Foundations of Computer Science
Lecture 22

Infinity

Size versus Cardinality: Comparing “Sizes”
Countable: Sets Which Are Not “Larger” Than N
Is There A Set “Larger” Than N? Cantor’s Diagonal Argument
Infinity and Computing
Precise statements, proofs and logic.

INDUCTION.

Recursively defined structures and Induction. (Data structures; PL)

Sums and asymptotics. (Algorithm analysis)

Number theory. (Cryptography; probability; fun)

Graphs. (Relationships/conflicts; resource allocation; routing; scheduling, . . . )

Counting. (Enumeration and brute force algorithms)

Probability. (Real world algorithms involve randomness/uncertainty)
  ▶ Inputs arrive in a random order;
  ▶ Randomized algorithms (primality testing, machine learning, routing, conflict resolution . . . )
Today: Infinity

   - Rationals are countable.

2. Uncountable
   - Infinite binary strings.

3. What does Infinity have to do with computing?
“Size” of a Set: Cardinality

You have 5 fingers on each hand.

You must know how to count.
“Size” of a Set: Cardinality

You have 5 fingers on each hand.

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You have an *equal* number of fingers on each hand.
“Size” of a Set: Cardinality

You have 5 fingers on each hand.
    You must know how to count.

You have an equal number of fingers on each hand.
    All you need is a correspondence.
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\[ A \quad B \]

not a function
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\begin{figure}
\centering
\begin{tabular}{cc}
\begin{tikzpicture}
  \node (A) at (0,0) {\textbf{A}};
  \node (B) at (2,0) {\textbf{B}};
  \draw[->,thick] (A) -- (B);
\end{tikzpicture}
& \begin{tikzpicture}
  \node (A) at (0,0) {\textbf{A}};
  \node (B) at (2,0) {\textbf{B}};
  \draw[->,thick] (A) -- (B);
\end{tikzpicture}
\\
\text{not a function} & \text{1-to-1; (injection, } A \xrightarrow{\text{inj}} B) \text{ implies } |A| \leq |B|\\
\end{tabular}
\end{figure}
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\[
\begin{align*}
A & \quad B \\
\text{not a function} & \\
\end{align*}
\]

\[
\begin{align*}
A & \quad B \\
\text{1-to-1;} & \\
\text{(injection, } A^\text{INJ} B) & \\
\text{implies } |A| \leq |B| & \\
\end{align*}
\]

\[
\begin{align*}
A & \quad B \\
\text{onto;} & \\
\text{(surjection, } A^\text{SUR} B) & \\
\text{implies } |A| \geq |B| & \\
\end{align*}
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\[
\begin{align*}
A & \quad B \\
\text{not a function} & \quad 1\text{-to-1;} & \quad A \overset{\text{inj}}{\longrightarrow} B \\
& \quad \text{(injection, } A \overset{\text{inj}}{\longrightarrow} B) \quad \text{implies } |A| \leq |B| \\
\text{onto;} & \quad (\text{surjection, } A \overset{\text{sur}}{\longrightarrow} B) \quad \text{implies } |A| \geq |B| \\
1\text{-to-1 and onto} & \quad (\text{bijection, } A \overset{\text{bij}}{\longrightarrow} B) \quad \text{implies } |A| = |B|
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$A \rightarrow B$

not a function

1-to-1;

(injection, $A \overset{\text{inj}}{\rightarrow} B$)

implies $|A| \leq |B|$

onto;

(surjection, $A \overset{\text{sur}}{\rightarrow} B$)

implies $|A| \geq |B|$

1-to-1 and onto

(bijection, $A \overset{\text{bij}}{\rightarrow} B$)

implies $|A| = |B|$

*Cardinality* ("size"). $|A|$, read "cardinality of $A$." (Number of elements for finite sets)
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\[
\begin{array}{c|c}
A & B \\
\hline
\text{not a function} & \\
\end{array}
\]

\[
\begin{array}{c|c}
A & B \\
\hline
1-to-1; & \text{(injection, } A^{\text{INJ}} \text{ B)} \\
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\begin{array}{c|c}
A & B \\
\hline
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**Cardinality** ("size"). \(|A|\), read “cardinality of \(A\).” (Number of elements for finite sets)

\[|A| \leq |B| \text{ iff there is an injection (1-to-1) from } A \text{ to } B, \text{ i.e., } f : A^{\text{INJ}} B.\]
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\[|A| > |B| \text{ iff there is no injection from } A \text{ to } B.\]
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A & \quad B \\
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\end{align*}
\]

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\[
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|A| & = |B| \text{ iff there is an bijection (1-to-1 and onto) from } A \text{ to } B, \text{ i.e., } f : A \overset{\text{BIJ}}{\rightarrow} B.
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\[
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\]

\[
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\]

\[
|A| \geq |B| \quad \text{iff there is a surjection (onto) from } A \text{ to } B, \text{ i.e., } f : A \text{ sur} \to B.
\]

\[
|A| = |B| \quad \text{iff there is a bijection (1-to-1 and onto) from } A \text{ to } B, \text{ i.e., } f : A \text{ bij} \to B.
\]

\[
|A| \leq |B| \quad \text{AND} \quad |B| \leq |A| \quad \to \quad |A| = |B|. \quad \text{(Cantor-Bernstein Theorem)}
\]
Finite sets: $|A| = n$ if and only if there is a bijection from $A$ to $\{1, \ldots, n\}$. 
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Infinite sets: The set $A$ is countable if $|A| \leq |\mathbb{N}|$. $A$ is “smaller than” $\mathbb{N}$. 
A Countable Set’s Cardinality Is At Most $|\mathbb{N}|$

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\[
\begin{align*}
A : & \quad \text{••} \quad \text{••} \quad \text{••} \quad \star \quad \star \quad \star \quad \star \quad \star \quad \blacksquare \quad \blacklozenge \quad \circ \quad \cdots \\
\mathbb{N} : & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19 \quad 20 \quad \cdots
\end{align*}
\]
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\[
A : \ldots \bullet \bullet \bullet \times \times \diamond \diamond \star \star \bigtriangleup \square \circ \circ \cdots \\
\mathbb{N} : 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 18 \ 19 \ 20 \ \cdots
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\[\begin{array}{cccccccccccccccccc}
A : & \bigcirc & \bullet & \heartsuit & \star & \bigstar & \blacktriangle & \square & \bigcirc & \cdots \\
\downarrow & & & & & & & & & \\
\mathbb{N} : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \cdots
\end{array}\]
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\begin{align*}
A : & \quad \circ \quad \bullet \quad \otimes \quad \boxtimes \quad \times \quad \lozenge \quad \ast \quad \blacklozenge \quad \triangle \quad \square \quad \blacksquare \quad \bigcirc \quad \cdots \\
\mathbb{N} : & \quad \begin{array}{ccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \cdots
\end{array}
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\[ A : \quad \begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc\n\]
\[ N : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19 \quad 20 \quad \cdots \]

Creator: Malik Magdon-Ismail

Infinity: 5 / 14

All Finite Sets are Countable →
A Countable Set’s Cardinality Is At Most $\vert \mathbb{N} \vert$

Finite sets: $\vert A \vert = n$ if and only if there is a bijection from $A$ to $\{1, \ldots, n\}$.

Infinite sets: The set $A$ is countable if $\vert A \vert \leq \vert \mathbb{N} \vert$. $A$ is “smaller than” $\mathbb{N}$.

To show that $A$ is countable you must find a 1-to-1 mapping from $A$ to $\mathbb{N}$.

\[ A : \quad \bullet \circ \star \times \diamond \star \bullet \circ \star \square \diamond \circ \cdots \]

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You cannot skip over any elements of $A$, but you might not use every element of $\mathbb{N}$. 
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To show that $A$ is countable you must find a 1-to-1 mapping from $A$ to $\mathbb{N}$.

To prove that a function $f : A \mapsto \mathbb{N}$ is an injection:
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To prove that a function $f : A \mapsto \mathbb{N}$ is an injection:

1: Assume $f$ is not an injection. (Proof by contradiction.)
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To prove that a function $f : A \mapsto \mathbb{N}$ is an injection:

1: Assume $f$ is not an injection. (Proof by contradiction.)
2: This means there is a pair $x, y \in A$ for which $x \neq y$ and $f(x) = f(y)$. 
A Countable Set’s Cardinality Is At Most $|\mathbb{N}|$

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To prove that a function $f : A \mapsto \mathbb{N}$ is an injection:
1: Assume $f$ is not an injection. (Proof by contradiction.)
2: This means there is a pair $x, y \in A$ for which $x \neq y$ and $f(x) = f(y)$.
3: Use $f(x) = f(y)$ to prove that $x = y$, a contradiction. Hence, $f$ is an injection.
$A = \{3, 6, 8\}$. To show $|A| \leq \mathbb{N}$, we give an injection from $A$ to $\mathbb{N}$,
All Finite Sets are Countable

\[ A = \{3, 6, 8\} \]. To show \(|A| \leq \mathbb{N}\), we give an injection from \(A\) to \(\mathbb{N}\),

\[
3 \mapsto 1 \quad 6 \mapsto 2 \quad 8 \mapsto 3.
\]
All Finite Sets are Countable

$A = \{3, 6, 8\}$. To show $|A| \leq \mathbb{N}$, we give an injection from $A$ to $\mathbb{N}$,

\[
3 \mapsto 1 \quad 6 \mapsto 2 \quad 8 \mapsto 3.
\]

For an arbitrary finite set $A = \{a_1, a_2, \ldots, a_n\}$, $\mathbb{N}$,

\[
a_1 \mapsto 1 \quad a_2 \mapsto 2 \quad a_3 \mapsto 3 \quad \cdots \quad a_n \mapsto n.
\]
Non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ are countable
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How can this be? $\mathbb{N}_0$ contains every element in $\mathbb{N}$ plus 0?
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To prove $|\mathbb{N}_0| \leq |\mathbb{N}|$, we give an injection $f : \mathbb{N}_0 \to \mathbb{N}$,
Non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \ldots \}$ are countable

How can this be? $\mathbb{N}_0$ contains every element in $\mathbb{N}$ plus 0?

To prove $| \mathbb{N}_0 | \leq | \mathbb{N} |$, we give an injection $f : \mathbb{N}_0 \rightarrow \mathbb{N}$,

$$f(x) = x + 1, \quad \text{for } x \in \mathbb{N}_0.$$
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To prove $|\mathbb{N}_0| \leq |\mathbb{N}|$, we give an injection $f : \mathbb{N}_0 \rightarrow \mathbb{N}$,

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Proof. Assume $f$ is not an injection. So, there are $x \neq y$ in $\mathbb{N}_0$ with $f(x) = f(y)$:
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**Proof.** Assume \( f \) is not an injection. So, there are \( x \neq y \) in \( \mathbb{N}_0 \) with \( f(x) = f(y) \):

\[
x + 1 = f(x) = f(y) = y + 1.
\]
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To prove $|\mathbb{N}_0| \leq |\mathbb{N}|$, we give an injection $f : \mathbb{N}_0 \hookrightarrow \mathbb{N}$,

$$f(x) = x + 1, \quad \text{for } x \in \mathbb{N}_0.$$  

**Proof.** Assume $f$ is not an injection. So, there are $x \neq y$ in $\mathbb{N}_0$ with $f(x) = f(y)$:

$$x + 1 = f(x) = f(y) = y + 1.$$  

That is $x + 1 = y + 1$ or $x = y$, which contradicts $x \neq y$.  

$\blacksquare$
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How can this be? $\mathbb{N}_0$ contains every element in $\mathbb{N}$ \textit{plus} 0?

To prove $|\mathbb{N}_0| \leq |\mathbb{N}|$, we give an injection $f : \mathbb{N}_0 \rightarrow \mathbb{N}$,

$$f(x) = x + 1,$$

for $x \in \mathbb{N}_0$.

\textit{Proof.} Assume $f$ is not an injection. So, there are $x \neq y$ in $\mathbb{N}_0$ with $f(x) = f(y)$:

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That is $x + 1 = y + 1$ or $x = y$, which contradicts $x \neq y$. \qed

Also, $|\mathbb{N}| \leq |\mathbb{N}_0|$ because $\mathbb{N} \subseteq \mathbb{N}_0$ \rightarrow $|\mathbb{N}_0| = |\mathbb{N}|$. (Cantor-Bernstein)
Non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ are countable

How can this be? $\mathbb{N}_0$ contains every element in $\mathbb{N}$ plus 0?

To prove $|\mathbb{N}_0| \leq |\mathbb{N}|$, we give an injection $f : \mathbb{N}_0 \overset{\text{inj}}{\rightarrow} \mathbb{N}$,

$$f(x) = x + 1, \quad \text{for } x \in \mathbb{N}_0.$$  

Proof. Assume $f$ is not an injection. So, there are $x \neq y$ in $\mathbb{N}_0$ with $f(x) = f(y)$:

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Also, $|\mathbb{N}| \leq |\mathbb{N}_0|$ because $\mathbb{N} \subseteq \mathbb{N}_0 \rightarrow |\mathbb{N}_0| = |\mathbb{N}|$. (Cantor-Bernstein)

\[\begin{align*}
\mathbb{N}_0 : & \quad 0 \uparrow 1 \uparrow 2 \uparrow 3 \uparrow 4 \uparrow 5 \uparrow 6 \uparrow 7 \uparrow 8 \uparrow 9 \uparrow \cdots \\
\mathbb{N} : & \quad 1 \downarrow 2 \downarrow 3 \downarrow 4 \downarrow 5 \downarrow 6 \downarrow 7 \downarrow 8 \downarrow 9 \downarrow 10 \downarrow \cdots
\end{align*}\]
Positive Even Numbers and Integers are Countable

\[ E = \{2, 4, 6, \ldots \} \]. Surely \(|E| = \frac{1}{2}|\mathbb{N}|\)?
Positive Even Numbers and Integers are Countable

$E = \{2, 4, 6, \ldots \}$. Surely $|E| = \frac{1}{2}|\mathbb{N}|$?

The bijection $f(x) = \frac{1}{2}x$ proves $|E| = |\mathbb{N}|$.
Positive Even Numbers and Integers are Countable

$E = \{2, 4, 6, \ldots \}$. Surely $|E| = \frac{1}{2} |\mathbb{N}|$?

The bijection $f(x) = \frac{1}{2} x$ proves $|E| = |\mathbb{N}|$

\[
\begin{align*}
E : & \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \quad 20 \quad \cdots \\
\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\
\mathbb{N} : & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad \cdots
\end{align*}
\]
Positive Even Numbers and Integers are Countable

\[ E = \{2, 4, 6, \ldots\} \]. Surely \(|E| = \frac{1}{2}|\mathbb{N}|\)?

The bijection \(f(x) = \frac{1}{2}x\) proves \(|E| = |\mathbb{N}|\)

\[
\begin{array}{cccccccccc}
E : & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & \cdots \\
\downarrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \cdots \\
\mathbb{N} : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
\end{array}
\]

\[ \mathbb{Z} = \{0, \pm1, \pm2, \ldots\} \]. \(|\mathbb{Z}| = |\mathbb{N}|\).
Positive Even Numbers and Integers are Countable

\[ E = \{2, 4, 6, \ldots\}. \text{ Surely } |E| = \frac{1}{2} |\mathbb{N}|? \]

The bijection \( f(x) = \frac{1}{2}x \) proves \( |E| = |\mathbb{N}| \)

\[
\begin{array}{cccccccccc}
E & : & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\mathbb{N} & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
\end{array}
\]

\[ \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}. \ |\mathbb{Z}| = |\mathbb{N}|. \]
Positive Even Numbers and Integers are Countable

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\[
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\[ E : \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad 18 \quad 20 \quad \cdots \]
\[ \mathbb{N} : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad \cdots \]

\[ \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \text{. } |\mathbb{Z}| = |\mathbb{N}|. \]

\[ \mathbb{N} : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad \cdots \]

Exercise. What is a mathematical formula for the bijection?
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list.
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list.
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list. What about \(\mathbb{Z}\)?
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \( E = \{2, 4, 6, \ldots\} \) is a list. What about \( \mathbb{Z} \)?

\[
\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots
\]

\( \leftarrow \) not a list
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list. What about \(\mathbb{Z}\)?

\[
\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots
\]

\[
0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots
\]

← not a list

← list
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list.  \(E = \{2, 4, 6, \ldots\}\) is a list.  What about \(\mathbb{Z}\)?

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\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots
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\[
0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots
\]

\(\leftarrow\) not a list

\(\leftarrow\) list
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list. What about \(\mathbb{Z}\)?

\[
\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots \quad \leftarrow \text{not a list}
\]

\[
0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots \quad \leftarrow \text{list}
\]
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list. What about \(\mathbb{Z}\)?

\[
\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots \quad \leftarrow \text{not a list}
\]

\[
0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots \quad \leftarrow \text{list}
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\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots \quad \leftarrow \text{not a list}
\]
\[
0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots \quad \leftarrow \text{list}
\]
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\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list. What about \(\mathbb{Z}\)?

\[
\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots\quad \leftarrow \text{not a list}
\]

\[
0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots\quad \leftarrow \text{list}
\]
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list. What about \(\mathbb{Z}\)?

\(-\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\) ← not a list

\(0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots\) ← list
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list. What about \(\mathbb{Z}\)?

\[
\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots
\]

\(0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots\)

\(\leftarrow\) not a list

\[
\cdots
\]

\(\leftarrow\) list

\[N_0: \{0, 1, 2, 3, 4, 5, \ldots\} \quad E: \{2, 4, 6, 8, 10, \ldots\} \quad \mathbb{Z}: \{0, +1, -1, +2, -2, +3, -3, +4, -4, \ldots\}\]
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list. What about \(\mathbb{Z}\)?

\[
\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots \quad \leftarrow \text{not a list}
\]

\[
0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots \quad \leftarrow \text{list}
\]

Different elements are assigned to different list-positions.
Every Countable Set Can Be “Listed”

\{3, 6, 8\} is a list. \(E = \{2, 4, 6, \ldots\}\) is a list. What about \(\mathbb{Z}\)?

\[
\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots
\]

\[
0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots
\]

\(\cdots\) ← not a list

\(\cdots\) ← list

\(1\) Different elements are assigned to different list-positions.

\(2\) Can determine the list-position of any element in the set.
Every Countable Set Can Be “Listed”

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\[
\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots \quad \leftrightarrow \text{not a list}
\]

\[
0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots \quad \leftrightarrow \text{list}
\]

\[\mathbb{N}\]

\[\mathbb{A}\]

\[\mathbb{E}\]

\[\mathbb{Z}\]

\begin{align*}
\mathbb{N}_0 : & \quad \{0, 1, 2, 3, 4, 5, \ldots\} \\
\mathbb{E} : & \quad \{2, 4, 6, 8, 10, \ldots\} \\
\mathbb{Z} : & \quad \{0, +1, -1, +2, -2, +3, -3, +4, -4, \ldots\}
\end{align*}

1. Different elements are assigned to different list-positions.
2. Can determine the list-position of \textit{any} element in the set. For \(\mathbb{Z}\),

\[
\text{list position of } z = \begin{cases} 
2z & z > 0; \\
2|z| + 1 & z \leq 0;
\end{cases}
\]
Union of Two Countable Sets is Countable
Union of Two Countable Sets is Countable

A and B are countable, so they can be listed.

\[ A = \{a_1, a_2, a_3, a_4, a_5, \ldots \} \quad B = \{b_1, b_2, b_3, b_4, b_5, \ldots \}. \]
A and $B$ are countable, so they can be listed.

\[ A = \{a_1, a_2, a_3, a_4, a_5, \ldots\} \quad B = \{b_1, b_2, b_3, b_4, b_5, \ldots\} \].

Here is a list for $A \cup B$

\[ A \cup B = \{a_1, a_2, a_3, a_4, a_5, \ldots, b_1, b_2, b_3, b_4, b_5, \ldots\} \].
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What is the list-position of \( b_1 \)?
Union of Two Countable Sets is Countable

A and B are countable, so they can be listed.

\[ A = \{a_1, a_2, a_3, a_4, a_5, \ldots \} \quad \text{and} \quad B = \{b_1, b_2, b_3, b_4, b_5, \ldots \}. \]

Here is a list for \( A \cup B \)

\[ A \cup B = \{a_1, a_2, a_3, a_4, a_5, \ldots, b_1, b_2, b_3, b_4, b_5, \ldots \}. \]

What is the list-position of \( b_1 \)? Cannot use “…” twice.
A and $B$ are countable, so they can be listed.

\[ A = \{a_1, a_2, a_3, a_4, a_5, \ldots \} \quad B = \{b_1, b_2, b_3, b_4, b_5, \ldots \}. \]

Here is a list for $A \cup B$

\[ A \cup B = \{a_1, a_2, a_3, a_4, a_5, \ldots, b_1, b_2, b_3, b_4, b_5, \ldots \}. \quad \times \]

What is the list-position of $b_1$? Cannot use “…” twice.

\[ A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, \ldots \}. \]
A and $B$ are countable, so they can be listed.

$$A = \{a_1, a_2, a_3, a_4, a_5, \ldots\} \quad B = \{b_1, b_2, b_3, b_4, b_5, \ldots\}.$$

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$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, \ldots\}.$$

list-position of $a_i$ is $2i - 1$;
Union of Two Countable Sets is Countable

$A$ and $B$ are countable, so they can be listed.

$A = \{a_1, a_2, a_3, a_4, a_5, \ldots\}$ \hspace{1cm} $B = \{b_1, b_2, b_3, b_4, b_5, \ldots\}$.

Here is a list for $A \cup B$

$A \cup B = \{a_1, a_2, a_3, a_4, a_5, \ldots, b_1, b_2, b_3, b_4, b_5, \ldots\}$. ×

What is the list-position of $b_1$? Cannot use “…” twice.

$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, \ldots\}$.

list-position of $a_i$ is $2i - 1$;
list-position of $b_i$ is $2i$. 
A and $B$ are countable, so they can be listed.

$A = \{a_1, a_2, a_3, a_4, a_5, \ldots\}$ \hspace{1cm} $B = \{b_1, b_2, b_3, b_4, b_5, \ldots\}$.

Here is a list for $A \cup B$

$A \cup B = \{a_1, a_2, a_3, a_4, a_5, \ldots, b_1, b_2, b_3, b_4, b_5, \ldots\}$. \times

What is the list-position of $b_1$? Cannot use “…” twice.

$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, \ldots\}$.

list-position of $a_i$ is $2i - 1$;
list-position of $b_i$ is $2i$.

**Pop Quiz.** Get a list of $\mathbb{Z}$ with $A = \{0, -1, -2, -3, \ldots\}$ and $B = \{1, 2, 3, \ldots\}$ using union.
Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

This is surprising because between any two rationals there is another (not true for $\mathbb{N}$).
Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

This is surprising because between any two rationals there is another (not true for $\mathbb{N}$).

<table>
<thead>
<tr>
<th>Q</th>
<th>$0$</th>
<th>$+1$</th>
<th>$-1$</th>
<th>$+2$</th>
<th>$-2$</th>
<th>$+3$</th>
<th>$-3$</th>
<th>$+4$</th>
<th>$-4$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{0}{1}$</td>
<td>$\frac{+1}{1}$</td>
<td>$\frac{-1}{1}$</td>
<td>$\frac{+2}{1}$</td>
<td>$\frac{-2}{1}$</td>
<td>$\frac{+3}{1}$</td>
<td>$\frac{-3}{1}$</td>
<td>$\frac{+4}{1}$</td>
<td>$\frac{-4}{1}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{0}{2}$</td>
<td>$\frac{+1}{2}$</td>
<td>$\frac{-1}{2}$</td>
<td>$\frac{+2}{2}$</td>
<td>$\frac{-2}{2}$</td>
<td>$\frac{+3}{2}$</td>
<td>$\frac{-3}{2}$</td>
<td>$\frac{+4}{2}$</td>
<td>$\frac{-4}{2}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>N 3</td>
<td>$\frac{0}{3}$</td>
<td>$\frac{+1}{3}$</td>
<td>$\frac{-1}{3}$</td>
<td>$\frac{+2}{3}$</td>
<td>$\frac{-2}{3}$</td>
<td>$\frac{+3}{3}$</td>
<td>$\frac{-3}{3}$</td>
<td>$\frac{+4}{3}$</td>
<td>$\frac{-4}{3}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{0}{4}$</td>
<td>$\frac{+1}{4}$</td>
<td>$\frac{-1}{4}$</td>
<td>$\frac{+2}{4}$</td>
<td>$\frac{-2}{4}$</td>
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<td>$\frac{-3}{4}$</td>
<td>$\frac{+4}{4}$</td>
<td>$\frac{-4}{4}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{0}{5}$</td>
<td>$\frac{+1}{5}$</td>
<td>$\frac{-1}{5}$</td>
<td>$\frac{+2}{5}$</td>
<td>$\frac{-2}{5}$</td>
<td>$\frac{+3}{5}$</td>
<td>$\frac{-3}{5}$</td>
<td>$\frac{+4}{5}$</td>
<td>$\frac{-4}{5}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$\vdots$</td>
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<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

Intuition suggests $|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}| \gg |\mathbb{N}|$. ✗
Rationals are Countable: \(| \mathbb{Q} | = | \mathbb{N} |\)

This is surprising because between any two rationals there is another (not true for \(\mathbb{N}\)).

<table>
<thead>
<tr>
<th>(\mathbb{Q})</th>
<th>0</th>
<th>+1</th>
<th>−1</th>
<th>+2</th>
<th>−2</th>
<th>+3</th>
<th>−3</th>
<th>+4</th>
<th>−4</th>
<th>⋮</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{0}{1})</td>
<td>(\frac{+1}{1})</td>
<td>(\frac{−1}{1})</td>
<td>(\frac{+2}{1})</td>
<td>(\frac{−2}{1})</td>
<td>(\frac{+3}{1})</td>
<td>(\frac{−3}{1})</td>
<td>(\frac{+4}{1})</td>
<td>(\frac{−4}{1})</td>
<td>⋮</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{0}{2})</td>
<td>(\frac{+1}{2})</td>
<td>(\frac{−1}{2})</td>
<td>(\frac{+2}{2})</td>
<td>(\frac{−2}{2})</td>
<td>(\frac{+3}{2})</td>
<td>(\frac{−3}{2})</td>
<td>(\frac{+4}{2})</td>
<td>(\frac{−4}{2})</td>
<td>⋮</td>
</tr>
<tr>
<td>(\mathbb{N})</td>
<td>3</td>
<td>(\frac{0}{3})</td>
<td>(\frac{+1}{3})</td>
<td>(\frac{−1}{3})</td>
<td>(\frac{+2}{3})</td>
<td>(\frac{−2}{3})</td>
<td>(\frac{+3}{3})</td>
<td>(\frac{−3}{3})</td>
<td>(\frac{+4}{3})</td>
<td>(\frac{−4}{3})</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{0}{4})</td>
<td>(\frac{+1}{4})</td>
<td>(\frac{−1}{4})</td>
<td>(\frac{+2}{4})</td>
<td>(\frac{−2}{4})</td>
<td>(\frac{+3}{4})</td>
<td>(\frac{−3}{4})</td>
<td>(\frac{+4}{4})</td>
<td>(\frac{−4}{4})</td>
<td>⋮</td>
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<tr>
<td>5</td>
<td>(\frac{0}{5})</td>
<td>(\frac{+1}{5})</td>
<td>(\frac{−1}{5})</td>
<td>(\frac{+2}{5})</td>
<td>(\frac{−2}{5})</td>
<td>(\frac{+3}{5})</td>
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<td>(\frac{−4}{5})</td>
<td>⋮</td>
</tr>
</tbody>
</table>

Intuition suggests \(| \mathbb{Q} | = | \mathbb{N} | \times | \mathbb{Z} | \gg | \mathbb{N} |\). ✗
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This is surprising because between any two rationals there is another (not true for $\mathbb{N}$).

<table>
<thead>
<tr>
<th>$\mathbb{Q}$</th>
<th>0</th>
<th>+1</th>
<th>−1</th>
<th>+2</th>
<th>−2</th>
<th>+3</th>
<th>−3</th>
<th>+4</th>
<th>−4</th>
<th>⋯</th>
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<td>1</td>
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<td>+1/1</td>
<td>−1/1</td>
<td>+2/1</td>
<td>−2/1</td>
<td>+3/1</td>
<td>−3/1</td>
<td>+4/1</td>
<td>−4/1</td>
<td>⋯</td>
</tr>
<tr>
<td>2</td>
<td>1/2</td>
<td>+1/2</td>
<td>−1/2</td>
<td>+2/2</td>
<td>−2/2</td>
<td>+3/2</td>
<td>−3/2</td>
<td>+4/2</td>
<td>−4/2</td>
<td>⋯</td>
</tr>
<tr>
<td>$\mathbb{N}$ 3</td>
<td>0/3</td>
<td>+1/3</td>
<td>−1/3</td>
<td>+2/3</td>
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Intuition suggests \(|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}| \gg |\mathbb{N}|\). ✗
Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

This is surprising because between any two rationals there is another (not true for $\mathbb{N}$).

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Rationals are Countable: \(| \mathbb{Q} | = | \mathbb{N} |\)

This is surprising because between any two rationals there is another (not true for \(\mathbb{N}\)).

| \(\mathbb{Q}\) | 0 | +1 | −1 | +2 | −2 | +3 | −3 | +4 | −4 | ⋮  
|---|---|---|---|---|---|---|---|---|---|---
| 1 | 0/1 | +1/1 | −1/1 | +2/1 | −2/1 | +3/1 | −3/1 | +4/1 | −4/1 | ⋮
| 2 | 0/2 | +1/2 | −1/2 | +2/2 | −2/2 | +3/2 | −3/2 | +4/2 | −4/2 | ⋮
| \(\mathbb{N}\) | 0/3 | +1/3 | −1/3 | +2/3 | −2/3 | +3/3 | −3/3 | +4/3 | −4/3 | ⋮  
| 4 | 0/4 | +1/4 | −1/4 | +2/4 | −2/4 | +3/4 | −3/4 | +4/4 | −4/4 | ⋮
| 5 | 0/5 | +1/5 | −1/5 | +2/5 | −2/5 | +3/5 | −3/5 | +4/5 | −4/5 | ⋮  
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮  

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</table>

Intuition suggests $|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}| \gg |\mathbb{N}|$. 😞
Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

This is surprising because between any two rationals there is another (not true for $\mathbb{N}$).

<table>
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| $\mathbb{Z}$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | $\frac{0}{2}$ | $\frac{+1}{2}$ | $\frac{−1}{2}$ | $\frac{+2}{2}$ | $\frac{−2}{2}$ | $\frac{+3}{2}$ | $\frac{−3}{2}$ | $\frac{+4}{2}$ | $\frac{−4}{2}$ | ... |

| $\mathbb{N}$ | 3 | $\frac{0}{3}$ | $\frac{+1}{3}$ | $\frac{−1}{3}$ | $\frac{+2}{3}$ | $\frac{−2}{3}$ | $\frac{+3}{3}$ | $\frac{−3}{3}$ | $\frac{+4}{3}$ | $\frac{−4}{3}$ | ... |

Intuition suggests $|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}| \gg |\mathbb{N}|$. 😞
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Rationals are Countable: \(|Q| = |N|\)

This is surprising because between any two rationals there is another (not true for \(N\)).

\[
\begin{array}{cccccccccc}
Q & 0 & +1 & -1 & +2 & -2 & +3 & -3 & +4 & -4 & \cdots \\
\hline
1 & 0 & \frac{+1}{1} & \frac{-1}{1} & \frac{+2}{1} & \frac{-2}{1} & \frac{+3}{1} & \frac{-3}{1} & \frac{+4}{1} & \frac{-4}{1} & \cdots \\
2 & 0 & \frac{+1}{2} & \frac{-1}{2} & \frac{+2}{2} & \frac{-2}{2} & \frac{+3}{2} & \frac{-3}{2} & \frac{+4}{2} & \frac{-4}{2} & \cdots \\
\hline
N & 3 & 0 & \frac{+1}{3} & \frac{-1}{3} & \frac{+2}{3} & \frac{-2}{3} & \frac{+3}{3} & \frac{-3}{3} & \frac{+4}{3} & \frac{-4}{3} & \cdots \\
4 & 0 & \frac{+1}{4} & \frac{-1}{4} & \frac{+2}{4} & \frac{-2}{4} & \frac{+3}{4} & \frac{-3}{4} & \frac{+4}{4} & \frac{-4}{4} & \cdots \\
5 & 0 & \frac{+1}{5} & \frac{-1}{5} & \frac{+2}{5} & \frac{-2}{5} & \frac{+3}{5} & \frac{-3}{5} & \frac{+4}{5} & \frac{-4}{5} & \cdots \\
\end{array}
\]

Intuition suggests

\(|Q| = |N| \times |Z| \gg |N| \).  

\[
Q = \left\{ \frac{0}{1}, \frac{+1}{1}, \frac{+1}{2}, \frac{0}{2}, \frac{0}{3}, \frac{+1}{3}, \frac{-1}{3}, \frac{-1}{2}, \frac{-1}{1}, \frac{+2}{1}, \frac{+2}{2}, \frac{+2}{3}, \frac{+2}{4}, \frac{-1}{4}, \frac{+1}{4}, \frac{0}{4}, \frac{0}{5}, \cdots \right\}
\]
Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

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<table>
<thead>
<tr>
<th>$\mathbb{Q}$</th>
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| 1 | 0 | $\frac{1}{1}$ | $\frac{-1}{1}$ | $\frac{+2}{1}$ | $\frac{-2}{1}$ | $\frac{+3}{1}$ | $\frac{-3}{1}$ | $\frac{+4}{1}$ | $\frac{-4}{1}$ | $\cdots$ |
| 2 | 0 | $\frac{1}{2}$ | $\frac{-1}{2}$ | $\frac{+2}{2}$ | $\frac{-2}{2}$ | $\frac{+3}{2}$ | $\frac{-3}{2}$ | $\frac{+4}{2}$ | $\frac{-4}{2}$ | $\cdots$ |
| 3 | 0 | $\frac{1}{3}$ | $\frac{-1}{3}$ | $\frac{+2}{3}$ | $\frac{-2}{3}$ | $\frac{+3}{3}$ | $\frac{-3}{3}$ | $\frac{+4}{3}$ | $\frac{-4}{3}$ | $\cdots$ |

Intuition suggests $|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}| \gg |\mathbb{N}|$. ✗

$\mathbb{Q} = \left\{ \frac{0}{1}, \frac{+1}{1}, \frac{-1}{2}, \frac{+2}{2}, \frac{0}{3}, \frac{+3}{3}, \frac{-1}{3}, \frac{-2}{3}, \frac{-1}{1}, \frac{+2}{1}, \frac{+2}{2}, \frac{+2}{3}, \frac{+2}{4}, \frac{-1}{4}, \frac{+1}{4}, \frac{0}{4}, \frac{0}{5}, \cdots \right\}$

$|\{\text{Rational Values}\}| \leq |\mathbb{Q}| \leq |\mathbb{N}|$.

**Exercise.** What is a mathematical formula for the list-position of $z/n \in \mathbb{Q}$?
Programs are Countable

Programs are finite binary strings.
Programs are finite binary strings. We show that all finite binary strings $\mathcal{B}$ are countable.
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$$\mathcal{B} = \{ \varepsilon \}$$
Programs are Countable

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$$\mathcal{B} = \{\varepsilon, 0, 1\}$$
Programs are Countable

Programs are finite binary strings. We show that all finite binary strings $\mathcal{B}$ are countable.

\[ \mathcal{B} = \{ \varepsilon, 0, 1, 00, 01, 10, 11 \} \]
Programs are Countable

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$$\mathcal{B} = \{ \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111 \}$$
Programs are Countable

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\[ \mathcal{B} = \{ \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots \} \]
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$$\mathcal{B} = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots\}$$ ←list

Pop Quiz. What is the list-position of 0110?
Programs are Countable

Programs are finite binary strings. We show that all finite binary strings $\mathcal{B}$ are countable.

$$\mathcal{B} = \{ \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots \}$$

Pop Quiz. What is the list-position of 0110?

Exercise. For the $(k + 1)$-bit string $b = b_k b_{k-1} \cdots b_1 b_0$, define the strings numerical value:

$$\text{value}(b) = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \cdots + b_{k-1} \cdot 2^{k-1} + b_k \cdot 2^k.$$  

Show:

$$\text{list-position of } b = 2^{\text{length}(b)} + \text{value}(b).$$
Programs are Countable

Programs are finite binary strings. We show that all finite binary strings \( \mathcal{B} \) are countable.

\[
\mathcal{B} = \{ \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots \} \quad \leftarrow \text{list}
\]

**Pop Quiz.** What is the list-position of 0110?

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\]

Show:

\[
\text{list-position of } b = 2^{\text{length}(b)} + \text{value}(b).
\]

\( \mathbb{N}_0, E, \mathbb{Z}, \mathbb{Q}, \mathcal{B} \) are countable, \ldots
Programs are Countable

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$\mathbb{N}_0, E, \mathbb{Z}, \mathbb{Q}, \mathcal{B}$ are countable,\ldots Is Everything Countable?
Infinite Binary Strings are Uncountable
Infinite Binary Strings are Uncountable

Cantor’s Diagonal Argument:
Infinite Binary Strings are Uncountable

Cantor’s Diagonal Argument: Assume there is a list of \textit{all} infinite binary strings.
Cantor’s Diagonal Argument: Assume there is a list of all infinite binary strings.

\[
\begin{align*}
  b_1 & : \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 
  b_2 & : \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 
  b_3 & : \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 
  b_4 & : \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 
  b_5 & : \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 
  b_6 & : \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 
  b_7 & : \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 
  b_8 & : \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 
  b_9 & : \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 
  b_{10} & : \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 
  \vdots
\end{align*}
\]
Infinite Binary Strings are Uncountable

Cantor’s Diagonal Argument: Assume there is a list of *all* infinite binary strings.

\[
\begin{align*}
  b_1: & \quad 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots \\
  b_2: & \quad 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \cdots \\
  b_3: & \quad 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \cdots \\
  b_4: & \quad 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \cdots \\
  b_5: & \quad 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots \\
  b_6: & \quad 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots \\
  b_7: & \quad 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \cdots \\
  b_8: & \quad 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \cdots \\
  b_9: & \quad 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots \\
  b_{10}: & \quad 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots \\
  \vdots
\end{align*}
\]

Consider the “diagonal string”

\[
b = 0000100101\cdots
\]
Cantor’s Diagonal Argument: Assume there is a list of all infinite binary strings.

\[
\begin{align*}
    b_1 & : 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \cdots \\
    b_2 & : 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ \cdots \\
    b_3 & : 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ \cdots \\
    b_4 & : 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ \cdots \\
    b_5 & : 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \cdots \\
    b_6 & : 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \cdots \\
    b_7 & : 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ \cdots \\
    b_8 & : 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ \cdots \\
    b_9 & : 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \cdots \\
    b_{10} & : 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ \cdots \\
\end{align*}
\]

Consider the “diagonal string”

\[ b = 0000100101 \cdots \]

Flip the bits,

\[ \bar{b} = 1111011010 \cdots \]
Cantor’s Diagonal Argument: Assume there is a list of all infinite binary strings.

\[
\begin{align*}
b_1: & \quad 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\b_2: & \quad 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \\b_3: & \quad 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \\b_4: & \quad 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \\b_5: & \quad 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\b_6: & \quad 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \\b_7: & \quad 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \\b_8: & \quad 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \\b_9: & \quad 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \\b_{10}: & \quad 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \\
& \vdots
\end{align*}
\]

Consider the “diagonal string”

\[
b = 0000100101 \cdots
\]

Flip the bits,

\[
\bar{b} = 1111011010 \cdots
\]

\(\bar{b}\) is not in the list (differs in the \(i\)th position from \(b_i\)), a contradiction.
Infinite Binary Strings are Uncountable

Cantor’s Diagonal Argument: Assume there is a list of all infinite binary strings.

\[
\begin{align*}
    b_1 & : 0 0 0 1 0 0 0 0 0 0 0 0 0 0 \\
    b_2 & : 0 0 1 1 0 1 0 0 1 0 0 0 0 1 \\
    b_3 & : 1 1 0 0 0 1 0 0 0 0 1 1 0 0 \\
    b_4 & : 1 0 1 0 0 1 0 0 0 0 1 0 0 0 \\
    b_5 & : 0 1 1 0 1 0 1 0 0 0 0 0 0 0 \\
    b_6 & : 0 1 0 1 1 0 0 0 1 0 0 0 0 0 \\
    b_7 & : 0 0 1 0 0 1 0 0 0 0 0 0 0 1 \\
    b_8 & : 0 0 1 0 1 1 0 1 0 0 0 1 0 0 \\
    b_9 & : 0 0 0 0 1 1 0 0 0 1 0 0 0 0 \\
    b_{10} & : 1 0 1 1 1 0 1 0 0 1 1 0 0 0 \\
\end{align*}
\]

Consider the “diagonal string”

\[
b = 0000100101 \cdots
\]

Flip the bits,

\[
\bar{b} = 1111011010 \cdots
\]

\(\bar{b}\) is not in the list (differs in the \(i\)th position from \(b_i\)), a contradiction.

Reals are Uncountable
Infinite Binary Strings are Uncountable

Cantor’s Diagonal Argument: Assume there is a list of all infinite binary strings.

\[ b_1: \quad 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots \]
\[ b_2: \quad 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \cdots \]
\[ b_3: \quad 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \cdots \]
\[ b_4: \quad 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \cdots \]
\[ b_5: \quad 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots \]
\[ b_6: \quad 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \cdots \]
\[ b_7: \quad 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \cdots \]
\[ b_8: \quad 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \cdots \]
\[ b_9: \quad 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \cdots \]
\[ b_{10}: \quad 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \cdots \]
\vdots

Consider the “diagonal string”

\[ b = 0000100101 \cdots \]

Flip the bits,

\[ \bar{b} = 1111011010 \cdots \]

\( \bar{b} \) is not in the list (differs in the \( i \)th position from \( b_i \)), a contradiction.

Reals are Uncountable
Every real has an infinite binary representation and every infinite binary string evaluates to a real number.
Infinite Binary Strings are Uncountable

Cantor’s Diagonal Argument: Assume there is a list of all infinite binary strings.

\[
\begin{align*}
b_1 &: 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 \\
b_2 &: 0 0 1 1 0 1 0 0 1 0 0 0 0 1 0 \\
b_3 &: 1 1 0 0 0 1 0 0 0 0 1 1 0 0 \\
b_4 &: 1 0 1 0 0 1 0 0 0 0 1 0 0 0 \\
b_5 &: 0 1 1 0 1 0 0 0 0 0 0 0 0 0 \\
b_6 &: 0 1 0 1 1 0 0 0 0 1 0 0 0 0 \\
b_7 &: 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1 \\
b_8 &: 0 0 1 0 1 1 0 1 0 0 0 0 1 0 0 \\
b_9 &: 0 0 0 0 1 1 0 0 0 0 1 0 0 0 0 \\
b_{10} &: 1 0 1 1 1 0 1 0 0 1 1 0 0 0 0 \[...
\end{align*}
\]

Consider the “diagonal string”

\[ b = 0000100101 \cdots \]

Flip the bits,

\[ \bar{b} = 1111011010 \cdots \]

\( \bar{b} \) is not in the list (differs in the \( i \)th position from \( b_i \)), a contradiction.

Reals are Uncountable

Every real has an infinite binary representation and every infinite binary string evaluates to a real number.

\[
e.g. \quad 0.00111111111111111111 \cdots = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^7} + \cdots = \frac{1}{2}.
\]
Infinite Binary Strings are Uncountable

Cantor’s Diagonal Argument: Assume there is a list of all infinite binary strings.

\[
\begin{align*}
  b_1: \quad & 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\
  b_2: \quad & 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \\
  b_3: \quad & 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \\
  b_4: \quad & 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \\
  b_5: \quad & 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\
  b_6: \quad & 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \\
  b_7: \quad & 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \\
  b_8: \quad & 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \\
  b_9: \quad & 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\
  b_{10}: \quad & 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \\
  \vdots & 
\end{align*}
\]

Consider the “diagonal string”

\[b = 0000100101 \cdots\]

Flip the bits,

\[\bar{b} = 1111011010 \cdots\]

\(\bar{b}\) is not in the list (differs in the \(i\)th position from \(b_i\)), a contradiction.

Reals are Uncountable

Every real has an infinite binary representation and every infinite binary string evaluates to a real number.

\[0.00111111111111111111111111111111111 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots = \frac{1}{2}.\]

That is \(|\{\text{reals in } [0, 1]\}| = |\{\text{infinite binary strings}\}| > |N|\).
Cantor took on the abstract beast Infinity. (1874)
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∼ 60 years later, Alan Turing asked the abstract question: What can we compute? (1936)
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~ 60 years later, Alan Turing asked the abstract question: What can we compute? (1936)

Every binary function $f$ on $\mathbb{N}$ corresponds to an infinite binary string $f(1)f(2)f(3)\cdots$. 
Cantor took on the abstract beast Infinity. (1874) 
\sim 60\text{ years later, Alan Turing asked the abstract question: What can we compute? (1936)}

Every binary function $f$ on $\mathbb{N}$ corresponds to a infinite binary string $f(1)f(2)f(3)\cdots$,

\[
\begin{array}{cccccccccc}
  n: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
  f(n): & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \cdots \\
\end{array}
\]
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\sim 60 \text{ years later, Alan Turing asked the abstract question: What can we compute? (1936)}

Every binary function \( f \) on \( \mathbb{N} \) corresponds to a infinite binary string \( f(1)f(2)f(3) \cdots \),

\[
\begin{array}{ccccccccccc}
  n: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
  f(n): & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \cdots \\
\end{array}
\]

Every program is a finite binary string.
Cantor took on the abstract beast Infinity. (1874)

\sim 60 \text{ years later, Alan Turing asked the abstract question: What can we compute?} \ (1936)

Every binary function $f$ on $\mathbb{N}$ corresponds to an infinite binary string $f(1)f(2)f(3) \cdots$,

\[ n: \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad \cdots \]
\[ f(n): \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad \cdots \]

Every program is a finite binary string. For example,

```
int main(); // a program that does nothing
```
Cantor took on the abstract beast Infinity. (1874)
∼ 60 years later, Alan Turing asked the abstract question: What can we compute? (1936)

Every binary function \( f \) on \( \mathbb{N} \) corresponds to an infinite binary string \( f(1)f(2)f(3) \cdots \),

\[
\begin{array}{cccccccccccc}
n: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
f(n): & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \cdots \\
\end{array}
\]

Every program is a finite binary string. For example,

```c
int main(); // a program that does nothing
```

is the finite binary string (ASCII code)

```
011010010110110011001101000010000001101101011000101101001011011100010101000010111011111
```
Cantor took on the abstract beast Infinity. (1874)

∼ 60 years later, Alan Turing asked the abstract question: What can we compute? (1936)

Every binary function \( f \) on \( \mathbb{N} \) corresponds to an infinite binary string \( f(1)f(2)f(3) \cdots \),

\[
\begin{array}{cccccccccccc}
n: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
f(n): & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \cdots \\
\end{array}
\]

Every program is a finite binary string. For example,

```c
int main(); // a program that does nothing
```

is the finite binary string (ASCII code)

```
0110100101101100101101100101000011011011001011010000011010010110110010110001101010111011
```

Programs \( \leftrightarrow \) Countable
Cantor took on the abstract beast Infinity. (1874)

∼ 60 years later, Alan Turing asked the abstract question: What can we compute? (1936)

Every binary function $f$ on $\mathbb{N}$ corresponds to an infinite binary string $f(1)f(2)f(3)\cdots$,

\[
\begin{array}{cccccccccccc}
n: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
f(n): & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \cdots \\
\end{array}
\]

Every program is a finite binary string. For example,

```
int main();    // a program that does nothing
```

is the finite binary string (ASCII code)

```
011010010110111001110100001011011011101100001011011111010000101101011111111110001010011110111
```

Programs $\leftarrow$ Countable
Functions $\leftarrow$ Uncountable
Cantor took on the abstract beast Infinity. (1874)

~ 60 years later, Alan Turing asked the abstract question: What can we compute? (1936)

Every binary function $f$ on $\mathbb{N}$ corresponds to an infinite binary string $f(1)f(2)f(3)\cdots$,

\[
\begin{array}{ccccccccccc}
 n: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
 f(n): & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \cdots \\
\end{array}
\]

Every program is a finite binary string. For example,

```c
int main(); // a program that does nothing
```

is the finite binary string (ASCII code)

```
011010010110111001110100001011010010110111000101000001010010011101110111101110111
```

Programs $\leftarrow$ Countable
Functions $\leftarrow$ Uncountable

$\Rightarrow$ $|\{\text{functions on } \mathbb{N}\}| \gg |\{\text{programs}\}|$
Cantor took on the abstract beast Infinity. (1874)

~ 60 years later, Alan Turing asked the abstract question: What can we compute? (1936)

Every binary function $f$ on $\mathbb{N}$ corresponds to an infinite binary string $f(1)f(2)f(3)\cdots$,

\[
n: \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad \cdots
\]
\[
f(n): \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad \cdots
\]

Every program is a finite binary string. For example,

```cpp
int main(); // a program that does nothing
```

is the finite binary string (ASCII code)

```text
011010010110110001100000110110101100001011010010110110000100000010100100111011101111101
```

Programs $\leftarrow$ Countable
Functions $\leftarrow$ Uncountable $\rightarrow$ $|\{\text{functions on } \mathbb{N}\}| \gg |\{\text{programs}\}|$

There are MANY MANY functions that cannot be computed by programs!
Are there interesting, useful functions that cannot be computed by programs?