Foundations of Computer Science
Lecture 20

Expected Value of a Sum

Linearity of Expectation
Iterated Expectation
Build-Up Expectation
Sum of Indicators
1. Sample average and expected value.

2. Definition of Mathematical expectation.

3. Examples: Sum of dice; Bernoulli; Uniform; Binomial; waiting time;


5. Law of Total Expectation.
Today: Expected Value of a Sum

1. Expected value of a sum.
   - Sum of dice.
   - Binomial.
   - Waiting time.
   - Coupon collecting.

2. Iterated expectation.

3. Build-up expectation.

4. Expected value of a product.

5. Sum of indicators.
Expected Value of a Sum

You expect to win twice as much from two lottery tickets as from one.
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Theorem (Linearity of Expectation). Let $X_1, X_2, \ldots, X_k$ be random variables and let $Z = a_1X_1 + a_2X_2 + \cdots + a_kX_k$ be a linear combination of the $X_i$. Then,
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$$\mathbb{E}[Z] = \mathbb{E}[a_1X_1 + a_2X_2 + \cdots + a_kX_k] = a_1 \mathbb{E}[X_1] + a_2 \mathbb{E}[X_2] + \cdots + a_k \mathbb{E}[X_k].$$
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E[Z] = E[a_1 X_1 + a_2 X_2 + \cdots + a_k X_k] = a_1 E[X_1] + a_2 E[X_2] + \cdots + a_k E[X_k].
$$

**Proof.**

$$
E[Z] = \sum_{\omega \in \Omega} \left( a_1 X_1(\omega) + a_2 X_2(\omega) + \cdots + a_k X_k(\omega) \right) \cdot P(\omega)
$$
Expected Value of a Sum

You expect to win twice as much from two lottery tickets as from one.

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= a_1 \sum_{\omega \in \Omega} X_1(\omega) \cdot P(\omega) + a_2 \sum_{\omega \in \Omega} X_2(\omega) \cdot P(\omega) + \cdots + a_k \sum_{\omega \in \Omega} X_k(\omega) \cdot P(\omega)
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**Proof.**

$$E[Z] = \sum_{\omega \in \Omega} \left( a_1 X_1(\omega) + a_2 X_2(\omega) + \cdots + a_k X_k(\omega) \right) \cdot P(\omega)$$

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$$= a_1 E[X_1] + a_2 E[X_2] + \cdots + a_k E[X_k].$$
Expected Value of a Sum

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Proof. \[
E[Z] = \sum_{\omega \in \Omega} (a_1X_1(\omega) + a_2X_2(\omega) + \cdots + a_kX_k(\omega)) \cdot P(\omega)
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\]

\[
= a_1 E[X_1] + a_2 E[X_2] + \cdots + a_k E[X_k].
\]

1. Summation can be taken inside or pulled outside an expectation.
2. Constants can be taken inside or pulled outside an expectation.

\[
E \left[ \sum_{i=1}^{k} a_iX_i \right] = \sum_{i=1}^{k} a_i E[X_i]
\]
Let $X$ be the sum of $n$ fair dice,
Sum of Dice

Let $\mathbf{X}$ be the sum of $n$ fair dice,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n,$$

where $\mathbf{X}_i$ is the value rolled by die $i$. 

Creator: Malik Magdon-Ismail

Expected Value of a Sum: 5/12

Expected Number of Successes →
Let $\mathbf{X}$ be the sum of $n$ fair dice,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n,$$

where $\mathbf{X}_i$ is the value rolled by die $i$.

$$\mathbb{E}[\mathbf{X}_i] = \frac{3}{2}.$$
Let $X$ be the sum of $n$ fair dice,

$$X = X_1 + \cdots + X_n,$$

where $X_i$ is the value rolled by die $i$.

$$\mathbb{E}[X_i] = 3\frac{1}{2}.$$ 

Linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n \times 3\frac{1}{2}.$$
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**Example.** The expected sum of 4 dice is $4 \times \frac{3}{2} = 14$.

**Exercise.** Compute the PDF for the sum of 4 dice and expected value from the PDF.
\( \mathbf{X} \) is the number of successes in \( n \) trials with success probability \( p \) per trial,
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\[
\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n
\]

Each \( \mathbf{X}_i \) is a Bernoulli and

\[
\mathbb{E}[\mathbf{X}_i] = p.
\]
**Expected Number of Successes in $n$ Coin Tosses**

$X$ is the number of successes in $n$ trials with success probability $p$ per trial,

$$X = X_1 + \cdots + X_n$$

Each $X_i$ is a Bernoulli and

$$\mathbb{E}[X_i] = p.$$ 

Linearity of expectation,

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] = n \times p.$$
Expected Waiting Time to $n$ Successes

$X$ is the waiting time for $n$ successes with success probability $p$. 
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$$X = \text{wait to 1st} + \frac{\text{wait from 1st to 2nd}}{X_2} + \frac{\text{wait from 2nd to 3rd}}{X_3} + \cdots + \frac{\text{wait from $(n-1)$th to $n$th}}{X_n}$$
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$X$ is the waiting time for $n$ successes with success probability $p$.

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X = X_1 + X_2 + X_3 + \cdots + X_n.
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\[
= X_1 + X_2 + X_3 + \cdots + X_n.
\]

Each $X_i$ is a waiting time to one success, so

\[
\mathbb{E}[X_i] = \frac{1}{p}.
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Expected Waiting Time to \( n \) Successes

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\[
= E[X_1] + E[X_2] + \cdots + E[X_n]
\]

\[
= \frac{1}{p} + \frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p} = n \times \frac{1}{p}.
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= \frac{1}{p} + \frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p} = n \times \frac{1}{p}.
\]

**Example.** If you are waiting for 3 boys, you have to wait 3-times as long as for 1 boy.

**Exercise.** Compute the expected square of the waiting time.
Coupon Collecting: Collecting the Flags

A pack of gum comes with a flag (169 countries). $X$ is the number of gum-purchases to get all the flags.
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\]

\[
X_1 \quad X_1 \quad X_1 \quad X_1
\]

\[
p_1 = \frac{n}{n} \quad p_2 = \frac{n-1}{n} \quad p_3 = \frac{n-2}{n} \quad p_n = \frac{n-(n-1)}{n}
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\[
\begin{align*}
\uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
X_1 & \quad X_1 & \quad X_1 & \quad X_1 \\
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$\mathbb{E}[X_i] = 1/p_i,$
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Linearity of expectation:

\[E[X] = n\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{1}\right) = nH_n \approx n(\ln n + 0.577).\]
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$n = 169 \rightarrow$ you expect to buy about 965 packs of gum. Lots of chewing!
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\( n = 169 \rightarrow \) you expect to buy about 965 packs of gum. Lots of chewing!

**Example.** Cereal box contains 1-of-5 cartoon characters. Collect all to get $2 rebate.

Expect to buy about 12 cereal boxes. If a cereal box costs $5, that’s a whopping 3\%\% discount.
Iterated Expectation

**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.
Iterated Expectation

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An example outcome is $(4; 2, 1, 2, 6)$ with $X_1 = 4$ and $X_2 = 11$: 
**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.

An example outcome is $(4; 2, 1, 2, 6)$ with $X_1 = 4$ and $X_2 = 11$:

$$
\mathbb{E}[X_2 \mid X_1] = X_1 \times 3^\frac{1}{2}.
$$

The RHS is a *function* of $X_1$, a random variable.
**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.

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\mathbb{E}[X_2 | X_1] = X_1 \times 3\frac{1}{2}.
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The RHS is a function of $X_1$, a random variable. Compute its expectation.

$$
\mathbb{E}[X_2] = \mathbb{E}_{X_1}[\mathbb{E}[X_2 | X_1]]
$$

(another version of total expectation)
**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.

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$$E[X_2] = E_{X_1}[E[X_2 | X_1]] = E[X_1] \times 3\frac{1}{2} \quad \text{(another version of total expectation)}$$
**Iterated Expectation**

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\mathbb{E}[X_2 \mid X_1] = X_1 \times 3^{\frac{1}{2}}.
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\mathbb{E}[X_2] = \mathbb{E}_{X_1}[\mathbb{E}[X_2 \mid X_1]] = \mathbb{E}[X_1] \times 3^{\frac{1}{2}} = 3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 12^{\frac{1}{2}}.
$$
**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.

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$$E[X_2 | X_1] = X_1 \times 3\frac{1}{2}.$$ 

The RHS is a function of $X_1$, a random variable. Compute its expectation.

$$E[X_2] = E_{X_1}[E[X_2 | X_1]]$$

$$= E[X_1] \times 3\frac{1}{2}$$

$$= 3\frac{1}{2} \times 3\frac{1}{2} = 12\frac{1}{2}.$$ 

**Exercise.** Justify this computation using total expectation with 6 cases:
Iterated Expectation

**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.

An example outcome is $(4; 2, 1, 2, 6)$ with $X_1 = 4$ and $X_2 = 11$:

$$\mathbb{E}[X_2 \mid X_1] = X_1 \times 3\frac{1}{2}.$$  

The RHS is a function of $X_1$, a random variable. Compute its expectation.

$$\mathbb{E}[X_2] = \mathbb{E}_{X_1}[\mathbb{E}[X_2 \mid X_1]] = \mathbb{E}[X_1] \times 3\frac{1}{2} = 3\frac{1}{2} \times 3\frac{1}{2} = 12\frac{1}{2}.$$  

**Exercise.** Justify this computation using total expectation with 6 cases:

$$\mathbb{E}[X_2] = \mathbb{E}[X_2 \mid X_1 = 1] \cdot \mathbb{P}[X_1 = 1] + \mathbb{E}[X_2 \mid X_1 = 2] \cdot \mathbb{P}[X_1 = 2] + \cdots + \mathbb{E}[X_2 \mid X_1 = 6] \cdot \mathbb{P}[X_1 = 6].$$
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}] . \]
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}] \]

The first child is either a boy or girl, so by total expectation,

\[
W(k, l) = \mathbb{E}[\text{waiting time | boy}] \times \frac{p}{1 + W(k-1, \ell)} + \mathbb{E}[\text{waiting time | girl}] \times \frac{1-p}{1 + W(k, \ell-1)}
\]

\[ \mathbb{E}[X] = 12.156 \]

Expected Value of a Sum: 10 / 12

Expected Value of a Product →
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

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The first child is either a boy or girl, so by total expectation,

\[
W(k, l) = \frac{\mathbb{E}[\text{waiting time } | \text{ boy}]}{1 + W(k-1, \ell)} \times p + \frac{\mathbb{E}[\text{waiting time } | \text{ girl}]}{1 + W(k, \ell-1)} \times (1-p)
\]

\[
= 1 + pW(k - 1, \ell) + (1 - p)W(k, \ell - 1).
\]
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}] . \]

The first child is either a boy or girl, so by total expectation,

\[ W(k, \ell) = \frac{\mathbb{E}[\text{waiting time } | \text{ boy}]}{1 + W(k-1, \ell)} \times p + \frac{\mathbb{E}[\text{waiting time } | \text{ girl}]}{1 + W(k, \ell-1)} \times (1-p) \]

\[ = 1 + pW(k-1, \ell) + (1-p)W(k, \ell-1) . \]

Base cases: \( W(k, 0) = k/p \) and \( W(0, \ell) = \ell/(1-p) \)

<table>
<thead>
<tr>
<th>( W(k, \ell) )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>\cdots</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>\cdots</td>
</tr>
<tr>
<td>( k )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>\vdots</td>
<td>2</td>
<td>4</td>
<td>\vdots</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>12</td>
<td>\cdots</td>
</tr>
</tbody>
</table>

\( \mathbb{E}[X] = 12.156 \)
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = E[\text{waiting time to } k \text{ boys and } \ell \text{ girls}]. \]

The first child is either a boy or girl, so by total expectation,

\[
W(k, \ell) = \mathbb{E}[\text{waiting time } | \text{ boy}] \times \mathbb{P}[\text{boy}] + \mathbb{E}[\text{waiting time } | \text{ girl}] \times \mathbb{P}[\text{girl}]
\]

\[
= 1 + pW(k - 1, \ell) + (1 - p)W(k, \ell - 1).
\]

Base cases: \( W(k, 0) = k/p \) and \( W(0, \ell) = \ell/(1 - p) \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>\cdots</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>\cdots</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>\times p</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>\cdots</td>
</tr>
<tr>
<td>2</td>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}] . \]

The first child is either a boy or girl, so by total expectation,

\[
W(k, \ell) = \mathbb{E} [\text{waiting time} \mid \text{boy}] \times p + \mathbb{E} [\text{waiting time} \mid \text{girl}] \times (1-p) \\
= 1 + pW(k-1, \ell) + (1-p)W(k, \ell-1).
\]

Base cases: \( W(k, 0) = k/p \) and \( W(0, \ell) = \ell/(1-p) \)

| \( W(k, \ell) \) | 0 | 1 \( \times p \) | 2 | 3 | 4 | 5 | 6 | 7 | \( \ell \) |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | \( \cdots \) |
| 1 | \( 2 \times (1-p) \) | +1 | 3 | 4.5 | 6.25 | 8.13 | 10.06 | 12.03 | 14.02 | \( \cdots \) |
| \( k \) | | | | | | | | | |
| 2 | | | | | | | | | |
| \( \vdots \) | | | | | | | | | |
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}] . \]

The first child is either a boy or girl, so by total expectation,

\[
W(k, \ell) = \mathbb{E}[\text{waiting time } | \text{ boy}] \times \mathbb{P}[\text{boy}] + \mathbb{E}[\text{waiting time } | \text{ girl}] \times \mathbb{P}[\text{girl}] \\
= \frac{1}{1 + W(k-1, \ell)} + p \times \frac{1}{1 + W(k, \ell-1)}.
\]

Base cases: \( W(k, 0) = k/p \) and \( W(0, \ell) = \ell/(1 - p) \)

\[
\begin{array}{ccccccccc}
W(k, \ell) & | & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\hline
0 & | & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & \cdots \\
1 & | & 2 & +1 & 3 & 4.5 & 6.25 & 8.13 & 10.06 & 12.03 & 14.02 & \cdots \\
2 & | & 4 & 4.5 & 5.5 & 6.88 & 8.5 & 10.28 & 12.16 & 14.09 & \cdots \\
\vdots & | & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]
Expected Value of a Product

\( \mathbf{X} \) is a single die roll:
Expected Value of a Product

\( X \) is a single die roll:

\[
E[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
\]
**Expected Value of a Product**

\( \mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}. \)

\[ \mathbb{E}[X^2] = \mathbb{E}[X \times X] \]
Expected Value of a Product

**X** is a single die roll:

\[
\mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15 \frac{1}{6}.
\]

\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X]
\]
Expected Value of a Product

\(X\) is a single die roll:

\[
\mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
\]

\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X] = (3\frac{1}{2})^2 = 12\frac{1}{4} \times .
\]
Expected Value of a Product

\( \mathbf{X} \) is a single die roll:

\[
\mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
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\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X] = (3\frac{1}{2})^2 = 12\frac{1}{4}. \times
\]

\( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) are independent die rolls:

\begin{center}
\begin{tabular}{c|cccccc}
\hline
Die 1 Value & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
Die 2 Value & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\end{tabular}
\end{center}
Expected Value of a Product

\( \mathbf{X} \) is a single die roll:

\[
E[\mathbf{X}^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
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\[
E[\mathbf{X}^2] = E[\mathbf{X} \times \mathbf{X}] = E[\mathbf{X}] \times E[\mathbf{X}] = \left(3\frac{1}{2}\right)^2 = 12\frac{1}{4}. \times
\]

\( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) are independent die rolls:

\[
E[\mathbf{X}_1 \mathbf{X}_2] = \frac{1}{36}(1+2+\cdots+6+2+4+\cdots+12+3+6+\cdots+18+\cdots+6+12+\cdots+36)
\]

\[
= \frac{441}{36} = 12\frac{1}{4}.
\]
Expected Value of a Product

\( \mathbf{X} \) is a single die roll:

\[
\mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
\]

\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X] = \left(\frac{3\frac{1}{2}}{2}\right)^2 = 12\frac{1}{4}. \quad \text{✗}
\]

\( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) are independent die rolls:

\[
\mathbb{E}[X_1 X_2] = \frac{1}{36}(1+2+\cdots+6+2+4+\cdots+12+3+6+\cdots+18+\cdots+6+12+\cdots+36)
\]
\[
= \frac{441}{36} = 12\frac{1}{4}.
\]

\[
\mathbb{E}[X_1 X_2] = \mathbb{E}[X_1] \times \mathbb{E}[X_2] = \left(\frac{3\frac{1}{2}}{2}\right)^2 = 12\frac{1}{4}. \quad \text{✓}
\]
**Expected Value of a Product**

**X** is a single die roll:

\[
\mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
\]

\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X] = (3\frac{1}{2})^2 = 12\frac{1}{4} \quad \text{✗}
\]

**X_1** and **X_2** are independent die rolls:

\[
\mathbb{E}[X_1X_2] = \frac{1}{36}(1+2+\ldots+6+2+4+\ldots+12+3+6+\ldots+18+\ldots+6+12+\ldots+36)
\]
\[
= \frac{441}{36} = 12\frac{1}{4}.
\]

\[
\mathbb{E}[X_1X_2] = \mathbb{E}[X_1] \times \mathbb{E}[X_2] = (3\frac{1}{2})^2 = 12\frac{1}{4} \quad \text{✓}
\]

---

**Expected value of a product XY.**

- **In general,** the expected product is **not** a product of expectations.
- **For independent** random variables, it is: \( \mathbb{E}[XY] = \mathbb{E}[X] \times \mathbb{E}[Y] \).
Sum of Indicators: Successes in a Random Assignment

\( X \) is the number of correct hats when 4 hats randomly land on 4 heads.
X is the number of correct hats when 4 hats randomly land on 4 heads.
$X$ is the number of correct hats when 4 hats randomly land on 4 heads.

- **hats:** 4 2 3 1
- **men:** 1 2 3 4

$X_1 = 0$  $X_2 = 1$  $X_3 = 1$  $X_4 = 0$
\( X \) is the number of correct hats when 4 hats randomly land on 4 heads.

\[
\begin{align*}
\text{hats:} & \quad 4 & \quad 2 & \quad 3 & \quad 1 \\
\text{men:} & \quad \text{①} & \quad \text{②} & \quad \text{③} & \quad \text{④} \\
X_1 & = 0 & X_2 & = 1 & X_3 & = 1 & X_4 & = 0 \\
\end{align*}
\]

\[
X = X_1 + X_2 + X_3 + X_4 = 2
\]
Sum of Indicators: Successes in a Random Assignment

\( X \) is the number of correct hats when 4 hats randomly land on 4 heads.

\[
\begin{align*}
\text{hats:} & & 4 & 2 & 3 & 1 \\
\text{men:} & & 1 & 2 & 3 & 4 \\
X_1 &= 0 & X_2 &= 1 & X_3 &= 1 & X_4 &= 0 \\
\end{align*}
\]

\[
X = X_1 + X_2 + X_3 + X_4 = 2
\]

\( X_i \) are Bernoulli with \( P[X_i = 1] = \frac{1}{4} \).
**Sum of Indicators: Successes in a Random Assignment**

**X** is the number of correct hats when 4 hats randomly land on 4 heads.

<table>
<thead>
<tr>
<th>hats:</th>
<th>4</th>
<th>2</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>men:</td>
<td>①</td>
<td>②</td>
<td>③</td>
<td>④</td>
</tr>
</tbody>
</table>

\[ X_1 = 0 \quad X_2 = 1 \quad X_3 = 1 \quad X_4 = 0 \]

\[ X = X_1 + X_2 + X_3 + X_4 = 2 \]

**X**\(_i\) are Bernoulli with \( P[X_i = 1] = \frac{1}{4} \). Linearity of expectation:

\[
E[X] = E[X_1] + E[X_2] + E[X_3] + E[X_4] = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 4 \times \frac{1}{4} = 1.
\]
Sum of Indicators: Successes in a Random Assignment

\( \mathbf{X} \) is the number of correct hats when 4 hats randomly land on 4 heads.

<table>
<thead>
<tr>
<th>hats:</th>
<th>4</th>
<th>2</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>men:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
X_1 &= 0 \\
X_2 &= 1 \\
X_3 &= 1 \\
X_4 &= 0
\end{align*}
\]

\[
\mathbf{X} = X_1 + X_2 + X_3 + X_4 = 2
\]

\( X_i \) are Bernoulli with \( P[X_i = 1] = \frac{1}{4} \). Linearity of expectation:

\[
\mathbb{E}[\mathbf{X}] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4] = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 4 \times \frac{1}{4} = 1.
\]

**Exercise.** What about if there are \( n \) people?

**Interesting Example (see text).** Apply sum of indicators to breaking of records.

**Instructive Exercise.** Compute the PDF of \( \mathbf{X} \) and the expectation from the PDF.