1. (15 points) Suppose each non-leaf node of an $m$-ary has exactly $m$ children — in other words, each node of the tree has either exactly 0 or exactly $m$ children. Suppose we build such a tree of height $h \geq 0$ having the minimum possible number of nodes. What is the number of nodes in this “minimal” tree? Prove your answer using mathematical induction.

**Solution:** The answer is $mh + 1$. The proof using mathematical induction is as follows.

**Basis case:** For $h = 0$, there is a single node in the tree — the root node. Since $m \cdot 0 + 1 = 1$, the basis case is proved.

**Induction step:** For $h > 0$, suppose each minimal tree of height $i$, $0 \leq i < h$, has $m \cdot i + 1$ nodes, and consider a minimal tree of height $h$. Call this tree $T$ and consider the subtrees of the root. $m - 1$ of these subtrees have just a single node, otherwise $T$ wouldn’t be minimal. The other subtree is a minimal tree of height $h - 1$. By the inductive hypothesis, it has $m(h - 1) + 1$ nodes. Adding to this the $m - 1$ individual nodes and 1 for the root implies that $T$ has $m(h - 1) + 1 + m - 1 + 1 = mh + 1$ nodes.