Announcements

• Homework 1 is due now.

• Homework 2 will be posted on the web site today.
  • It is due Thursday, Feb. 12 at 10am in class.
Can you switch the order of quantifiers?

\[ \forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y) \]

- Yes! The order in which \( x \) and \( y \) are chosen does not matter.

- \[ \forall x \exists y P(x, y) \equiv \exists y \forall x P(x, y) \]
  - No!
  - For domain of real numbers:
    \[ \forall x \exists y (x+y=0) \text{ TRUE} \]
    \[ \exists x \forall y (x+y=0) \text{ FALSE} \]
More about Quantifiers

Can you distribute over logical connectives?

\[ \forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x) \quad \text{Yes!} \]

\[ \forall x (P(x) \to Q(x)) \not\equiv \forall x P(x) \to \forall x Q(x) \quad \text{No!} \]

For two statements to be logically equivalent, they must have the same truth values for any choice of predicates and domains.
Let $P(x)$ denote “$x$ is a cat”. Let $Q(x)$ denote “$x$ has fur”. Let the domain be all animals.

\[
\forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)
\]

\[
\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)
\]
Do We Need the Uniqueness Quantifier?

- Construct a logical statement that is equivalent to $\exists!x \ P(x)$
  “There exists exactly one $x$ such that $P(x)$”
- Hint: use nested quantifiers
Negation with Nested Quantifiers

• Recall DeMorgan’s Laws for Quantifiers.

\[-\forall x P(x) \equiv \exists x \neg P(x) \quad \neg \exists x P(x) \equiv \forall x \neg P(x)\]
Logic Programming

• Prolog (from Programming in Logic) is a programming language developed in the 1970s by researchers in AI.
• Prolog programs include Prolog facts and Prolog rules.
• As an example of a set of Prolog facts consider the following:

  instructor(chan, math273).
  instructor(patel, ee222).
  instructor(grossman, cs301).
  enrolled(kevin, math273).
  enrolled(juna, ee222).
  enrolled(juana, cs301).
  enrolled(kiko, math273).
  enrolled(kiko, cs301).

• Here the predicates instructor(p,c) and enrolled(s,c) represent that professor p is the instructor of course c and that student s is enrolled in course c.
Logic Programming (cont)

- In Prolog, names beginning with an uppercase letter are variables.
- If we have a predicate \( teaches(p,s) \) representing “professor \( p \) teaches student \( s \),” we can write the rule:
  \[
  teaches(P,S) :- 
  instructor(P,C),
  enrolled(S,C).
  \]
- This Prolog rule can be viewed as equivalent to the following statement in logic (using our conventions for logical statements).
  \[
  \forall p \forall c \forall s \ (I(p,c) \land E(s,c) \implies T(p,s))
  \]
LOGICAL ARGUMENTS
Arguments in Propositional Logic

• An **argument** is a sequence of propositional statements.
  • All but the last proposition are called **premises**.
  • The last proposition is called the **conclusion**.

• Example:
  If you have a valid login, then you can access your account info.
  You have a valid login.
  
  \[ \therefore \]
  You can access your account info.

• An argument is valid if the truth of all its premises combined implies the conclusion is true.
Argument Forms for Propositional Logic

• Instead of using truth table, we determine whether an argument is valid using established rules of inference for argument forms.

• An argument is a sequence of propositions.

• An argument form is a sequence of propositional variables.

If you have a valid login, then you can access your account info.
You have a valid login.

You can access your account info.

\[
p \rightarrow q
\]

\[
p
\]

\[
q
\]

\[
\therefore
\]

Argument Form
Valid Arguments

\[ p \rightarrow q \]

\[ \begin{array}{c}
p \\
\hline
\rightarrow
\end{array} \]

\[ q \]

\[ \therefore \]

• Rewrite as proposition \(((p \rightarrow q) \land p) \rightarrow q\)

• For an argument to be valid, this proposition must be a tautology.

\[ ((p \rightarrow q) \land p) \rightarrow q \equiv T \]

• Rules of inference are valid argument forms.
  • Argument is valid no matter what propositions are substituted for the propositional variables.
Rules of Inference: Modus Ponens

\[ p \rightarrow q \]

\[ p \]

\[ \therefore q \]

Corresponding Tautology:

\[ ((p \rightarrow q) \land p) \rightarrow q \]

Example:
Let \( p \) be “It is snowing.”
Let \( q \) be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”
“It is snowing.”

“Therefore, I will study discrete math.”
Rules of Inference: Modus Tollens

\[ p \rightarrow q \]
\[ \neg q \]
\[ \therefore \neg p \]

Corresponding Tautology: \((\neg q \land (p \rightarrow q)) \rightarrow \neg p\)

**Example:**
Let \( p \) be “It is snowing.”
Let \( q \) be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”
“I will not study discrete math.”

“Therefore, it is not snowing.”
Rules of Inference: Hypothetical Syllogism

$\begin{align*}
   p \rightarrow q \\
   q \rightarrow r \\
   \therefore p \rightarrow r
\end{align*}$

Corresponding Tautology:

$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example:

Let $p$ be “It snows.”
Let $q$ be “I will study discrete math.”
Let $r$ be “I will get an A.”
Rules of Inference: Disjunctive Syllogism

\[ p \lor q \quad \neg p \]
\[ \therefore \quad q \]

**Corresponding Tautology:**
\[ (\neg p \land (p \lor q)) \rightarrow q \]

**Example:**
Let \( p \) be “I will study discrete math.”
Let \( q \) be “I will study English literature.”
Rules of Inference: Addition

\[
\begin{align*}
    p \\
    \hline
    \Rightarrow p \lor q
\end{align*}
\]

Corresponding Tautology:

\[ p \rightarrow (p \lor q) \]

Example:
Let \( p \) be “I will study discrete math.”
Let \( q \) be “I will visit Las Vegas.”
Rules of Inference: Simplification

\[ \frac{p \land q}{\therefore q} \]

Corresponding Tautology:
\[ (p \land q) \rightarrow p \]

Example:
Let \( p \) be “I will study discrete math.”
Let \( q \) be “I will study English literature.”
Rules of Inference: Conjunction

\[
p
q
\]

\[
\therefore p \land q
\]

Corresponding Tautology:

\[
((p) \land (q)) \rightarrow (p \land q)
\]

Example:
Let \( p \) be “I will study discrete math.”
Let \( q \) be “I will study English literature.”
Rules of Inference: Resolution

\[ \neg p \lor r \]
\[ p \lor q \]
\[ \therefore q \lor r \]

**Corresponding Tautology:**

\[ (((\neg p \lor r) \land (p \lor q)) \rightarrow (q \lor r)) \]

**Example:**
Let \( p \) be “I will study discrete math.”
Let \( r \) be “I will study English literature.”
Let \( q \) be “I will study databases.”
A Valid Argument?

• “If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics. Therefore, you did every problem in this book.”
• Let $p$ denote “You did every problem in this book.”
• Let $q$ denote “You learned discrete mathematics.”

$$p \rightarrow q$$

What is the argument form:

$$\begin{array}{c}
q \\
\hline
p
\end{array}$$

• Is this a valid argument? Is a tautology?
  • NO: If $p$ is false and $q$ is true, then proposition is false.

• This is the fallacy of affirming the conclusion.
Another Fallacy

- Another common incorrect argument is the **fallacy of denying the hypothesis**.

\[ p \rightarrow q \]

\[ \neg p \]

\[ \therefore \neg q \]

- Is this a tautology?

\[ (p \rightarrow q) \land \neg p ) \rightarrow \neg q \]

- Let \( p \) denote “You did every problem in this book.”
- Let \( q \) denote “You learned discrete mathematics.”

- It is not correct to assume that if you did not do every problem in this book, then you did not learn discrete mathematics. You may have learned it another way.
Quick Quiz

Are the following arguments valid? Why or why not?

• “If $n$ is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.”

• If $n$ is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n \leq 3$. Then $n^2 \leq 9$. 
Using Rules of Inference to Build Valid Arguments

• If your problem structure matches a well-known argument form, you can apply that form directly.

• Sometimes, we are not so lucky…..
  • We may need to apply several rules of inference to show that an argument is valid.
  • Combine rules of inference we learned last week with these argument forms.
Using the Rules of Inference to Build Valid Arguments

• A **valid argument** is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.

• A valid argument takes the following form:

\[ S_1 \]
\[ S_2 \]
\[ \ldots \]
\[ S_n \]

\[ \therefore \quad C \]
Valid Argument - Example

Show that the following is a valid argument:

\[
p \land (p \rightarrow q) \quad :\quad q
\]
It’s not sunny today, and it’s colder than yesterday. We’ll go swimming only if it’s sunny. If we don’t go swimming, we’ll take a canoe trip. If we take a canoe trip, we’ll be home by sunset. Therefore, we’ll be home by sunset.

• Propositions:
  • p: “It’s sunny today”
  • q: “It’s colder than yesterday”
  • r: “We’ll go swimming”
  • s: “We’ll take a canoe trip”
  • t: “We’ll be home by sunset”
A Valid Argument?

\[
\begin{align*}
\neg p & \land q \\
r & \rightarrow p \\
\neg r & \rightarrow s \\
s & \rightarrow t \\
\therefore & t
\end{align*}
\]
Handling Quantified Statements

• Valid arguments for quantified statements are just like valid arguments for propositional statements.

• Each statement is either a premise or follows from previous statements by rules of inference which include:
  • Rules of Inference for Propositional Logic
  • Rules of Inference for Quantified Statements
Universal Instantiation (UI)

\[ \forall x P(x) \quad \therefore P(c) \]

**Example:**
P(x) denotes “x is cute”.
The domain is all dogs. 
c is some element of the domain. 
Let’s call c “Taz”.

“All dogs are cute.”
“Therefore, Taz is cute.”
Universal Generalization (UG)

\[ \frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)} \]

Used often implicitly in mathematical proofs.
Existential Instantiation (EI)

\[ \exists x P(x) \]
\[ \therefore P(c) \text{ for some element } c \]

**Example:**

P(x) denotes “x got an A in the class”
The domain is all people in the class.

“There is someone who got an A in the course.”
“Let’s call her c and say that c got an A”
Existential Generalization (EG)

\[
P(c) \text{ for some element } c \\
\therefore \exists x P(x)
\]

**Example:**
Let’s call \( c \) “Michelle”.
\( P(x) \) denotes “\( x \) got an A in the class”
The domain is all people in the class.

“Michelle got an A in the class.”
“Therefore, someone got an A in the class.”
Using Rules of Inference

• Construct a valid argument to show that
  “Jerry has four legs” is a consequence of the premises:
  “Every mouse has four legs” and “Jerry is a mouse”.

Let $M(x)$ denote “$x$ is a mouse.”
Let $L(x)$ denote “$x$ has four legs.”
Let the domain be all creatures.
The Socrates Example: Revisited Again

“All men are mortal. Socrates is a man. Does it follow that “Socrates is mortal?”

- Let the domain be all creatures.
- Let $\text{Man}(x)$ denote “$x$ is a man”
- Let $\text{Mortal}(x)$ denote “$x$ is mortal”

- Premises: $\forall x (\text{Man}(x) \rightarrow \text{Mortal}(x))$
  $\text{Man}(\text{Socrates})$
- Conclusion: $\text{Mortal}(\text{Socrates})$

- Is this a valid argument?
Universal Modus Ponens

Universal Modus Ponens combines Universal Instantiation and Modus Ponens into one rule.

\[ \forall x (P(x) \rightarrow Q(x)) \]

\[ P(a), \text{ where } a \text{ is a particular element in the domain} \]

\[ \therefore Q(a) \]

This rule could be used in the Socrates example.
MATHEMATICAL PROOFS
Valid Arguments vs. Mathematical Proofs

- The valid arguments we just did are examples of formal proofs.
  - Each statement follows logically from preceding statements.
  - Only one rule of inference is used per step.
- Formal proofs can be extremely long and hard to follow (for humans).

- Proofs of mathematical theorems for human consumption are often informal proofs.
  - *though people often call these proofs “formal proofs”
- In informal proofs: more than one rule of inference may be used per step, steps may be skipped, premises (axioms) may not be clearly identified.
- Informal proofs must still be logically correct.
Good Problems to Review

- Section 1.6: 3, 13, 15, 19