FINAL: 180 Minutes

Last Name: _______________________
First Name: ______________________
RIN: _____________________________
Section: __________________________

Answer ALL questions. You may use two single sided 8\(\frac{1}{2}\) × 11 crib sheets.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

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1. Circle at most one answer per question. 10 points for each correct answer and -5 points for each incorrect answer (blank answer is 0 points). Don’t guess!

(a) \( P(n) \) is a predicate \( (n \in \mathbb{N}) \). \( P(1), P(2), P(3) \) are true, and \( P(n) \Rightarrow P(n + 4) \) is true for \( n \geq 1 \). For which \( n \) can we be sure \( P(n) \) is true?

A All \( n \geq 1 \) except multiples of 2.
B All \( n \geq 1 \) except multiples of 4.
C All \( n \geq 1 \)
D Only \( n = 1, 2, 3 \).

(b) Of the following five sets, list all that are countable (\( A \) is countable if \( \mathbb{N} \xrightarrow{\text{surj}} A \)):
(I) Prime numbers; (II) Rational numbers; (III) Integers; (IV) Even numbers; (V) Infinite binary strings.

A I and III.
B I and II and III and IV.
C I and III and V.
D II and III and IV.

(c) A class with 25 students needs to choose a representative committee which is a subset of 5 students. How many different committees can be formed?

A \( 25^5 \).
B \( \frac{25!}{20! \times 5!} \).
C \( \frac{25!}{5!} \).
D \( 25 \times 24 \times 23 \times 22 \times 21 = \frac{25!}{20!} \).

(d) A friendship network has 7 people and each person has at least 1 friend. 6 of the people have exactly two friends. How many friends can the 7th person have? Give all possibilities.

A The seventh person could have either 2 or 4 friends.
B The seventh person could have either 2 or 4 or 6 friends.
C The seventh person could have either 1 or 2 or 3 friends.
D The seventh person could have any number of friends that is greater than 1.

(e) Compute the summation \( (0 + 1) + (1 + 2) + (2 + 4) + (3 + 8) + \cdots + (10 + 2^{10}) = \sum_{i=0}^{10} (i + 2^i) \)

A 2048.
B 2102.
C 1078.
D 2200.
(f) You have a known fact that 0 = 0 and all the standard operations of algebra you learned in high-school math. Which of the following is a valid proof that 7 = 7:

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<tr>
<td>1</td>
<td>7 = 7</td>
<td>7 ≠ 7</td>
<td>0 = 0</td>
</tr>
<tr>
<td>2</td>
<td>7 − 7 = 7 − 7</td>
<td>7 − 7 ≠ 7 − 7</td>
<td>0 + 7 = 0 + 7</td>
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<tr>
<td>3</td>
<td>0 = 0 ✓</td>
<td>0 ≠ 0 !FISHY</td>
<td>7 = 7 ✓</td>
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→ 7 = 7

A I & II & III.  B II & III.  C I & II  D I & III.

(g) Let \( f(n) = \sum_{i=1}^{n} i \) and \( g(n) = 2^{3\log_2 n} \). What is the big-Oh relationship between \( f \) and \( g \)?

A \( f(n) = O(g(n)) \) and \( g(n) = O(f(n)) \).

B \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

C \( f(n) \neq O(g(n)) \) and \( g(n) = O(f(n)) \).

D \( f(n) \neq O(g(n)) \) and \( g(n) \neq O(f(n)) \).

(h) You independently generate the ten bits of a binary sequence \( b_1 b_2 \cdots b_{10} \) with \( P[b_i = 0] = \frac{1}{2} \). Compute the probability that the sequence is sorted from low to high. For example 0000111111 is sorted.

A \( \frac{10}{1024} \)

B \( \frac{11}{1024} \)

C \( \frac{20}{1024} \)

D \( \frac{12}{1024} \)

(i) \( x_1, x_2, x_3 \) are non-negative integers. Compute the number of different solutions to \( x_1 + x_2 + x_3 = 100 \). (For example two different solutions are 1 + 2 + 97 = 100 and 97 + 1 + 2 = 100.)

A 10302

B 5151

C 4949

D 5050

(j) For the automaton on the right, which input string is accepted? (Strings are processed from left to right.)

A 010101

B 0101011

C 01010110

D 010101100
2 Proofs

1. Prove that for all integers \( n \geq 1 \): \( n^{2^n} \leq 3^n \)

2. Prove that every odd natural number is the difference of two square numbers.
3 Finite Automaton with a Random Input String

The automaton to the right processes a random binary string $b_1b_2\ldots b_n$ of length $n$ generated as follows: you independently generate each bit $b_i$ with $P[b_i = 1] = p$ and $P[b_i = 0] = 1 - p$. Show that the probability that the string is accepted is

$$P[\text{random input string is accepted}] = 1 - (1 - p)^n - np(1 - p)^{n-1}.$$ 

[Hints: (i) Figure out a simple property of a string for it to be accepted. (ii) Binomial distribution.]
4 Probability and Expectation

(a) You independently roll 3 fair dice $D_1, D_2, D_3$ and let $S = D_1 + D_2 + D_3$ be the sum. Compute:

(i) $P[S = 8]$

(ii) $P[S = 8 | D_1 = 1]$

(iii) Compute the expectation and variance of $S$.

(b) You toss a fair coin independently until you get two heads in a row. Let $X$ be the number of tosses. Compute $E[X]$ using the law of total expectation:

(i) Consider the 3 cases $T, HT, HH$ for how the tosses may start and show that

$$E[X] = \frac{1}{2}(1 + E[X]) + \frac{1}{4}(2 + E[X]) + \frac{1}{2}.$$

(ii) Use (i) to show that $E[X] = 6$. 


This problem is about the language \( L \) generated by the CFG:

\[
S \rightarrow 1T | 0T \\
T \rightarrow 1T1 | 0T0 | \epsilon
\]

(a) Is the string 1010010 in \( L \)? If yes then give a derivation or parse tree; if no then explain why.

(b) Prove that the length of every string in \( L \) is odd.
6 Turing Machine

(a) What is the difference between a Turing-recognizable language and a Turing-decidable language?

(b) Consider the arithmetic task of squaring, which corresponds to the language \( L = \{0^n \# 0^{n^2} | n \geq 1\} \).

(i) Circle the simplest model of computing that you think solves the problem \( L \):

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<tr>
<th>Finite Automaton</th>
<th>Context Free Grammar</th>
<th>Turing Machine</th>
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(ii) Give your machine from (i) that solves \( L \) (for a TM, a high level description will do).