Announcements

• Homework 3 is due now.
• Exam 1 is this Thursday in class – more info at the end of lecture today.

• Late policy change for homework 3 only.
  • You can turn it in in this Wednesday’s recitation for 50% credit.
  • I will post solutions at 5pm, after last recitation.
  • No homework will be accepted on Thursday.
• **Theorem**: If \( n \) is a composite number, then \( n \) has a prime divisor less than or equal to \( \sqrt{n} \).

• We will show that, if \( n \) is a composite number, then \( n \) has a divisor \( d \) that is less than or equal to \( \sqrt{n} \).

• By the Fundamental Theorem of Arithmetic, it follows that \( d \) is either prime or the product of primes, so \( n \) must have a prime divisor that is less than or equal to \( \sqrt{n} \).
• Lemma: \((n \text{ is a composite number}) \rightarrow (n \text{ has a divisor } d \leq \sqrt{n})\)
Checking if a number is prime

- **Theorem**: If \( n \) is a composite integer, then \( n \) has a prime divisor less than or equal to \( \sqrt{n} \).

- Show that 101 is prime.
SETS

Section 2.1
Introduction

• Sets are one of the basic building blocks of discrete mathematics.
  • Important for counting.
  • Used for representing relations between objects.
    • * Graphs

• Set theory is an important branch of mathematics.
  • Many different systems of axioms have been used to develop set theory.

• We are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.
  • Naïve set theory describes the subject in natural language.
Sets

- A set is an unordered collection of objects.
  - the students enrolled in this class
  - the chairs in this room
- The objects in a set are called the elements, or members of the set. A set is said to contain its elements.
- The notation $a \in A$ denotes that $a$ is an element of the set $A$.
- If $a$ is not a member of $A$, we write $a \notin A$. 
Defining a Set

- We can describe a set by listing all of its elements. This is called the **roster method**.
  - $S = \{a, b, c, d\}$
- Order does not matter in a set
  - $S = \{a, b, c, d\} = \{b, c, a, d\}$
- Each distinct object is either a member or not; listing an object more than once does not change the set.
  - $S = \{a, b, c, d\} = \{a, a, a, a, b, c, b, c, d\}$
- Ellipses (...) may be used to describe a set without listing all of the members when the pattern is clear.
  - $S = \{a, b, c, d, ..., z\}$
Roster Method

- \( V = \{a, e, i, o, u\} \)
  - Set of all vowels in the English alphabet

- \( O = \{1, 3, 5, 7, 9\} \)
  - Set of all odd positive integers less than 10

- \( S = \{..., -3, -2, -1\} \)
  - Set of all integers less than 0

- \( S = \{1, ..., 100\} \)
  - Ambiguous – Don’t do this.
Some Important Sets

\( N = \textit{natural numbers} = \{0,1,2,3,\ldots\} \)

\( Z = \textit{integers} = \{-\ldots,-3,-2,-1,0,1,2,3,\ldots\} \)

\( Z^+ = \textit{positive integers} = \{1,2,3,\ldots\} \)

\( R = \text{set of real numbers} \)

\( R^+ = \text{set of positive real numbers} \)

\( C = \text{set of complex numbers} \)

\( Q = \text{set of rational numbers} \)
Set Builder Notation

• Can also describe a set using **set builder notation**. Specify the property or properties that all members must satisfy:

  \[ S = \{ x \mid x \text{ is a positive integer less than 100} \} \]

  \[ O = \{ x \mid x \text{ is an odd positive integer} \} \]

• Can also use predicates in set builder notation:
  Example: \( P(x) \): \( \exists y \in \mathbb{Z} \) such that \( y^2 = x \)
  • What is \( S = \{ x \mid P(x) \} \)?
  • The set of all perfect squares.
Interval Notation

We can also define a set using interval notation.

\[
[a,b] = \{ x \mid a \leq x \leq b \}
\]
\[
[a,b) = \{ x \mid a \leq x < b \}
\]
\[
(a,b] = \{ x \mid a < x \leq b \}
\]
\[
(a,b) = \{ x \mid a < x < b \}
\]

*closed interval*  \([a,b]\)

*open interval*  \((a,b)\)
Universal Set and Empty Set

• The **universal set** $U$ is the set containing everything currently under consideration.
  • Sometimes implicit.
  • Sometimes explicitly stated.
  • Contents depend on the context.

• The **empty set** is the set with no elements. Denoted by $\emptyset$ or {}.
Some things to remember

• Sets can be elements of sets.
  \[ \{\{1, 2, 3\}, a, \{b, c\}\} \]
  \[ \{\text{N, Z, Q, R}\} \]

• The empty set is different from a set containing the empty set.
  \[ \emptyset \neq \{\emptyset\} \]
Set Equality

**Definition:** Two sets are equal if and only if they have the same elements.

- We write $A = B$ if the set $A$ is equal to the set $= B$
  
  \[
  \{1,3,5\} = \{3, 5, 1\} \\
  \{1,5,5,5,3,3,1\} = \{1,3,5\}
  \]
Subsets

**Definition:** The set $A$ is a **subset** of $B$, if and only if every element of $A$ is also an element of $B$.

- The notation $A \subseteq B$ denotes “$A$ is a subset of $B$”.

- Every set is a subset of itself, i.e., $S \subseteq S$, for every set $S$.

- The empty set is a subset of every set, i.e., $\emptyset \subseteq S$, for every set $S$.

Why?
Proper Subsets

• We say $A$ is a proper subset of $B$, written $A \subset B$, if $A \subseteq B$ and $A \neq B$.

• Example: $\mathbb{Z} \subset \mathbb{R}$
Quick Quiz

• True or false?

1. $0 \in \emptyset$  \hspace{1cm} False
2. $\{0\} \subset \{0\}$  \hspace{1cm} False
3. $\emptyset \in \{0\}$  \hspace{1cm} False
4. $\{\emptyset\} \subseteq \emptyset$  \hspace{1cm} False
5. $\emptyset \subseteq \emptyset$  \hspace{1cm} True
Another look at Equality of Sets

• Recall that two sets $A$ and $B$ are *equal*, denoted by $A = B$, iff

$$\forall x (x \in A \leftrightarrow x \in B)$$

• Using logical equivalences, $A = B$ iff

$$\forall x [(x \in A \to x \in B) \land (x \in B \to x \in A)]$$

• This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$
Set Cardinality

• The **cardinality** of a finite set $A$, denoted by $|A|$, is the number of distinct elements of $A$.

• If there are exactly $n$ distinct elements in $S$ where $n$ is nonnegative integer, we say that $S$ is **finite**. Otherwise it is **infinite**.
  • The set of integers is an example of an infinite set.

• What is the cardinality of the following sets?
  1. $|\{a, e, i, o, u\}| = 5$
  2. $|\{1,2,1,3\}| = 3$
  3. $|\emptyset| = 0$
  4. $|\{\emptyset\}| = 1$
Power Sets

• The **power set** of a set $A$ is the set of all subsets of $A$.
  • Let $A = \{a,b\}$. What is $\mathcal{P}(A)$?

• If a set has $n$ elements, what is the cardinality of its power set?
Tuples

- Sometimes, the order of a collection is important.
- The **ordered n-tuple** \((a_1,a_2,\ldots,a_n)\) is the ordered collection that has \(a_1\) as its first element and \(a_2\) as its second element, \(\ldots\), and \(a_n\) as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
  - The ordered tuples \((a,b)\) and \((c,d)\) are equal if and only if \(a = c\) and \(b = d\).
- 2-tuples are called **ordered pairs**.
Cartesian Product

• The **Cartesian product** of two sets $A$ and $B$, denoted $A \times B$, is the set of ordered pairs $(a,b)$ where $a \in A$ and $b \in B$.

• Example:
  
  $A = \{a,b\} \quad B = \{1,2,3\}$

• If $|A| = m$ and $|B| = n$, what is the cardinality of $A \times B$?
Cartesian Product (cont.)

- The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for $i = 1, \ldots, n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

- What is $A \times B \times C$ where $A = \{0, 1\}$, $B = \{1\}$ and $C = \{0, 1, 2\}$?

- The notation $A^2$ is used to denote $A \times A$.
  - Can also write $A^3, A^4$, etc.
Truth Sets of Quantifiers

• Given a predicate $P$ and a domain $D$, the truth set of $P$ is the set of elements in $D$ for which $P(x)$ is true. The truth set of $P(x)$ is denoted by

$$\{x \in D | P(x)\}$$

• Let $D$ be the integers
  • Let $P(x)$ be “$|x| = 1$”. What is the truth set?

• Let $Q(x)$ be “$x^2 < 3$”. What is the truth set?
SET OPERATIONS

Section 2.2
Union

• The **union** of the sets $A$ and $B$ is denoted by $A \cup B$
Intersection

- The **intersection** of sets $A$ and $B$ is denoted by $A \cap B$.

Venn Diagram for $A \cap B$

- If the intersection is empty, then $A$ and $B$ are said to be **disjoint**.
The Cardinality of the Union of Sets

• What is the cardinality of $A \cup B$?

Venn Diagram for $A, B, A \cap B, A \cup B$

• This can be generalized to the union of more than two sets.
  • It is called the **inclusion-exclusion principle**.
Difference

- Let $A$ and $B$ be sets. The **difference** of $A$ and $B$, denoted $A - B$, is the set containing the elements of $A$ that are not in $B$.
- $A - B$ is sometimes denoted $A \setminus B$

Venn Diagram for $A - B$
Complement

- If $A$ is a set, then the complement of $A$ (with respect to $U$), is the set $U - A$. It is denoted by $\bar{A}$.
  - The complement of $A$ is sometimes denoted by $A^c$.

Venn Diagram for Complement
• Example: Prove that $A \cap B = \overline{A \cup \overline{B}}$

\[
A \cap B = \exists x \mid x \in A \cap B \ \exists \\
= \exists x \mid x \in A \cap B \ \text{def of complement} \\
= \exists x \mid \neg (x \in A \cap B) \ \text{def of } \neg \\
= \exists x \mid \neg (x \in A \land x \in B) \ \text{def of intersection} \\
= \exists x \mid \neg (x \in A) \lor \neg (x \in B) \ \text{De Morgan's Law} \\
= \exists x \mid x \notin A \lor x \notin B \ \text{def. of } \neg \\
= \exists x \mid x \in \overline{A} \lor x \in \overline{B} \ \text{def. of complement} \\
= \exists x \mid x \in \overline{A} \cup \overline{B} \ \text{def. of union} \\
QED
\]

De Morgan's Laws for Sets
Exam 1 - logistics

• The exam will last 100 minutes.
• There will be approximately 8 problems.
• You can bring a 2-sided 8 ½ by 11 inch sheet of notes.
  • It can be typed or handwritten.
• Logical equivalences and rules of inference will be provided (see course web site).
• For students who get special accommodations, I will email you today with your exam time and location.
• The TAs will go over review problems in recitation on Wednesday.
  • Send email to me if you have requests.
Exam 1 - content

• Everything covered in lecture and homework is fair game (including today’s material).

• Questions will be similar to those on the homework and the “good problems to review”.
Good Problems to Review

- Section 2.1: 5, 7, 9, 11, 23, 25, 27, 33
- Section 2.2: 11, 13, 29, 35