus $B$

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"$A > B$" $\mathcal{E}_1 = \{\text{ outcomes }\}$

"Sum $> 8$" $\mathcal{E}_2 = \{\text{ outcomes }\}$

"$B < 9$" $\mathcal{E}_3 = \{\text{ outcomes }\}$

Combining events using logical connectors corresponds to set operations:

"$A > B$" $\lor$ "Sum $> 8$" $\mathcal{E}_1 \cup \mathcal{E}_2 = \{\text{ outcomes }\}$

"$A > B$" $\land$ "Sum $> 8$" $\mathcal{E}_1 \cap \mathcal{E}_2 = \{\text{ outcomes }\}$

$\neg (" A > B")$ $\mathcal{E}_1^c = \{\text{ outcomes }\}$

"$A > B$" $\rightarrow$ "$B < 9$" $\mathcal{E}_1 \subseteq \mathcal{E}_3$

Important: Exercise 15.9. Sum rule, complement, inclusion-exclusion, union, implication and intersection bounds.
Uniform Probability Space : Probability $\sim$ Size

$$P(\omega) = \frac{1}{|\Omega|}$$
Uniform Probability Space: Probability $\sim$ Size

\[ P(\omega) = \frac{1}{|\Omega|} \]
\[ \mathbb{P}[\mathcal{E}] = \frac{|\mathcal{E}|}{|\Omega|} = \frac{\text{number of outcomes in } \mathcal{E}}{\text{number of possible outcomes in } \Omega}. \]
Uniform Probability Space: Probability $\sim$ Size

$$P(\omega) = \frac{1}{|\Omega|} \quad \mathbb{P}[\mathcal{E}] = \frac{|\mathcal{E}|}{|\Omega|} = \frac{\text{number of outcomes in } \mathcal{E}}{\text{number of possible outcomes in } \Omega}.$$ 

Toss a coin 3 times:

<table>
<thead>
<tr>
<th>Outcome</th>
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<tbody>
<tr>
<td>HHH</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>HHT</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>HTH</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>HTT</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>THH</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>THT</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>TTH</td>
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Uniform Probability Space: Probability $\sim$ Size

$$P(\omega) = \frac{1}{|\Omega|} \quad \mathbb{P}[\mathcal{E}] = \frac{|\mathcal{E}|}{|\Omega|} = \frac{\text{number of outcomes in } \mathcal{E}}{\text{number of possible outcomes in } \Omega}.$$ 

Toss a coin 3 times:

$$\mathbb{P}[\text{“2 heads”}] = \frac{\text{number of sequences with 2 heads}}{\text{number of possible sequences in } \Omega} = \binom{3}{2} \times \frac{1}{8} = \frac{3}{8}. $$
Uniform Probability Space : Probability $\sim$ Size

\[ P(\omega) = \frac{1}{|\Omega|} \quad \text{and} \quad \mathbb{P}[\mathcal{E}] = \frac{|\mathcal{E}|}{|\Omega|} = \frac{\text{number of outcomes in } \mathcal{E}}{\text{number of possible outcomes in } \Omega}. \]

Toss a coin 3 times:

\[
\begin{array}{c}
\text{Toss 1} \\
\text{Toss 2} \\
\text{Toss 3} \\
\text{Outcome} \\
\text{Probability}
\end{array}
\]

\[
\begin{array}{cccccc}
\text{HHH} & \text{HHT} & \text{HTH} & \text{HTT} & \text{THH} & \text{THT} & \text{TTH} & \text{TTT} \\
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8}
\end{array}
\]

\[
\mathbb{P}[\text{"2 heads"}] = \frac{\text{number of sequences with 2 heads}}{\text{number of possible sequences in } \Omega} = \left(\frac{3}{2}\right) \times \frac{1}{8} = \frac{3}{8}.
\]

Practice: Exercise 15.10.

1. You roll a pair of regular dice. What is the probability that the sum is 9?
2. You toss a fair coin ten times. What is the probability that you obtain 4 heads?
3. You roll die A ten times. Compute probabilities for: 4 sevens? 4 sevens and 3 sixes? 4 sevens or 3 sixes?
52 card deck has 4 suits (♠, ♥, ♦, ♣) and 13 ranks in a suit (A,K,Q,J,T,9,8,7,6,5,4,3,2).

Randomly deal 5-cards: each set of 5 cards is equally likely → uniform probability space.

number of possible outcomes = \( \binom{52}{5} \) possible hands.
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**Full house:** 3 cards of one rank and 2 of another. How many full-houses?
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**Full house:** 3 cards of one rank and 2 of another. How many full-houses?
To construct a full house, specify \((\text{rank}_3, \text{ suits}_3, \text{rank}_2, \text{ suits}_2)\). Product rule:
52 card deck has 4 suits (♠, ♥, ♦, ♣) and 13 ranks in a suit (A,K,Q,J,T,9,8,7,6,5,4,3,2).

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\[
\# \text{ full houses} = 13 \times \binom{4}{3} \times 12 \times \binom{4}{2}
\]
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**Full house:** 3 cards of one rank and 2 of another. How many full-houses?
To construct a full house, specify \((\text{rank}_3, \text{suits}_3, \text{rank}_2, \text{suits}_2)\). Product rule:

\[
\# \text{ full houses} = 13 \times \binom{4}{3} \times 12 \times \binom{4}{2} \quad \rightarrow \quad P[\text{“FullHouse”}] = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} \approx 0.00144;
\]
Poker: Probabilities of Full House and Flush

52 card deck has 4 suits (♠, ♥, ♦, ♣) and 13 ranks in a suit (A,K,Q,J,T,9,8,7,6,5,4,3,2).

Randomly deal 5-cards: each set of 5 cards is equally likely → uniform probability space.

number of possible outcomes = \( \binom{52}{5} \) possible hands.

**Full house:** 3 cards of one rank and 2 of another. How many full-houses?
To construct a full house, specify (rank\(_3\), suits\(_3\), rank\(_2\), suits\(_2\)). Product rule:

\[
\# \text{ full houses} = 13 \times \binom{4}{3} \times 12 \times \binom{4}{2} \quad \rightarrow \quad P[\text{“FullHouse”}] = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} \approx 0.00144;
\]

**Flush:** 5 cards of same suit. How many flushes?
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To construct a flush, specify \((\text{suit}, \text{ranks})\). Product rule:
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\# \text{ flushes} = 4 \times \binom{13}{5} \quad \rightarrow \quad \mathbb{P}[\text{“Flush”}] = \frac{4 \times \binom{13}{5}}{\binom{52}{5}} \approx 0.00198;
\]

Full house is rarer. That’s why full house beats flush.
Toss a Coin Until Heads: Infinite Probability Space

Toss 1

\[ \frac{1}{2} \]

T

\[ \frac{1}{2} \]

H

H

\[ \frac{1}{2} \]
Toss a Coin Until Heads: Infinite Probability Space

Creator: Malik Magdon-Ismail
Probability: 13 / 14
Game: First Person To Toss H Wins →
Toss a Coin Until Heads: Infinite Probability Space

Game: First Person To Toss H Wins

Probability: 13/14

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Game: First Person To Toss H Wins

Outcome Probability

\[
\begin{array}{ccccccccc}
\text{Toss 1} & \text{Toss 2} & \text{Toss 3} & \text{Toss 4} & \text{Toss 5} & \text{Toss 6} & \cdots & \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \\
\text{H} & \text{TH} & \text{TTH} & \text{TTTH} & \text{TTTTH} & \text{TTTTTH} & \cdots & \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots & \\
\end{array}
\]
Toss a Coin Until Heads: Infinite Probability Space

\[ \Omega = \left\{ \omega \right\} \]

\[ P(\omega) = \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4, \left(\frac{1}{2}\right)^5, \left(\frac{1}{2}\right)^6, \ldots, \left(\frac{1}{2}\right)^{i+1}, \ldots \]

\[
\begin{array}{|c|cccccccc|}
\hline
# Tosses & 1 & 2 & 3 & 4 & 5 & 6 & \ldots & i + 1 & \ldots \\
\hline
\end{array}
\]
Toss a Coin Until Heads: Infinite Probability Space

<table>
<thead>
<tr>
<th>Toss 1</th>
<th>Toss 2</th>
<th>Toss 3</th>
<th>Toss 4</th>
<th>Toss 5</th>
<th>Toss 6</th>
<th>⋮</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>⋮</td>
</tr>
<tr>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>⋮</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>⋮</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>$H$</th>
<th>$TH$</th>
<th>$T^2H$</th>
<th>$T^3H$</th>
<th>$T^4H$</th>
<th>$T^5H$</th>
<th>⋮</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(\omega)$</td>
<td>$\frac{1}{2}$</td>
<td>$(\frac{1}{2})^2$</td>
<td>$(\frac{1}{2})^3$</td>
<td>$(\frac{1}{2})^4$</td>
<td>$(\frac{1}{2})^5$</td>
<td>$(\frac{1}{2})^6$</td>
<td>⋮</td>
</tr>
<tr>
<td># Tosses</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>⋮</td>
</tr>
</tbody>
</table>

Sum of outcome probabilities:

$$\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + (\frac{1}{2})^4 + \cdots = \sum_{i=1}^{\infty} (\frac{1}{2})^i = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1. \checkmark$$
Game: First Person To Toss H Wins. Always Go First

- Toss 1
- Toss 2
- Toss 3
- Toss 4
- Toss 5
- Toss 6

<table>
<thead>
<tr>
<th>(\Omega)</th>
<th>H</th>
<th>TH</th>
<th>T(^2)H</th>
<th>T(^3)H</th>
<th>T(^4)H</th>
<th>T(^5)H</th>
<th>...</th>
<th>T(^i)H</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(\omega))</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})^2</td>
<td>(\frac{1}{2})^3</td>
<td>(\frac{1}{2})^4</td>
<td>(\frac{1}{2})^5</td>
<td>(\frac{1}{2})^6</td>
<td>...</td>
<td>(\frac{1}{2})^{i+1}</td>
<td>...</td>
</tr>
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Outcome probabilities:

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<th>...</th>
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<th>...</th>
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<td>...</td>
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Game: First Person To Toss H Wins. Always Go First

The event “YouWin” is $\mathcal{E} = \{H, T^2H, T^4H, T^6H, \ldots \}$. 

Creator: Malik Magdon-Ismail

Probability: 14 / 14
Game: First Person To Toss H Wins. Always Go First

The event “YouWin” is \( \mathcal{E} = \{H, T\cdot^2H, T\cdot^4H, T\cdot^6H, \ldots\} \).

\[
P[\text{“YouWin”}] = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^7 + \cdots = \frac{1}{2} \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i = \frac{1}{2} \frac{1}{1 - \frac{1}{4}} = \frac{2}{3}.
\]

Your odds improve by a factor of 2 if you go first (vs. second).