Deviations from the Mean

Measuring Deviations from the mean
Variance and Standard deviation.
Bounding Deviations: Markov and Chebychev.
The 3-σ Rule.
1. Expected Value of a Sum is the Sum of the Expectations:

\[ \mathbb{E} \left[ \sum_i a_i X_i \right] = \sum_i a_i \mathbb{E} [X_i]. \]

2. Examples: Dice sum; Binomial; Coupon collector; Random walk.

3. Expected value of a product is the product of expected values?
   Example: let \( R_1, R_2 \) be rolls of independent dice. \( \mathbb{E}[R_1^2] \neq \mathbb{E}[R_1]^2; \mathbb{E}[R_1 R_2] = \mathbb{E}[R_1] \mathbb{E}[R_2]. \)

4. Is \( \mathbb{E}[1/X] = 1/ \mathbb{E} [X]? \)

5. Example to introduce variance: expected value is not all there is to the game.
Today: Variance

1. \( \text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] \).
   Example with variance of a dice roll.

2. \( \text{var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \).
   Examples: Binary, Uniform distributions. \( \text{var}(aX) = a^2 \text{var}(X); \text{var}(X + a) = \text{var}(X) \).
   Law of total expectation gives: \( \text{var}[T] = \frac{1-p}{p^2} \) (tries to success).

3. Independent random variables:
   \[ \text{var}(X_1 + X_2 + \cdots + X_n) = \text{var}(X_1) + \text{var}(X_2) + \cdots + \text{var}(X_n). \]

4. Examples: Binomial distribution.

5. Markov: \( \mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t} \) for a positive random variable \( X \).

6. Chebyshev:
   \[ \mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\text{var}(X)}{t^2} \rightarrow X = \mathbb{E}[X] \pm 3\sigma, \]
   with probability at least \( 8/9 \): the 3-\( \sigma \) rule.