Announcements

• Homework 1 is posted on the web site.
  • It is due Thursday, Feb. 5 at 10am in class
  • Must be typed and stapled.

• Recitation problem set is posted on the web site.
  • Solutions will only be given in recitation.
Predicate Logic

- A propositional function is a statement that describes a property of one or more variables
  - E.g. “$x^3 > 0$”
- The statement can be denoted by a propositional function.
  - E.g. $P(x)$ denotes “$x^3 > 0$”

- A propositional function is not the same thing as a proposition.
  - A propositional function is neither true nor false
  - A proposition is true or false

- A propositional function becomes a proposition when
  - Its variables are assigned values
  - OR the function is **bound** by a quantifier
Quantifiers

• The universal quantifier
  \( \forall x \, P(x) \) is true if \( P(x) \) is true for every \( x \) in the domain

• The existential quantifier
  \( \exists x \, P(x) \) is true if \( P(x) \) is true for at least one \( x \) in the domain
Evaluating Predicates with Universal Quantifiers

What if we wanted to write a program to evaluate $\forall x \ P(x)$?

To evaluate $\forall x \ P(x)$ for finite domain,

- Loop through all $x$ in the domain
- If at a step, $P(x)$ is false
  then $\forall x \ P(x)$ is false (can terminate loop early)
- If at every step, $P(x)$ is true
  then $\forall x \ P(x)$ is true
Evaluating Predicates with Existential Quantifiers

What if we wanted to write a program to evaluate $\exists x \ P(x)$ for a finite domain?

- Loop through all $x$ in the domain

- If at some step, $P(x)$ is true then $\exists x \ P(x)$ is true (loop terminates early)

- If the loop ends without finding an $x$ for which $P(x)$ is true, then $\exists x \ P(x)$ is false
Quantifiers and Empty Domains

What is truth value for quantified propositions when the domain is the empty set?

• $\forall x \ P(x)$ is always true.
  • To be true, need $P(x)$ to be true for every $x$ in the domain.
  • Trivially satisfied if domain is empty.

• $\exists x \ P(x)$ is always false.
  • To be true, need $P(x)$ to be true for some $x$ in the domain.
  • Since the domain is empty, this is impossible.
Uniqueness Quantifier

• \( \exists!x \ P(x) \) means that \( P(x) \) is true for one and only one \( x \) in the universe of discourse.

• We say
  • “There is a unique \( x \) such that \( P(x) \).”
  • “There is one and only \( x \) such that \( P(x) \).”
  • “There exists exactly one \( x \) such that \( P(x) \).”

• Let \( P(x) \) denote \( \text{“} x^2 = 4 \text{”} \).

  • If the domain is the positive integers, is \( \exists!x \ P(x) \) true or false?
  • If the domain is all integers, is \( \exists!x \ P(x) \) true or false?
Precedence of Quantifiers

• The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

• For example, $\forall x \ P(x) \lor Q(x)$ means $(\forall x \ P(x)) \lor Q(x)$
  • $\forall x \ (P(x) \lor Q(x))$ means something different.

• Unfortunately, often people write $\forall x \ P(x) \lor Q(x)$ when they mean $\forall x \ (P(x) \lor Q(x))$. 
Translating from English to Logic

• Translate the following sentence into predicate logic: “Every student in this class has taken a course in Java.”

• First decide on the domain.

• If the domain is all students in this class, define \( J(x) \) denoting “\( x \) has taken a course in Java”

• But if the domain is all people, also define \( S(x) \) denoting “\( x \) is a student in this class”
Translating from English to Logic

Translate the following into predicate logic:
“Some student in this class has taken a course in Java.”

First decide on the domain
• If the domain is all students in this class:

• If the domain is all people:
More Translation

Let $L(x)$ denote “$x$ is a lion.”
Let $C(x)$ denote “$x$ drinks coffee.”
Let $F(x)$ denote “$x$ is fierce”
The domain is all creatures.

- Translate the following into predicate logic.
  - All lions are fierce.
  - Some lions drink coffee.
  - Some fierce creatures do not drink coffee.
And Some More Translation

Let C(x) denote “x is a cat”
Let F(x) denote “x has fur”
Let W(x) denote “x is weird looking”
The domain is all animals.

- Translate the following into predicate logic:
  - Some cats do not have fur.
  - All cats without fur are weird-looking.
Translating Logic into English

Let $C(x)$ denote “$x$ is a cat”
Let $F(x)$ denote “$x$ has fur”
Let $W(x)$ denote “$x$ is weird looking”
The domain is all animals.

- Translate the following from logic to English

\[ \exists x (C(x) \land \neg F(x) \land \neg W(x)) \]

\[ \forall x (W(x) \rightarrow \neg C(x)) \]
The Socrates Example: Revisited

“All men are mortal. Socrates is a man.
Does it follow that “Socrates is mortal?”

• Let the domain be all people.
• Let Man(x) denote “x is a man”
• Let Mortal(x) denote “x is mortal”
• Translation:
Equivalences in Predicate Logic

• Statements involving predicates and quantifiers are **logically equivalent** if and only if they have the same truth value
  • for every predicate substituted into these statements and
  • for every domain used for the variables in the expressions.

• E.g  \( \forall x \, \neg \neg P(x) \equiv \forall x \, P(x) \)
Negating Quantified Expressions

- Let $P(x)$ denote “$x$ has taken a philosophy course.”
- The domain is students in this class.
- Consider $\forall x \ P(x)$
  - “Every student in this class has taken a philosophy course.”
- Negating the original statement gives $\neg \forall x \ P(x)$
  - “It is not the case that every student in this class has taken a philosophy course.”
  - This implies that “There is a student in this class who has not taken a philosophy course.”

- $\neg \forall x \ P(x)$ and $\exists x \ \neg P(x)$ are logically equivalent.
Negating Quantified Expressions (cont.)

- Now consider $\exists x \ Q(x)$
  - “There is a student in this class who has taken a music course.”

- Negating the original statement gives $\neg \exists x \ Q(x)$
  - “It is not the case that there is a student in this class who has taken a music course.”
  - This implies that “Every student in this class has not taken a music course.”

- $\neg \exists x \ Q(x)$ and $\forall x \ \neg Q(x)$ are logically equivalent.
De Morgan’s Laws for Quantifiers

$$\neg\forall x P(x) \equiv \exists x \neg P(x) \quad \neg\exists x P(x) \equiv \forall x \neg P(x)$$

Remember DeMorgan’s Laws for Propositional Logic

$$\neg(p \land q) \equiv \neg p \lor \neg q \quad \neg(p \lor q) \equiv \neg p \land \neg q$$
Translating with Negation

Let $K(x)$ denote “$x$ has a cat”
Let $D(x)$ denote “$x$ has a dog”
Let $F(x)$ denote “$x$ has a ferret”
Domain: students in this class

• Translate:
  • No student in this class has a cat, a dog, and a ferret.
Nested Quantifiers

• Sometimes, we need a quantifier inside a quantified statement. E.g. “For every real number, there is a real number that is larger than it.” The domain is the real numbers.

\[ \forall x \exists y \ (y > x) \]

• Literally, “For every real number x, there exists a real number y such that y is greater than x.”.

• Can define propositional function of two variables, i.e., \( P(x,y) \) denotes “\( y > x \)”.
  • Logical statement is \( \forall x \exists y \ P(x, y) \)

• *Can have different domains for x and y.*
Examples with Nested Quantifiers

• Let \( P(x) \) be “\( x = 0 \)”
• Let \( Q(x,y) \) be “\( x + y = x - y \)”
• The domain for \( x \) and \( y \) is the integers.

• Are the following true or false?
  • \( \forall y \, Q(1,y) \)
  • \( \forall x \, \forall y \, Q(x,y) \)
  • \( \forall x \, \exists y \, Q(x,y) \)
  • \( \forall x \, \forall y \, P(y) \rightarrow Q(x,y) \)
Translating English to Logic

- Translate: “Every movie actor has either been in a movie with Kevin Bacon or has been in a movie with someone who has been in a movie with Kevin Bacon.”

- $M(x,y)$ denotes “$x$ has been in a movie with $y$”
- The domain is all movie actors.
Translating Mathematical Statements into Logic

Translate “The sum of two positive integers is always positive.” into a logical expression.

• First, rewrite the sentence (in English) to make the domains and quantifiers explicit:
  • “For every two integers x and y, if both x and y are positive, then the sum of x and y is positive.”

• Then, translate into logic:
  • How do you translate “x is a positive number”?
  • How do you translate “the sum of x and y is positive”?
  • Full translation: (domain is the integers)
Translating back to English

• Translate the following to English. The domain is the real numbers.

\[ \exists x \ \forall y \ (xy = y) \]

• “There is a multiplicative identity for the real numbers.”

\[ \forall x \ \forall y \ (((x < 0) \land (y < 0)) \rightarrow (xy > 0)) \]

• “The product of two negative numbers is positive.”

\[ \forall x \ \forall y \ \exists z \ (x + y = z) \]

• “The real numbers are closed under addition.”
Order of quantifiers

Can you switch the order of quantifiers?
Good Problems to Review

• Section 1.4: 7, 9, 11, 13, 15, 35
• Section 1.5: 9, 11, 25, 27, 31